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Faedo-Galerkin method for heat equation

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Abstract

This paper is devoted to existence of the heat equation in bounded domains. we prove the existence, uniqueness, stability and regularity, the equation with a homegeneous Dirichlet condition by Faedo-Galerkin's method.

AMS subject classification:
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1. Introduction

The one-dimensional diffusion equation is the linear second order partial differential equation $u_t - Du_{xx} = f$ where $u = u(x, t)$, x is a real space variable, t a time variable and D a positive constant, called diffusion coefficient. In space dimension $n > 1$, that is when $x \in \mathbb{R}^n$, the diffusion equation reads

$$u_t - D\Delta u = f \tag{1.1}$$

When $f \equiv 0$ the equation is said to be homogeneous. A common example of diffusion is given by heat conduction in a solid body. Conduction comes from molecular collision, transferring heat by kinetic energy, without macroscopic material movement. If the medium is homogeneous and isotropic with respect to the heat propagation, the evolution of the temperature is described by equation (1.1); f represents the intensity of an external distributed source. For this reason equation (1.1) is also known as the heat equation. On the other hand equation (1.1) constitutes a much more general diffusion model, where by diffusion we mean, for instance, the transport of a substance due to the molecular motion of the surrounding medium. In this case, u could represent the concentration of a polluting material or of a solute in a liquid or a gas (dye in a liquid, smoke in the atmosphere) or even a probability density. We may say that the diffusion equation unifies at a macroscopic scale a variety of phenomena, that look quite different when observed at a microscopic scale. Through equation (1.1) and some of its variants.

In this paper, we consider the heat aquation and the Dirichlet condition

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(x,t) & in\ \Sigma = \Omega\times]0,T[\\ u(x,0) = g(x) & in\ \Omega \\ u(\sigma,x) = 0 & on\ \Gamma = \partial\Omega\times]0,T[\end{cases} \tag{1.2}$$

Let Ω is a bounded open set in $\mathbb{R}^n$, $[0,T]$ is a finite time interval, $f(x,t)$ real functions.

We want to find a weak formulation. Let us proceed formally. As we did several times, we multiply the diffusion equation by a smooth function $v = v(x)$, vanishing at the boundary of Ω, and integrate over Ω. We find

$$\int_\Omega u_t(x,t)v(x)dx - \int_\Omega \Delta u(x,t)v(x)dx = \int_\Omega f(x,t)v(x)dx$$

Integrating by parts the second term[4], we get

$$\int_\Omega u_t(x,t)v(x)dx + \int_\Omega \nabla u(x,t)\nabla v(x)dx = \int_\Omega f(x,t)v(x)dx \tag{1.3}$$

First of all, since we are dealing with evolution equations, it is convenient to adopt the point of Spaces Involving Time, and consider.

2. Preliminaries

$u = u(x,t)$ as a function of t with values into a suitable Hilbert space V:

$$u : [0,T] \to V.$$

When we adopt this convention, we write $u(t)$ instead of $u(x,t)$ and $\dot{u}$ instead of u_t. Accordingly, we write $f(t)$ instead of $f(x,t)$. With these notations, (1.3) becomes

$$\int_\Omega \dot{u}v(x)dx + \int_\Omega \nabla u(t)\nabla v(x)dx = \int_\Omega f(t)v(x)dx \tag{2.4}$$

The homogeneous Dirichlet condition, i.e. $u(t) = 0$ on $\partial\Omega$ for $t \in [0,T]$, suggests that the natural space for $u(t)$ is $V = H_0^1(\Omega)$, at least for a.e. $t \in [0,T]$. As usual, in $H_0^1(\Omega)$ we use the inner product

$$(w,v)_1 = (\nabla w, \nabla v)_0$$

with corresponding norm $\|.\|_1$. Thus, the second integral in (1.3) may be written in the form

$$(\nabla u(t), \nabla v)_0$$

Also, it would seem to be appropriate that $\dot{u} \in L^2(\Omega)$, looking at the first integral. This however is not coherent with the choice $u(t) \in H_0^1(\Omega)$, since we have $\Delta u(t) \in H^{-1}(\Omega)$ and

$$\dot{u}(t) = \Delta u(t) + f(t) \tag{2.5}$$

from the diffusion equation. Thus, we deduce that $H^{-1}(\Omega)$ is the natural space for $\dot{u}$ as well. Consequently, the first integral in (2.4) has to be interpreted as

$$\langle \dot{u}(t), \nabla v \rangle_*$$

where $\langle ., . \rangle_*$ denotes the pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. A reasonable hypothesis on f is $f \in L^2(\Sigma)$, which in the new notations becomes. Also
$f \in L^2(0, T; V^*)$ is fine;$f \in L^2(0, T; L^2(\Omega))$ Coherently, from (2.5) we require $u \in L^2(0, T; H_0^1(\Omega))$ and $\dot{u} \in L^2(0, T; H^{-1}(\Omega))$.Now, we know that $u \in C(0, T; H_0^1(\Omega))$ so that the initial condition $u(0) = g$ makes perfect sense if we choose $g \in L^2(\Omega)$. The above arguments motivate the following definition. Consider the Hilbert triplet (V, H, V^*), where $V = H_0^1(\Omega)$,$H = L^2(\Omega)$ and $V^* = H^{-1}(\Omega)$. Recall that Poincaré's inequality holds in

$$V : \|v\|_0 \leq c\|v\|_1$$

Finally, let

$$a(w, v) = (\nabla w, \nabla v)_0.$$

Definition 2.1. A function $u \in L^2(0, T; V)$ is called weak solution of problem (1.2) if $\dot{u} \in L^2(0, T; V^*)$ and:
1. for every $v \in V$,

$$\langle \dot{u}(t), v \rangle_* + a(u(t), v) = (f(t), v)_0 \qquad a.e.t \in [0, T]. \tag{2.6}$$

2. $u(0) = g$.

Remark 2.2. Equation (2.6) may be interpreted in the sense of distributions. To see this, observe that, for every $v \in V$, the real function

$$w(t) = \langle \dot{u}(t), v \rangle_*$$

is a distribution in $\mathfrak{D}'(0, T)$ and[7]

$$\langle \dot{u}(t), v \rangle_* = \frac{d}{dt}(u(t), v)_0 \qquad \in \mathfrak{D}'(0, T). \tag{2.7}$$

This means that, for every $\varphi \in \mathfrak{D}(0, T)$, we have

$$\int_0^T \langle \dot{u}(t), v \rangle_* \varphi(t) dt = -\int_0^T (u(t), v)_0 \dot{\varphi}(t) dt$$

In fact, since $u(t) \in V$, by Bochner's Theorem

$$\left(u, \int_0^T f(t) dt \right)_V = \int_0^T (u, f(t))_V dt, \quad \forall u \in V$$

and the definition of u, we may write

$$\int_0^T \langle \dot{u}(t), v\rangle_* \varphi(t)dt = \langle \int_0^T \dot{u}(t)\varphi(t)dt, v\rangle_* = \langle -\int_0^T u(t)\dot{\varphi}(t)dt, v\rangle_*$$

On the other hand $\int_0^T u(t)\dot{\varphi}(t)dt \in V$ so that

$$\left\langle -\int_0^T u(t)\dot{\varphi}(t)dt, v\right\rangle_* = \left(-\int_0^T u(t)\dot{\varphi}(t)dt, v\right)_0 = -\int_0^T (\dot{u}(t), v)_0\dot{\varphi}(t)dt$$

Thus $w \in L^1_{loc}(0,T) \subset \mathfrak{D}'(0,T)$ and (2.7) is true. As a consequence, equation (2.6) may be written in the form

$$\frac{d}{dt}(u(t), v)_0 + a(u(t), v) = (f, v)_0$$
$$\text{in the sense of distributions in } \mathfrak{D}'(0,T), \text{ for all } v \in V. \tag{2.8}$$

Remark 2.3. We leave it to the reader to check that if a weak solution u is smooth i.e. $u \in C^{2,1}(\Sigma)$, then u is a classical solution.

Theorem 2.4. Let $\{x_k\} \subset H$ such that $x_k \rightharpoonup x$. Then

1) $\{x_k\}$ is bounded,

2) $\|x\| \leq \lim\limits_{k\to\infty} inf \|x_k\|$.

Proof. See [2]. ■

Theorem 2.5. Let $u \in L^2(0,T;V)$, with $\dot{u} \in L^2(0,T;V^*)$. Then

a) $u \in C([0,T];H)$ and,

$$\max_{0\leq t\leq T} \| u(t) \|_H \leq \{\|u\|_{L^2(0,T;V)} + \|\dot{u}\|_{L^2(0,T;V^*)}\}$$

b) If also $v \in L^2(0,T;V)$ and $\dot{v} \in L^2(0,T;V^*)$, the following integration by parts formula holds:

$$\int_s^t \{\langle \dot{u}(r), v(r)\rangle_* + \langle u(r), \dot{v}(r)\rangle_*\}dr = (u(t), v(t))_H - (u(s), v(s))_H$$

for all $s, t \in [0,T]$.

Proof. See [3]. ■

*Proposition*prop6Let $\{u_k\} \subset L^2(0, T; V)$, weakly convergent to u. Assume that

$$\sup_{0 \le t \le T} \|u_k(t)\|_V \le C$$

with C independent of k. Then, also,

$$\sup_{0 \le t \le T} \|u_(t)\|_V \le C.$$

Proof. See [9]. ■

3. Existence, uniqueness, regularity and stability

We want to show that problem (1.2) has exactly one weak solution, which depends continuously on the data in a suitable norm. Although there are variants of the Lax-Milgram Theorem perfectly adapted to solve evolution problems, we shall use the so-called Faedo-Galerkin method [5], also more convenient for numerical approximations.

Let us describe the main strategy. We divide into three steps. In Step 1 solution of the approximate problem. In Step 2 a priori estimates for u_n are derived. In Step 3 passage to limits, derive the regularity of $\dot{u}$ and justify the initial condition.

Step 1: Solution of the approximate problem: We select a sequence of smooth functions $\{wk\}_{k=1}^{\infty}$ constituting

$$\text{an orthogonal basis in } \; V = H_0^1(\Omega)$$

and

$$\text{an orthogonal basis in } \; L^2(\Omega).$$

In particular, we can write $\sum_{k=1}^{\infty} g_k w_k$ where $g_k = (g, w_k)_0$ and the series converges in $L^2(\Omega)$.

We construct the sequence of finite-dimensional subspaces

$$V_n = spam\{w_1, w_2, \dots, w_n\}.$$

Clearly

$$V_n \subset V_{n+1} \;\text{ and }\; \overline{\cup V_n} = V$$

For n fixed, let

$$u_n(t) = \sum_{k=1}^{\infty} c_k(t) w_k, \quad G_n = \sum_{k=1}^{n} g_k w_k. \tag{3.9}$$

We solve the following approximate problem:
Determine $u_n \in H^1(0, T, V)$, satisfying, for every $s = 1, \dots, n$,

$$\begin{cases} (\dot{u}_n(t), w_s)_0 + a(u_n(t), w_s) = (f(t), w_s)_0 \quad a.e\, t \in [0, T] \\ u_n(0) = G_n \end{cases} \tag{3.10}$$

Note that the differential equation in (3.10) is true for each element of the basis w_s, $s = 1, \dots, n$, if and only if it is true for every $v \in V_n$. Moreover, since $\dot{u}_n \in L^2(0, T; V)$, we have

$$(\dot{u}_n(t), v)_0 = \langle \dot{u}_n(t), v \rangle_*.$$

The following lemma holds:

Lemma 3.1. For all n, there exists a unique solution u_n of problem (3.10). In particular, since $u_n \in H^1(0, T; V_n)$, we have $u_n \in C([0, T]; Vm)$.

Proof. Since $w_1, \dots, w_n$ are mutually orthonormal in $L^2(\omega)$, we have

$$(\dot{u}_n(t), w_s)_0 = \left(\sum_{k=1}^{n} \dot{c}_k(t) w_k, w_s \right)_0 = \dot{c}_s(t)$$

Also $w_1, \dots, w_n$ is an orthogonal system in V_n, hence

$$a\left(\sum_{k=1}^{n} c_k(t) w_k, w_s \right) = (\nabla w_s, \nabla w_s)_0 c_s(t) = \|\nabla w_s\|_0^2 c_s(t)$$

Let

$$F_s(t) = (f(t), w_s), \quad \mathbf{F}_n(t) = (F_1(t), \dots, F_n(t))$$

Let

$$\mathbf{C}_n(t) = (C_1(t), \dots C_n(t), \quad g_n = (g_1, \dots, g_n)$$

If we introduce the diagonal matrix

$$\mathbf{W} = diag\left\{ \|\nabla w_1\|_0^2, \|\nabla w_2\|_0^2, \dots, \|\nabla w_n\|_0^2 \right\}$$

of order n, problem (3.10) is equivalent to the following system of n uncoupled linear ordinary differential equations, with constant coefficients:

$$\dot{\mathbf{C}}_n(t) = -\mathbf{W}\mathbf{C}_n(t) + \mathbf{F}_n(t) \quad a.e.\, t \in [0, T] \tag{3.11}$$

with initial condition

$$C_n(0) = g_n$$

Since $\mathbf{F} \in L^2(0, T; \mathbb{R}^n)$, there exists a unique solution $\mathbf{C} \in H^1(0, T; \mathbb{R}^n)$. From

$$u_n(t) = \sum_{k=1}^{n} c_k(t) w_k$$

we deduce that $u_n \in H^1(0, T; V_n)$. ■

Remark 3.2. We have chosen a basis $\{w_k\}$ orthonormal in L^2 and orthogonal in H_0^1 because with respect to this base, the Laplace operator becomes a diagonal operator, as it is reflected by the approximate problem (3.11). However, the method works using any countable basis for both spaces. Problem (3.10) becomes

$$\dot{\mathbf{C}}_n(t) = -\mathbf{M}^{-1}\mathbf{W}\mathbf{C}_n(t) + \mathbf{M}^{-1}\mathbf{F}_n(t) \quad a.e.\, t \in [0, T].$$

where, since $w_1, \ldots, w_n$ is a basis in V_m, the matrix M is positive, hence non singular.

$$M = (M_{sk}),\, M_{sk} = (w_s, w_k)_0, \quad W = (W_{sk}),\, W_{sk} = (\nabla w_s, \nabla w_k)_0.$$

This is particularly important in the numerical implementation of the method, where, in general, the elements of the basis in V_n are not mutually orthogonal.

Step 2 a priori estimates for u_n.
Our purpose is to show that we can extract from the sequence of Galerkin approximations $\{u_n\}$ a subsequence converging in some sense to a solution of problem (1.2). This is a typical compactness problem in Hilbert spaces. The key tool is theorem 2.4:

$$\|x\| \leq \lim_{k\to\infty} inf \|x_k\|. \tag{3.12}$$

Thus, what we need is to show that suitable Sobolev norms of um can be estimated by suitable norms of the data, and the estimates are independent of n. Moreover, these estimates must be powerful enough in order to pass to the limit as $n+\infty$ in the approximating equation

$$(\dot{u}_n, v)_0 + (\nabla u_n, \nabla v)_0 = (f, v)_0.$$

In our case we will be able to control the norms of $u_n\, in\, L^\infty(0, T; H)$ and $L^2(0, T; V)$, and the norm of $\dot{u}_n$ in $L^2(0, T; V^*.)$, that is the norms

$$\max_{t\in[0,T]} \|u_n(t)\|_0, \quad \int_\Omega \|u_n(t)\|_1^2 dt \text{ and } \int_\Omega \|\dot{u}_n(t)\|_*^2.$$

Thus, let $u_n(t) = \displaystyle\sum_{k=1}^{n} c_k(t) w_k$ be the solution of problem (3.10). ■

Theorem 3.3. (Estimate of u_n). For every $t \in [0, T]$, the following estimate holds:

$$\|u_n(t)\|_0^2 + \int_0^t \|u_n(s)\|_1^2 ds \leq \|g\|_0^2 + c^2 \int_0^t \|f(s)\|_0^2 ds \tag{3.13}$$

Proof. Multiplying equation (3.10) by $c_k(t)$ and summing for $k = 1, \ldots, n$, we get

$$(\dot{u}_n(t), u_n(t))_0 + a(u_n(t), u_n(t)) = (f(t), u_n(t))_0. \tag{3.14}$$

for a.e. $t \in [0, T]$. Now, note that

$$(\dot{u}_n(t), u_n(t))_0 = \frac{1}{2}\frac{d}{dt}\|u_n(t)\|_0^2, \quad a.e\ t \in (0, T)$$

and

$$a(u_n(t), u_n(t)) = \|\nabla u_n(t)\|_0^2 = \|u_n(t)\|_1^2.$$

From the inequalities of Schwarz and Poincar'e and the elementary inequality[1]

$$(f(t), u_n(t))_0 \leq \|f(t)\|_0\|u_n(t)\|_0 \leq C\|f(t)\|_0\|u_n(t)\|_1$$
$$\leq \frac{c^2}{2}\|f(t)\|_0^2 + \frac{1}{2}\|u_n(t)\|_1^2.$$

Thus, from (3.12) we obtain

$$\frac{d}{dt}\|u_n(t)\|_0^2 + \|u_n(t)\|_1^2 \leq c^2\|f(t)\|_0^2.$$

We now integrate over $(0, t)$, since $u_n(0) = G_n$ and observing that $\|G_n\|_0^2 \leq \|g\|_0^2$

$$\int_0^t \frac{d}{dt}\|u_n(s)\|_0^2 ds + \int_0^t \|u_n(s)\|_1^2 ds \leq c^2 \int_0^t \|f(s)\|_0^2 ds.$$

$$\|u_n(t)\|_0^2 - \|u_n(0)\|^2 + \int_0^t \|u_n(s)\|_1^2 ds \leq c^2 \int_0^t \|f(s)\|_0^2 ds.$$

by the orthogonality of $w_1, \ldots, w_n$ in $L^2(\Omega)$, we may write:

$$\|u_n(t)\|_0^2 + \int_0^t \|u_n(s)\|_1^2 ds \leq \|G_n\|_0^2 + c^2 \int_0^t \|f(s)\|_0^2 ds \tag{3.15}$$
$$\leq \|g\|_0^2 + c^2 \int_0^t \|f(s)\|_0^2 ds$$

which is (3.13). ■

We now give an estimate of the norm of $\dot{u}$ in $L^2(0, T; V^*)$.

Theorem 3.4. (Estimate of $\dot{u}_n$). The following estimate holds:

$$\int_0^T \|\dot{u}_n(t)\|_*^2 dt \leq 2\|g\|_0^2 + 4c^2 \int_0^T \|f(t)\|_0^2 dt \tag{3.16}$$

Proof. Let $v \in V$ and write $v = w + z$ where $w \in V_n$ and $z \in V_n^{\perp}$.
We have $\|w\|_1 \leq \|v\|_1$ Let $v = w$ in problem (3.10); this yields

$$(\dot{u}_n(t), v)_0 = (\dot{u}_n(t), w)0 = -a(u_n(t), w) + (f(t), w)_0.$$

[1] $|ab| \leq \frac{a^2}{2} + \frac{b^2}{2}$

Since

$$|a(u_n(t), w)| \leq \|u_n(t)\|_1 \|w\|_1$$

we infer, using the Schwarz and Poincaré inequalities,

$$\begin{aligned}
|(\dot{u}_n(t), w)_0| &\leq \|u_n(t)\|_1 \|w\|_1 + \|f(t)\|_0 \|w\|_1 \\
&\leq \{\|u_n(t)\|_1 + c\|f(t)\|_0\} \|w\|_1 \\
&\leq \{\|u_n(t)\|_1 + c\|f(t)\|_0\} \|v\|_1
\end{aligned}$$

Then, by the definition of norm in V^*, we may write

$$\|\dot{u}_n(t)\|_* \leq \|u_n(t)\|_1 + c\|f(t)\|_0$$

Squaring both sides and integrating over $(0, t)$ we get

$$\int_0^t \|\dot{u}_n(s)\|_* ds \leq 2 \int_0^t \|u_n(s)\|_1 ds + 2c^2 \int_0^t \|f(s)\|_0 ds$$

Using (3.13) to estimate $2 \int_0^t \|u_n(s)\|_1 ds$, we easily obtain (3.14). ■

Step 3. Passage to limits

We call u_n a Galerkin approximation of the solution u. We show that $\{u_n\}$ and $\{\dot{u}_n\}$ are bounded in $L^2(0, T; V)$ and $L^2(0, T; V^*)$, respectively.

Then, the weak compactness Theorem (2.4) implies that a subsequence $\{u_{m_k}\}$ converges weakly in $L^2(0, T; V)$ to some element u, while $\{\dot{u}_{m_k}\}$ converges weakly in $L^2(0, T; V^*)$ to $\dot{u}$.

Theorems 3.3 and 3.4 show that the sequence of Galerkin's approximations $\{u_n\}$ is bounded in $L^\infty(0, T; V)$, hence in $L^2(0, T; V)$, while $\{\dot{u}_n\}$ is bounded in $L^2(0, T; V^*)$. We now use the compactness Theorem 2.4 and deduce that there exists a subsequence, which for simplicity we still denote by $\{u_n\}$, such that, as $n \to \infty$,

$$u_n \to u \quad \text{weakly in } L^2(0, T; V)$$

and

$$\dot{u}_n \to \dot{u} \quad \text{weakly in } L^2(0, T; V^*)$$

This u is the unique solution of problem (1.2). Precisely:

Theorem 3.5. Let $f \in L^2(0, T; L^2(\Omega))$ and $g \in L^2(\Omega)$. Then, u is the unique solution of problem (1.2). Moreover

$$\|u(t)\|_0^2 + \int_0^T \|u(t)\|_1^2 dt \leq \|g\|_0^2 + 2c^2 \int_0^T \|f(t)\|_0^2 dt \tag{3.17}$$

for every $t \in [0, T]$, and

$$\int_0^T \|\dot{u}(t)\|_1^2 dt \leq 2\|g\|_0^2 + 4c^2 \int_0^T \|f(t)\|_0^2 dt \tag{3.18}$$

3.1. Existence

Proof. To say that $u_n \rightharpoonup u$, weakly in $L^2(0, T; V)$ as $n \to +\infty$, means that

$$\int_0^T (\nabla u_n(t), \nabla v(t))_0 dt \to \int_0^T (\nabla u(t), \nabla v(t))_0 dt$$

for all $v \in L^2(0, T; V)$. Similarly, $\dot{u}_n \to \dot{u}$, weakly in $L^2(0, T; V^*)$, means that

$$\int_0^T (\dot{u}_n(t), v(t))_0 dt = \int_0^T \langle \dot{u}_n(t), v(t)\rangle_* \to \int_0^T \langle \dot{u}(t), v(t)\rangle_* dt$$

for all $v \in L^2(0, T; V)$.

We want to use these properties to pass to the limit as $n \to +\infty$ in problem (3.10), keeping in mind that the test functions have to be chosen in V_n.
Fix $v \in L^2(0, T; V)$; we may write

$$v(t) = \sum_{k=1}^{\infty} b_k(t) w_k \text{ with the series convergent in } \; V, \; \text{ for } a.e.\, t \in [0, T].$$

Let

$$v_N(t) = \sum_{k=1}^{N} b_k(t) w_k$$

and keep N fixed, for the time being. If $n \geq N$, then $v_N \in L^2(0, T; V_n)$. Multiplying equation (3.10) by $b_k(t)$ and summing for $k = 1, \ldots, N$, we get

$$(\dot{u}_n(t), v_N(t))_0 + (\nabla u_n(t), \nabla v_N(t))_0 = (f(t), v_N(t))_0.$$

An integration over $(0, T)$ yields

$$\int_0^T \{(\dot{u}_n, v_N)_0 + (\nabla u_n, \nabla v_N)_0\} dt = \int_0^T (f, v_N)_0 dt \tag{3.19}$$

Thanks to the weak convergence of u_n and $\dot{u}_n$ in their respective spaces, we can let $n \to +\infty$. Since

$$\int_0^T (\dot{u}_n, v_N)_0 dt = \int_0^T \langle \dot{u}_n, v_N\rangle_* dt \to \int_0^T \langle \dot{u}, v_N\rangle_* dt$$

We obtain

$$\int_0^T \{\langle \dot{u}, v_N\rangle_* + (\nabla u, \nabla v_N)_0\} dt = \int_0^T (f, v_N)_0 dt$$

Now, let $N \to +\infty$ observing that $v_N \to v$ in $L^2(0, T; V)$ and in particular weakly in this space as well. We obtain

$$\int_0^T \{\langle \dot{u}, v\rangle_* + (\nabla u, \nabla v)_0\} dt = \int_0^T (f, v)_0 dt \tag{3.20}$$

Then, (3.20) is valid for all $v \in L^2(0, T; V)$. This entails

$$\langle \dot{u}(t), v\rangle_* + (\nabla u(t), \nabla v)_0 = (f(t), v)_0$$

for all $v \in V$ and a.e. $t \in [0, T]$. Therefore u satisfies (2.6).

From Theorem 2.5, we know that $u \in C([0, T]; H)$. It remains to check that $u(t)$ satisfies the initial condition $u(0) = g$. Let $v \in C^1([0, T]; V)$ with $v(T) = 0$. Integrating by parts (see Theorem 2.5, b)), we obtain

$$\int_0^T (\dot{u}_n, v_N)_0 dt = (G_n, v_N(0))_0 - \int_0^T (u_n, \dot{v}_N)_0 dt$$

so that, from (3.19) we find

$$-\int_0^T \{(u_n, \dot{v}_N)_0 + (\nabla u_n, \nabla v_N)_0\} dt = -(G_n, v_N(0))_0 + \int_0^T (f, v_N)_0 dt.$$

Let first $n \to +\infty$ and then $N \to +\infty$ we get

$$-\int_0^T \{(u, \dot{v})_0 + (\nabla u, \nabla v)_0\} dt = -(g, v(0))_0 + \int_0^T (f, v)_0 dt. \tag{3.21}$$

On the other hand, integrating by parts in formula (3.18) (see again Theorem 2.5, b)) we find

$$-\int_0^T \{(u, \dot{v})_0 + (\nabla u, \nabla v)_0\} dt = -(u(0), v(0))_0 + \int_0^T (f, v)_0 dt. \tag{3.22}$$

Subtracting (3.21) from (3.22), we deduce

$$(u(0), v(0))_0 = (g, v(0))_0$$

and the arbitrariness of $v(0)$ forces $u(0) = g$. ∎

3.2. Uniqueness

Let u_1 and u_2 be weak solutions of the same problem. Then, $w = u_1 - u_2$ is a weak solution of

$$\langle \dot{w}(t), v\rangle_* + (\nabla w(t), \nabla v)_0 = 0$$

for all $v \in V$ and a.e. $t \in [0, T]$, with initial data $w(0) = 0$. Choosing $v = w(t)$ we have

$$\langle \dot{w}(t), w(t)\rangle_* + (\nabla w(t), \nabla w(t))_0 = 0$$

or

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_0^2 = -\|w(t)\|_1^2$$

whence, since $\|w(0)\|_0^2 = 0$

$$\|w(t)\|_0^2 = -\int_0^T \|w(t)\|_1^2 dt \le 0$$

which entails $w(t) = 0$ for all $t \in [0, T]$. This gives uniqueness of the weak solution.

3.3. Stability

Letting $n \to +\infty$ in (3.11) and (3.14) we get, using Proposition 2.6,

$$\|u\|^2_{L^\infty(0,T;H)} + \|u\|^2_{L^2(0,T;V)} \le \|g\|_0^2 + c^2 \int_0^T \|f(t)\|_0^2 dt$$

and

$$\|\dot{u}\|^2_{L^2(0,T;V^*)} \le 2\|g\|_0^2 + 4c^2 \int_0^T \|f(t)\|_0^2 dt$$

which give (3.15) and (3.16).

Remark 3.6. As a by-product of the above proof, we deduce that, if $f = 0$, u satisfies the equation

$$\frac{d}{dt}\|w(t)\|_0^2 = -2\|w(t)\|_1^2 \le 0$$

which shows the dissipative nature of the diffusion equation.

3.4. Regularity

The regularity of the solution improves with the regularity of the data. Precisely, we have:

Theorem 3.7. Let Ω be a C^2- domain and u be the weak solution of problem (1.2). If $g \in V$, then $u \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; V)$ and $\dot{u} \in L^2(0, T; H)$.

Moreover

$$\|u\|_{L^2(0,T;H^2)} + \|u\|_{L^\infty(0,T;V)} + \|\dot{u}\|_{L^2(0,T;H)} \le C\{\|g\|_V + \|f\|_{L^2(0,T;H)}\} \quad (3.23)$$

Proof. Multiplying equation (3.10) by $\dot{c}_k(t)$ and summing for $k = 1, \ldots, n$, we get

$$\|\dot{u}_n(t)\|_0^2 + (\nabla u_n(t), \nabla \dot{u}_n(t))_0 = (f(t), \dot{u}_n(t))_0 \quad (3.24)$$

for a.e. $t \in [0, T]$. Now, note that

$$(\nabla u_n(t), \nabla \dot{u}_n(t))_0 = \frac{1}{2}\frac{d}{dt}\|\nabla u_n(t)\|_0^2 \quad a.e\, t \in (0, T)$$

and that, from Schwarz's inequality

$$(f(t), \dot{u}_n(t))_0 \le \|f(t)\|_0 \|\dot{u}_n(t)\|_0 \le \frac{1}{2}\|f(t)\|_0^2 + \frac{1}{2}\|\dot{u}_n(t)\|_0^2.$$

From this inequality and (3.22), we infer

$$\frac{d}{dt}\|\nabla u_n(t)\|_0^2 + \|\dot{u}_n(t)\|_0^2 \le \|f(t)\|_0^2 \quad a.e\, t \in (0, T)$$

An integration over $(0, t)$ yields

$$\|\nabla u_n(t)\|_0^2 + \int_0^t \|\dot{u}_n(s)\|_0^2 ds \leq \int_0^t \|f(t)\|_0^2 ds + \|g_n\|_1^2 \tag{3.25}$$

Passing to the limit as $n \to +\infty$ along an appropriate subsequence, we deduce that the same estimate holds for u and therefore, that $u \in L^\infty(0, T; V)$ and $\dot{u} \in L^2(0, T; H)$. In particular, we may write (2.6) in the form:

$$(\nabla u(t), \nabla v)_0 = (f(t) - \dot{u}(t), v)_0 \quad a.e.\, t \in [0, T]$$

for all $v \in V$.

Now, the regularity theory (see []) implies that $u(t) \in H^2(\Omega)$ for a.e. $t \in [0, T]$ and that

$$\|u(t)\|_{H^2(\Omega)}^2 \leq C(\Omega)\{\|g\|_1^2 + \|f(t)\|_0^2 + \|\dot{u}\|_0^2\}.$$

Integrating and using (3.25), we obtain $u \in L^2(0, T; H^2(\Omega)$ and the estimate (3.23). ■

References

[1] R. Adams, Sobolev Spaces. Academic Press, New York, 1975.

[2] H. Brezis, Analyse Fonctionnelle. Masson, 1983.

[3] R. Dautray and J. L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology, Vol. 1-5. Springer-Verlag, Berlin Heidelberg, 1985.

[4] A. Friedman, Partial Differential Equations of parabolic Type. Prentice-Hall, Englewood Cliffs, 1964.

[5] J.-L. Lions, Quelques methodes de resolution des problemes aux limites nonlineaires, Dunod-Gauthier-Villars, Paris, 1969.

[6] C.V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum Press, NewYork, 1992.

[7] L. Schwartz, Th´eorie des Distributions. Hermann, Paris, 1966.

[8] J. Smoller, Shock Waves and Reaction-Diffusion Equations, second ed Springer, Berlin, 1994.

[9] A. Yagi, Abstract Parabolic Evolution Equations and their Applications, Springer-Verlag Berlin Heidelberg 2010.

[10] K. Yoshida, Functional Analysis. Springer-Verlag, Berlin Heidelberg, 1965.