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Invariant submanifolds of special trans-Sasakian structure

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Abstract

The object of present paper is to find necessary and sufficient conditions for invariant submanifolds of special trans-Sasakian structure to be totally geodesic.

AMS subject classification:
Keywords: (ε, δ)Trans-Sasakian manifold, second fundamental form, invariant submanifold, totally geodesic, semi parallel.

1. Introduction

Invariant submanifolds of a contact manifold have been a major area of research for long time since the concept was borrowed from complex geometry. A submanifold of a contact manifold is said to be totally geodesic if every geodesic in that submanifold is also geodesic in the ambient manifold. There is well known result of Kon that an invariant submanifold of a Sasakian manifold is totally geodesic, provided the second fundamental form of the immersion is covariantly constant [11].

The concept of (ε)-Sasakian manifolds was introduced by A.Bejancu and K.L.Duggal [1] and further investigation was taken up by Xufend and Xiaoli[16] and Rakesh kumar et al.[12]. De and Sarkar [6] introduced and studied conformally flat,Weyl semisymmetric, ϕ-recurrent (ε)-Kenmotsu manifolds. In [1], the authors obtained Riemannian curvature tensor of (ε)-Sasakian manifolds and established relations among different curvatures. H.G.Nagraja et al.[9] introduced (ε, δ)-trans-Sasakian structures which generalizes both (ε)-Sasakian manifolds and (ε)-Kenmotsu manifolds and this structure is called special trans-Sasakian structure(STS). In this paper we studied invariant submanifolds of 3-dimensional (ε, δ)-trans-Sasakian structure $[(STS)_3]$.

2. Preliminaries

Let $(\overline{M}, g)$ be an almost contact metric manifold of dimension $(2n+1)$ equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\phi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$\phi\xi = 0, \eta \circ \phi = 0. \tag{2.3}$$

An almost contact metric manifold $\overline{M}$ is called an (ε)-almost contact metric manifold if

$$g(\xi, \xi) = \varepsilon, \tag{2.4}$$

$$\eta(X) = \varepsilon g(X, \xi), \tag{2.5}$$

$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \forall X, Y \in TM, \tag{2.6}$$

where $\varepsilon = g(\xi, \xi) = \pm 1$. An almost contact metric structure (ϕ, ξ, η, g) on a connected manifold M is called (ε, δ)-trans-Sasakian manifold structure. If $(M \times R, J, G)$ belongs to the class W_4 [8], where J is the almost complex structure on $M \times R$ defined by

$$J(X, f d/dt) = (\phi X - f\xi, \eta(X)d/dt) \tag{2.7}$$

for all vector fields X on M and smooth functions f on $\overline{M} \times R$ and G is the product metric on $\overline{M} \times R$.This may be expressed by the condition [3]

$$(\overline{\nabla}_X\phi)Y = \alpha[g(X, Y)\xi - \varepsilon\eta(Y)X] + \beta[g(\phi X, Y)\xi - \delta\eta(Y)\phi X], \tag{2.8}$$

holds for some smooth functions α and β on $\overline{M}$ and $\varepsilon = \pm 1, \delta = \pm 1$. For $\beta = 0,\alpha = 1$, an (ε, δ)-trans-Sasakian manifold reduces to an (ε)-Sasakian and for $\alpha = 0, \beta = 1$ it reduces to a (δ)-Kenmotsu manifold. Then from (2.8), it is easy to see that

$$(\overline{\nabla}_X\xi) = -\varepsilon\alpha\phi X - \beta\delta\phi^2 X, \tag{2.9}$$

$$(\overline{\nabla}_X \eta)Y = -\alpha g(Y, \phi X) + \varepsilon\delta\beta g(\phi X, \phi Y). \tag{2.10}$$

In a (2n+1) dimensional (ε, δ)-trans-Sasakian manifold (STS) we also have the following [9], [13]:

$$h(X, \xi) = 0. \tag{2.11}$$

$$\begin{aligned} R(X, Y)\xi &= \varepsilon((Y\alpha)\phi X - (X\alpha)\phi Y) + (\beta^2 - \alpha^2)(\eta(X)Y - \eta(Y)X) \\ &\quad - \delta((X\beta)\phi^2 Y - (Y\beta)\phi^2 X) + 2\varepsilon\delta\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) \\ &\quad + 2\alpha\beta(\delta - \varepsilon)g(\phi X, Y)\xi. \end{aligned} \tag{2.12}$$

$$S(X, \xi) = -(\phi X)\alpha + ((n-1)(\varepsilon\alpha^2 - \beta^2\delta) - (\xi\beta))\eta(X) - (2n-1)(X\beta), \tag{2.13}$$

where S is the Ricci tensor of type (0,2) and R curvature tensor of type (1,3). For constants α and β, we have

$$R(X, Y)\xi = (\beta^2 - \alpha^2)[\eta(X)Y - \eta(Y)X] \tag{2.14}$$

$$\xi\alpha + 2\alpha\beta\delta = 0. \tag{2.15}$$

In a 3-dimensionl (ε, δ)-trans-Sasakian manifold $[(STS)_3]$, the curvature tensor R and Ricci tensor S are given by [10],[18]

$$\begin{aligned} R(X, Y)Z =& (2A - \frac{r}{2})(g(Y, Z)X - g(X, Z)Y) + B(g(Y, Z)\eta(X) - g(X, Z)\eta(Y))\xi \\ &+ B\eta(Z)(\eta(Y)X - \eta(X)Y), \end{aligned} \tag{2.16}$$

$$R(X, Y)\xi = [(2A - \frac{r}{2})\varepsilon + B][\eta(Y)X - \eta(X)Y] \tag{2.17}$$

where $\varepsilon\delta = 1$, $A = \left(\frac{r}{2} - (\alpha^2 - \beta^2)\right)$, $B = \left(3(\alpha^2 - \beta^2) - \varepsilon\frac{r}{2}\right)$ and r is the scalar curvature. From (2.16), we have

$$S(X, Y) = \left[2\left(2A - \frac{r}{2}\right) + B\varepsilon\right] g(X, Y) + B(3 - 2\varepsilon)\eta(X)\eta(Y). \tag{2.18}$$

3. submanifolds of an almost contact metric manifold

Let M be a submanifold of a contact manifold $\overline{M}$. We denote ∇ and $\overline{\nabla}$ the Levi-Civita connections of M and $\overline{M}$ respectively, and $T^{\perp}(M)$ the normal bundle of M. Then Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{3.1}$$

$$\overline{\nabla}_X N = \nabla^{\perp}_X N - A_N X \tag{3.2}$$

for any $X, Y \in TM$. $\nabla^{\perp}$ is the connection in the normal bundle, h is the second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form h and the shape operator A related by

$$g(h(X, Y), N) = g(A_N X, Y). \tag{3.3}$$

From (3.1) we have

$$\overline{\nabla}_X \xi = \nabla_X \xi + h(X, \xi). \tag{3.4}$$

Let M and $\overline{M}$ be two Riemannian or semi-Riemannian manifolds, $f : M \to \overline{M}$ be an immersion, h be the second fundamental form and $\overline{\nabla}$ be the vander-Warden-Bortolotti connection of M. An immersion is said to be semiparallel if

$$\overline{R}(X, Y) \cdot h = \overline{\nabla}_X \overline{\nabla}_Y - \overline{\nabla}_Y \overline{\nabla}_X - \overline{\nabla}_{[X,Y]} \tag{3.5}$$

In [2] Arslan et.al defined and studied submanifolds satisfying the conditions

$$\overline{R}(X, Y) \cdot \overline{\nabla} h = 0 \tag{3.6}$$

for all vector fields X,Y tangent to M and such manifolds are called 2-semiparallel.

The 3-dimensional Weyl-projective curvature tensor P is defined as

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2}[g(Y, Z)QX - g(X, Z)QY] \tag{3.7}$$

By virtue of (2.17) and (3.7)

$$P(\xi, Y)Z = B\left[1 - \frac{\varepsilon}{2}\right] g(Y, Z)\xi + B\xi \left\{(1 - \varepsilon)\eta(Y)\eta(Z) - \frac{(3 - 2\varepsilon)}{2}[\eta(Y) + 1]\right\} \tag{3.8}$$

$$P(\xi, Y)\xi = B\left[(\varepsilon - 1)\eta(Y)\xi - \frac{(3 - 2\varepsilon)}{2}\xi\right] \tag{3.9}$$

4. Some basic properties of invariant submanifolds of special trans-Sasakian structure

Definition 4.1. A submanifold M of a $(STS)_3$-manifold $\overline{M}$ is said to be invariant if the structure vector field ξ is tangent to M at every point of M and ϕX is tangent to M for any vector field X tangent to M at every point of M, that is $\phi(TN) \subset TN$ at every point of M. The submanifold M of $(STS)_3$-manifold $\overline{M}$ is called totally geodesic if $h(X, Y) = 0$ for any $X, Y \in \Gamma(TN)$.

For the second fundamental form h, the covariant derivative of h is defined by

$$(\overline{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X (h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{4.1}$$

for any vector fields X, Y, Z tangent to M. Then $\overline{\nabla}h$ is a normal bundle valued tensor of type (0,3) and is called the third fundamental form of M, $\overline{\nabla}$ is called the Vander-Waerden-Bortolotti connection of $\overline{M}$. *i.e.*, $\overline{\nabla}$ is the connection in $TN \oplus T^{\perp}N$ built with ∇ and$\overline{\nabla}$. If $\overline{\nabla}h = 0$, then M is to have parallel second fundamental form [17]. From the Gauss and Weingarten formulae we obtain

$$\tilde{R}(X, Y)Z = R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X, \tag{4.2}$$

where $\tilde{R}(X, Y)Z$ denotes the tangential part of the curvature tensor of the submanifold. From (3.5), we get

$$(\overline{R}(X, Y)Z \cdot h)(Z, U) = R^{\perp}(X, Y)h(Z, U) - h(R(X, Y)Z, U) - h(Z, R(X, Y)U) \tag{4.3}$$

for all vector fields X, Y, Z and U, where

$$R^{\perp}(X, Y) = [\nabla^{\perp}_X, \nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]} \tag{4.4}$$

and $\overline{R}$ denotes the curvature tensor of $\overline{\nabla}$. In the similar manner we can write

$$\begin{aligned}(\overline{R}(X, Y) \cdot \overline{\nabla}h)(Z, U, W) =& R^{\perp}(X, Y)(\overline{\nabla}h)(Z, U, W) - (\overline{\nabla}h)(R(X, Y)Z, U, W)\\ &- (\overline{\nabla}h)(Z, R(X, Y)U, W) - (\overline{\nabla}h)(Z, U, R(X, Y)W)\end{aligned} \tag{4.5}$$

for all fields X, Y, Z, U and W tangent to M and $(\overline{\nabla}h)(Z, U, W) = \overline{\nabla}_Z h)(U, W)$. Again for Weyl-projective curvature tensor $\overline{P}$ we have

$$(\overline{P}(X, Y)\cdot h)(Z, U) = R^{\perp}(X, Y)h(Z, U) - h(P(X, Y)Z, U) - h(Z, P(X, Y)U) \tag{4.6}$$

For the second fundamental form h, the covariant derivative of h is defined by

$$(\overline{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{4.7}$$

for any vector fields X, Y, Z tangent to M. Then $\overline{\nabla}h$ is a normal bundle valued tensor of type (0,3) and is called the third fundamental form of M, $\overline{\nabla}$ is called the Vander-Waerden-Bortolotti connection of $\overline{M}$. *ie.*, $\overline{\nabla}$ is the connection in $TN \oplus T^{\perp}N$ built with ∇ and$\overline{\nabla}$. If $\overline{\nabla}h = 0$, then M is to have parallel second fundamental form[17]. From the Gauss and Weingarten formulae we obtain

$$\tilde{R}(X, Y)Z = R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X, \tag{4.8}$$

where $\tilde{R}(X, Y)Z$ denotes the tangential part of the curvature tensor of the submanifold. From (3.5), we get

$$(\overline{R}(X, Y)Z \cdot h)(Z, U) = R^{\perp}(X, Y)h(Z, U) - h(R(X, Y)Z, U) - h(Z, R(X, Y)U) \tag{4.9}$$

for all vector fields X, Y, Z and U, where

$$R^{\perp}(X, Y) = [\nabla^{\perp}_X, \nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]} \tag{4.10}$$

and $\overline{R}$ denotes the curvature tensor of $\overline{\nabla}$. In the similar manner we can write

$$\begin{aligned}(\overline{R}(X, Y) \cdot \overline{\nabla}h)(Z, U, W) =& R^{\perp}(X, Y)(\overline{\nabla}h)(Z, U, W) - (\overline{\nabla}h)(R(X, Y)Z, U, W) \\ &- (\overline{\nabla}h)(Z, R(X, Y)U, W) - (\overline{\nabla}h)(Z, U, R(X, Y)W)\end{aligned} \tag{4.11}$$

for all fields X, Y, Z, U and W tangent to M and $(\overline{\nabla}h)(Z, U, W) = \overline{\nabla}_Z h)(U, W)$. Again for Weyl-projective curvature tensor $\overline{P}$ we have

$$(\overline{P}(X, Y) \cdot h)(Z, U) = R^{\perp}(X, Y)h(Z, U) - h(P(X, Y)Z, U) - h(Z, P(X, Y)U) \tag{4.12}$$

and

$$\begin{aligned}(\overline{P}(X, Y) \cdot \overline{\nabla}h)(Z, U, W) =& R^{\perp}(X, Y)(\overline{\nabla}h)(Z, U, W) - (\overline{\nabla}h)(P(X, Y)Z, U, W) \\ &- (\overline{\nabla}h)(Z, P(X, Y)U, W) - (\overline{\nabla}h)(Z, U, P(X, Y)W)\end{aligned} \tag{4.13}$$

5. Invariant submanifolds of $(STS)_3$-manifolds satisfying $Q(h, R) = 0$

Theorem 5.1. In an Invariant submanifolds of $(STS)_3$-manifolds satisfying $Q(h, R) = 0$ with $\alpha \neq \beta$ if and only if it is totaly geodesic.

Proof. Let an Invariant submanifolds of $(STS)_3$-manifolds satisfying $Q(h, R) = 0$, Therefore,

$$\begin{aligned} 0 &= Q(h, R)(X, Y, Z; U, V) \\ &= ((U \wedge_h V, R)(X, Y)Z) \\ &= R((U \wedge_h V)X, Y)Z - R(X, (U \wedge_h V)Y)Z - R(X, Y)(U \wedge_h V)Z, \end{aligned} \tag{5.1}$$

where $(U \wedge V)$is defined by

$$(U \wedge V)W = h(V, W)U - h(U, W)V \tag{5.2}$$

Using (5.2) in (5.1), we have

$$\begin{aligned} &- h(V, X)R(U, Y)Z + h(U, X)R(, Y)Z - h(V, Y)R(X, U)Z \\ &+ h(U, Y)R(X, V)Z - h(V, Z)R(X, V)U + h(U, Z)R(X, Y)V = 0. \end{aligned} \tag{5.3}$$

Putting $Z = \xi$ in (5.3)

$$\begin{aligned} &- h(V, X)R(U, Y)\xi + h(U, X)R(V, Y)\xi - h(V, Y)R(X, U)\xi \\ &+ h(U, Y)R(X, V)\xi - h(V, \xi)R(X, V)U + h(U, \xi)R(X, Y)V = 0. \end{aligned} \tag{5.4}$$

Again put $V = \xi$ in (5.4), we have

$$h(U, X)R(\xi, Y)\xi + h(U, Y)R(X, \xi)\xi = 0. \tag{5.5}$$

This implies that

$$(3 - 2\varepsilon)\left\{h(U, X)[\eta(Y)\eta(W) - g(Y, W)] + h(U, Y)[g(X, W) - \eta(X)\eta(W)]\right\}. \tag{5.6}$$

Contracting Y and W, we get

$$h(U, X) = 0, \quad \text{provided } \alpha \neq \beta, \tag{5.7}$$

hence the proof. ∎

6. Invariant submanifolds of $(STS)_3$-manifolds satisfying $Q(S, h) = 0$

Theorem 6.1. In an Invariant submanifolds of $(STS)_3$-manifolds satisfying $Q(S, h) = 0$ with $B \neq (r - 4A)$ if and only if totally geodesic.

Proof. Let an invariant submanifolds of $(STS)_3$-manifolds satisfying $Q(h, R) = 0$, therefore,

$$\begin{aligned} 0 &= Q(S, h)(X, Y : U, V) \\ &= -h(U \wedge_S V)X, Y) - h(X, (U \wedge_S V)Y), \end{aligned} \tag{6.1}$$

where

$$h(U \wedge_S V)W = S(V, W)U - S(U, W)V. \tag{6.2}$$

Using (6.2) in (6.1), we have

$$-S(V, X)h(U, Y) + S(U, X)h(V, Y) - S(V, Y)h(X, U) + S(U, Y)h(X, V) = 0 \tag{6.3}$$

Putting $U = Y = \xi$ in (6.3), we obtain

$$S(\xi, \xi)h(X, V) = 0. \tag{6.4}$$

It implies that

$$h(X, V) = 0, \quad \text{provide } B \neq (r - 4A). \tag{6.5}$$

Hence the proof. ∎

7. 2-semiparallel invariant submanifolds of $(STS)_3$-manifolds

Theorem 7.1. Let M be an invariant submanifold of a $(STS)_3$ manifold $\overline{M}$ then M is 2-semiparallel if and only if $h(Y, \phi Z) = \frac{\beta}{\alpha} h(Y, Z)$ provided $\alpha \neq \beta$.

Proof. Let M be an invariant submanifold of a $(STS)_3$ manifold $\overline{M}$ then M is 2-semiparallel then from (4.11) we get

$$R^{\perp}(X, Y)(\overline{\nabla} h)(Z, U, W) - (\overline{\nabla} h)(R(X, Y)Z, U, W) - (\overline{\nabla} h)(Z, R(X, Y)U, W) - (\overline{\nabla} h)(Z, U, R(X, Y)W) = 0. \tag{7.1}$$

Putting $X = U = \xi$ in (7.2), we obtain

$$R^{\perp}(\xi, Y)(\overline{\nabla} h)(Z, \xi, W) - (\overline{\nabla} h)(R(\xi, Y)Z, \xi, W) - (\overline{\nabla} h)(Z, R(\xi, Y)\xi, W) - (\overline{\nabla} h)(Z, \xi, R(\xi, Y)W) = 0. \tag{7.2}$$

By virtue of (2.9), (2.17), (2.11) and (4.7), we have the following

$$\begin{aligned}(\overline{\nabla} h)(Z, \xi, W) &= (\overline{\nabla}_Z h)(\xi, W)\\ &= \overline{\nabla}^{\perp}_Z(h(\xi, W)) - h(\nabla_Z \xi, W) - h(\xi, \nabla_Z W)\\ &= \varepsilon\alpha h(\phi Z, W) - \beta\delta h(Z, W)\end{aligned} \tag{7.3}$$

$$\begin{aligned}(\overline{\nabla} h)(R(\xi, Y)Z, \xi, W) &= (\overline{\nabla}_{R(\xi,Y)Z} h)(\xi, W)\\ &= \nabla^{\perp}_{R(\xi,Y)Z}(h(\xi, W)) - h(\nabla_{R(\xi,Y)Z}\xi, W) - h(\xi, \nabla_{R(\xi,Y)Z} W)\\ &= -(3 - 2\varepsilon)(\alpha^2 - \beta^2)\eta(Z)[\varepsilon\alpha h(\phi Y, W) - \beta\delta h(Y, W)]\end{aligned} \tag{7.4}$$

$$\begin{aligned}(\overline{\nabla} h)(Z, R(\xi, Y)\xi, W) &= (\overline{\nabla}_Z h)(R(\xi, Y)\xi, W)\\ &= \nabla^{\perp}_Z(h(R(\xi, Y)\xi, W)) - h(\nabla_Z R(\xi, Y)\xi, W)\\ &\quad - h(R(\xi, Y)\xi, \nabla_Z W)\\ &= (3 - 2\varepsilon)(\alpha^2 - \beta^2)[-\overline{\nabla}_Z h(Y, W) + h(\nabla_Z Y, W)\\ &\quad - \eta(Y)h(\nabla_Z \xi, W) + h(Y, \nabla_Z W)\end{aligned} \tag{7.5}$$

and

$$\begin{aligned}(\overline{\nabla} h)(Z, \xi, R(\xi, Y)W) &= (\overline{\nabla}_Z h)(\xi, R(\xi, Y)W)\\ &= \nabla^{\perp}_Z(h(\xi, R(\xi, Y)W)) - h(\nabla_Z \xi, R(\xi, Y)W)\\ &\quad - h(\xi, \nabla_Z R(\xi, Y)W)\\ &= -(3 - 2\varepsilon)(\alpha^2 - \beta^2)\eta(W)[\varepsilon\alpha h(\phi Z, Y) - \beta\delta h(Z, Y)].\end{aligned} \tag{7.6}$$

Using (7.3),(7.4),(7.5),(7.6) in (7.2), we get

$$\begin{aligned}&R^{\perp}(\xi, Y)[\varepsilon\alpha h(\phi Z, W) - \beta\delta h(Z, W)] + [(3-2\varepsilon)(\alpha^2 - \beta^2)]\\&\big[\eta(Z)(\varepsilon\alpha h(\phi Y, W) - \beta\delta h(Y, W)) + \overline{\nabla}_Z h(Y, W) - h(\nabla_Z Y, W)\\&+\eta(Y)h(\nabla_Z \xi, W) - h(Y, \nabla_Z W) + \eta(W)[\varepsilon\alpha h(\phi Z, Y) - \beta\delta h(Z, Y)]\big] = 0.\end{aligned} \tag{7.7}$$

Putting $W = \xi$ in (7.7), we obtain

$$h(Y, \phi Z) = \frac{\beta\delta}{\varepsilon\alpha} h(Y, Z), \tag{7.8}$$

if $(\alpha^2 - \beta^2) \neq 0$. Hence the proof. ■

8. Invariant submanifolds of $(STS)_3$-manifolds satisfying $\overline{P}(X, Y) \cdot h = 0$ and $\overline{P}(X, Y) \cdot \overline{\nabla} h = 0$

Theorem 8.1. Let M be an invariant submanifold of a $(STS)_3$-manifold $\overline{M}$ such that $r \neq 4A$, then $\overline{P}(X, Y) \cdot h = 0$ holds on M if and only if M is totally geodesic.

Proof. Let M be an invariant submanifold of $(STS)_3$-manifold $\overline{M}$ satisfying $\overline{P}(X, Y) \cdot h = 0$. We have from (4.12) that

$$R^{\perp}(X, Y)h(Z, U) - h(P(X, Y)Z, U) - h(Z, P(X, Y)U) = 0. \tag{8.1}$$

Setting $X = U = \xi$ in (8.1) and using (3.8)and (2.11), we obtain

$$h(Z, P(\xi, Y)\xi)) = 0. \tag{8.2}$$

By virtue of (3.9) it follows from (8.2) that

$$\left[\frac{\varepsilon r}{2} - 2A\varepsilon\right] h(Z, Y) = 0 \tag{8.3}$$

It implies that

$$h(Z, Y) = 0, \quad \text{provided } r \neq 4A. \tag{8.4}$$

Hence the proof. ■

Theorem 8.2. Let M be an invariant submanifold of a $(STS)_3$-manifold $\overline{M}$ such that $r \neq 4A$ and $\beta \neq 0$, then $\overline{P}(X, Y) \cdot \overline{\nabla} h = 0$ holds on M if and only if M is totally geodesic.

Proof. Let M be an invariant submanifold of $(STS)_3$-manifold $\overline{M}$ satisfying $\overline{P}(X, Y) \cdot \overline{\nabla} h = 0$. We have from (4.13) that

$$\begin{aligned}&R^{\perp}(X, Y)(\overline{\nabla} h)(Z, U, W) - (\overline{\nabla} h)(P(X, Y)Z, U, W)\\&- (\overline{\nabla} h)(Z, P(X, Y)U, W) - (\overline{\nabla} h)(Z, U, P(X, Y)W) = 0.\end{aligned} \tag{8.5}$$

Putting $X = U = \xi$ in (8.5), we obtain

$$\begin{aligned} &R^{\perp}(\xi, Y)(\overline{\nabla} h)(Z, \xi, W) - (\overline{\nabla} h)(P(\xi, Y)Z, \xi, W) \\ &- (\overline{\nabla} h)(Z, P(\xi, Y)\xi, W) - (\overline{\nabla} h)(Z, \xi, P(\xi, Y)W) = 0. \end{aligned} \tag{8.6}$$

By virtue of (2.11),(4.7),(3.8) and (3.9), we get

$$\begin{aligned} (\overline{\nabla} h)(P(\xi, Y)Z, \xi, W) &= (\overline{\nabla}_{P(\xi,Y)Z} h)(\xi, W) \\ &= \nabla^{\perp}_{P(\xi,Y)Z}(h(\xi, W)) - h(\nabla_{P(\xi,Y)Z}\xi, W) - h(\xi, \nabla_{P(\xi,Y)Z} W) \\ &= -h(\nabla_{P(\xi,Y)Z}\xi, W) \\ &= \frac{-\beta}{2}(r - 4A)h(Y, W)\eta(Z). \end{aligned} \tag{8.7}$$

$$\begin{aligned} (\overline{\nabla} h)(Z, P(\xi, Y)\xi, W) &= (\overline{\nabla}_Z h)(P(\xi, Y)\xi, W) \\ &= \nabla^{\perp}_Z (h(P(\xi, Y)\xi, W) - h(\nabla_Z P(\xi, Y)\xi, W) \\ &- h(P(\xi, Y)\xi, \nabla_Z W) \\ &= -B[(\varepsilon - 1)\eta(Y) - \frac{(3 - 2\varepsilon)}{2}]h(\nabla_Z \xi, W). \end{aligned} \tag{8.8}$$

and

$$\begin{aligned} (\overline{\nabla} h)(Z, \xi, P(\xi, Y)W) &= (\overline{\nabla}_Z h)(\xi, P(\xi, Y)W) \\ &= \nabla^{\perp}_Z (h(\xi, P(\xi, Y)W)) - h(\nabla_Z \xi, P(\xi, Y)W) \\ &- h(\xi, \nabla_Z P(\xi, Y)W) \\ &= -h(\nabla_Z \xi, P(\xi, Y)W) = 0. \end{aligned} \tag{8.9}$$

In view of (8.7),(8.8)and(8.9), we have from (8.6) that

$$\begin{aligned} &R^{\perp}(\xi, Y)[\varepsilon\alpha h(\phi Z, W) - \beta\delta h(Z, W)] + \frac{\beta}{2}(r - 4A)\eta(Z)h(Y, W) \\ &+ B[(\varepsilon - 1)\eta(Y) - \frac{(3 - 2\varepsilon)}{2}]h(\nabla_Z \xi, W) = 0. \end{aligned} \tag{8.10}$$

Putting $Z = \xi$ in (8.10), we obtain

$$h(Y, W) = 0, \quad \text{provided } \beta \neq 0 and\ r \neq 4A. \tag{8.11}$$

Hence the proof. ∎

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