Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 4 (2017), pp. 1171–1182

http://www.ripublication.com/gjpam.htm

Measure zero stability problem for alternative Jensen functional equations

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Abstract

In this paper, using the Baire category theorem we investigate the Hyers-Ulam stability problem of alternative Jensen functional equations on a set of Lebesgue measure zero. As a consequence, we obtain asymptotic behaviors of the equations.

AMS subject classification: 39B22.
Keywords: Hyers-Ulam stability, Jensen equation, alternative Jensen functional equation, restricted domain.

1. Introduction

Throughout the paper, we denote by $\mathbb{R}$, X and Y be the set of real numbers, a real normed space and a real Banach space, respectively, $d > 0$ and $\epsilon \geq 0$ be fixed. A

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mapping $f : X \to Y$ is called *the alternative Jensen functional equations* if f satisfies one of the equations

$$f(x+y) + f(x-y) + 2f(-x) = 0, \tag{1.1}$$

$$f(x+y) - f(x-y) + 2f(-y) = 0, \tag{1.2}$$

$$2f\left(-\frac{x+y}{2}\right) + f(x) + f(y) = 0, \tag{1.3}$$

$$2f\left(-\frac{x-y}{2}\right) + f(x) - f(y) = 0 \tag{1.4}$$

for all $x, y \in X$. A mapping $f : X \to Y$ is called *an additive mapping* if f satisfies $f(x+y) - f(x) - f(y) = 0$ for all $x, y \in X$. The stability problems for functional equations have been originated by Ulam in 1940 (see [32]). One of the first assertions to be obtained is the following result, essentially due to Hyers [18] that gives an answer to the question of Ulam.

Theorem 1.1. Let $\epsilon > 0$ be fixed. Suppose that $f : X \to Y$ satisfies the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x,\ y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \le \epsilon$$

for all $x \in X$.

The terminology *domain* means a subset of $X \times X$. Let $f : X \to Y, d > 0$ and $\epsilon \ge 0$. Among the numerous results on Ulam-Hyers stability theorem for functional equations (e.g. [18], [20], [21], [25], [27], [28], [31]) there are various interesting results which deal with the stability of functional equations in restricted domains([2], [3], [5], [7], [14], [16], [22], [25]). In particular, J. M. Rassias and M. J. Rassias prove the Ulam-Hyers stability of the alternative Jensen functional equation (1.1) $\sim$ (1.4), i.e., we consider the alternative Jensen functional inequalities

$$\|f(x+y) + f(x-y) + 2f(-x)\| \le \epsilon, \tag{1.5}$$

$$\|f(x+y) - f(x-y) + 2f(-y)\| \le \epsilon, \tag{1.6}$$

$$\left\|2f\left(-\frac{x+y}{2}\right) + f(x) + f(y)\right\| \le \epsilon, \tag{1.7}$$

$$\left\|2f\left(-\frac{x-y}{2}\right) + f(x) - f(y)\right\| \le \epsilon \tag{1.8}$$

for all $x, y \in X$ with $\|x\| + \|y\| \ge d$. As main results they obtained the following.

Theorem 1.2. Suppose that $f : X \to Y$ satisfies the functional inequality (1.5) for all $x, y \in X$ with $\|x\| + \|y\| \geq d$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq 8\epsilon + \|f(0)\|$$

for all $x \in X$.

Theorem 1.3. Suppose that $f : X \to Y$ satisfies the functional inequality (1.6) for all $x, y \in X$ with $\|x\| + \|y\| \geq d$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq 21\epsilon + \|f(0)\|$$

for all $x \in X$.

Theorem 1.4. Suppose that $f : X \to Y$ satisfies the functional inequality (1.7) for all $x, y \in X$ with $\|x\| + \|y\| \geq d$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq 20\epsilon + 7\|f(0)\|$$

for all $x \in X$.

Theorem 1.5. Suppose that $f : X \to Y$ satisfies the functional inequality (1.8) for all $x, y \in X$ with $\|x\| + \|y\| \geq d$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq 24\epsilon + 4\|f(0)\|$$

for all $x \in X$.

It is very natural to ask if the restricted domain $\Omega_d := \{(x, y) : \|x\| + \|y\| \geq d\}$ in Theorem 1.2 $\sim$ 1.5 can be replaced by a smaller subset $\Gamma_d \subset \Omega_d$ (e.g., a subset of measure 0 in a measure space X) (see [14]).

In this paper we first consider the Ulam-Hyers stability of the alternative Jensen functional equations (1.1) $\sim$ (1.4) in restricted domains $\Omega \subset X \times X$ satisfying the condition (C):

$$\begin{aligned} &\text{Let } (\gamma_j, \lambda_j) \in \mathbb{R}^2,\ j = 1, 2, \ldots, r,\ \text{ with } \gamma_j^2 + \lambda_j^2 \neq 0 \\ &\qquad\qquad \text{for all } j = 1, 2, \ldots, r,\ \text{ be given.} \\ (\mathrm{C}) \quad &\text{For any } p_j,\ q_j \in X,\ j = 1, 2, \ldots, r,\ \text{ there exists } t \in X \text{ such that} \\ &\qquad\qquad \{(p_j + \gamma_j t,\ q_j + \lambda_j t) : j = 1, 2, \ldots, r\} \subset \Omega. \end{aligned}$$

Secondly, constructing a subset Γ_d of $R_d := \{(x, y) \in \mathbb{R}^2 : |x| + |y| \geq d\}$ of 2-dimensional Lebesgue zero satisfying the condition (C) we obtain measure zero stability problems of the alternative Jensen functional equations (1.1) $\sim$ (1.4) when $X = \mathbb{R}$.

Finally, as consequences of the results we also prove that if $f : \mathbb{R} \to Y$ satisfy the asymptotic condition

$$\|f(x+y)+f(x-y)+2f(-x)\| \to 0, \tag{1.9}$$

$$\|f(x+y)-f(x-y)+2f(-y)\| \to 0, \tag{1.10}$$

$$\left\|2f\left(-\frac{x+y}{2}\right)+f(x)+f(y)\right\| \to 0, \tag{1.11}$$

$$\left\|2f\left(-\frac{x-y}{2}\right)+f(x)-f(y)\right\| \to 0 \tag{1.12}$$

as $|x|+|y| \to \infty$ only for (x, y) in a set of Lebesgue measure zero in $\mathbb{R}$.

2. Abstract approach

Throught this section we assume that $\Omega \subset X \times X$ satisfies the condition (C). In this section we prove the Ulam-Hyers stability of (1.5) $\sim$ (1.8) in Ω. As main results we prove the following.

Theorem 2.1. Suppose that $f : X \to Y$ satisfies the inequality

$$\|f(x+y)+f(x-y)+2f(-x)\| \le \epsilon \tag{2.1}$$

for all $(x, y) \in \Omega$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)-f(0)\| \le 2\epsilon \tag{2.2}$$

for all $x \in X$.

Proof. Let $D(x, y) = f(x+y)+f(x-y)+2f(-x)$. Then we have the following inequality

$$\begin{aligned} &\|f(x+y)-f(x)-f(y)+f(0)\| \\ &\le \left\|\frac{1}{2}D(-x-y, x-y+t)-\frac{1}{2}D(-x, x+t)-\frac{1}{2}D(-y, -y+t)+\frac{1}{2}D(0, t)\right\| \le 2\epsilon \end{aligned} \tag{2.3}$$

for all $x, y, t \in X$. Since Ω satisfies the condition (C), it follows from (2.1) that for given $x, y \in X$, there exist $t \in X$ such that

$$\begin{gathered} \|D(-x-y, x-y+t)\| \le \epsilon, \ \|D(-x, x+t)\| \le \epsilon, \\ \|D(-y, -y+t)\| \le \epsilon, \ \|D(0, t)\| \le \epsilon. \end{gathered}$$

By theorem 1.1, there exist additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)-f(0)\| \le 2\epsilon \tag{2.4}$$

for all $x \in X$. This completes the proof. ■

Theorem 2.2. Suppose that $f : X \to Y$ satisfies the inequality

$$\|f(x+y) - f(x-y) + 2f(-y)\| \le \epsilon \tag{2.5}$$

for all $(x, y) \in \Omega$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{3}{2}\epsilon \tag{2.6}$$

for all $x \in X$.

Proof. Let $D(x, y) = f(x+y) - f(x-y) + 2f(-y)$. Then we have the following inequality

$$\begin{aligned} &\|f(x+y) - f(x) - f(y)\| \\ &\le \left\| \frac{1}{2}D(x-y+t, -x-y) - \frac{1}{2}D(x+t, -x) - \frac{1}{2}D(-y+t, -y) \right\| \le \frac{3}{2}\epsilon \end{aligned} \tag{2.7}$$

for all $x, y, t \in X$. Since Ω satisfies the condition (C), it follows from (2.5) that for given $x, y \in X$, there exist $t \in X$ such that

$$\begin{aligned} &\|D(x-y+t, -x-y)\| \le \epsilon, \ \|D(x+t, -x)\| \le \epsilon, \\ &\|D(-y+t, -y)\| \le \epsilon. \end{aligned}$$

By theorem 1.1, there exist additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{3}{2}\epsilon \tag{2.8}$$

for all $x \in X$. This completes the proof. ■

Theorem 2.3. Suppose that $f : X \to Y$ satisfies the inequality

$$\left\| 2f\left(-\frac{x+y}{2}\right) + f(x) + f(y) \right\| \le \epsilon \tag{2.9}$$

for all $(x, y) \in \Omega$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \le 2\epsilon \tag{2.10}$$

for all $x \in X$.

Proof. Let $D(x, y) = 2f\left(-\frac{x+y}{2}\right) + f(x) + f(y)$. Then we have the following inequality

$$\begin{aligned} &\|f(x+y) - f(x) - f(y) + f(0)\| \\ &\le \left\| \frac{1}{2}D(-2x-t, -2y+t) - \frac{1}{2}D(-2x-t, t) - \frac{1}{2}D(-t, -2y+t) + \frac{1}{2}D(-t, t) \right\| \le 2\epsilon \end{aligned} \tag{2.11}$$

for all $x, y, t \in X$. Since Ω satisfies the condition (C), it follows from (2.9) that for given $x, y \in X$, there exist $t \in X$ such that

$$\|D(-2x-t,-2y+t)\| \le \epsilon,\ \|D(-2x-t,t)\| \le \epsilon,\ \|D(-t,-2y+t)\| \le \epsilon,$$
$$\|D(-t,t)\| \le \epsilon.$$

By theorem 1.1, there exist additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \le 2\epsilon \tag{2.12}$$

for all $x \in X$. This completes the proof. ■

Theorem 2.4. Suppose that $f : X \to Y$ satisfies the inequality

$$\left\| 2f\left(-\frac{x-y}{2}\right) + f(x) - f(y) \right\| \le \epsilon \tag{2.13}$$

for all $(x, y) \in \Omega$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \le 2\epsilon \tag{2.14}$$

for all $x \in X$.

Proof. Let $D(x, y) = 2f\left(-\frac{x-y}{2}\right) + f(x) - f(y)$. Then we have the following inequality

$$\begin{aligned} &\|f(x+y) - f(x) - f(y) - f(0)\| \\ &\le \left\| \frac{1}{2}D(-2x+t, 2y+t) - \frac{1}{2}D(-2x+t, t) - \frac{1}{2}D(t, 2y+t) + \frac{1}{2}D(t,t) \right\| \le 2\epsilon \end{aligned} \tag{2.15}$$

for all $x, y, t \in X$. Since Ω satisfies the condition (C), it follows from (2.13) that for given $x, y \in X$, there exist $t \in X$ such that

$$\|D(-2x+t, 2y+t)\| \le \epsilon,\ \|D(-2x+t, t)\| \le \epsilon,\ \|D(t, 2y+t)\| \le \epsilon,\ \|D(t,t)\| \le \epsilon.$$

By theorem 1.1, there exist additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \le 2\epsilon \tag{2.16}$$

for all $x \in X$. This completes the proof. ■

It is obvious that the set $\{(x, y) \in X^2 : \|x\| + \|y\| \ge d\}$ satisfies the condition (C). Thus, as direct consequences of Theorem 2.1 $\sim$ Theorem 2.4 we obtain the refined results of Theorem 1.2 $\sim$ Theorem 1.5.

Corollary 2.5. Let $d > 0$. Suppose that $f : X \to Y$ satisfies the inequality (1.5) for all $(x, y) \in X$ with $\|x\| + \|y\| \geq d$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \leq 2\epsilon$$

for all $x \in X$.

Corollary 2.6. Let $d > 0$. Suppose that $f : X \to Y$ satisfies the inequality (1.6) for all $(x, y) \in X$ with $\|x\| + \|y\| \geq d$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{3}{2}\epsilon$$

for all $x \in X$.

Corollary 2.7. Let $d > 0$. Suppose that $f : X \to Y$ satisfies the inequality (1.7) for all $(x, y) \in X$ with $\|x\| + \|y\| \geq d$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \leq 2\epsilon$$

for all $x \in X$.

Corollary 2.8. Let $d > 0$. Suppose that $f : X \to Y$ satisfies the inequality (1.8) for all $(x, y) \in X$ with $\|x\| + \|y\| \geq d$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \leq 2\epsilon$$

for all $x \in X$.

3. Stability of the equations in restricted domain of Lebesque measure zero

In this section, constructing some sets $\Gamma \subset \mathbb{R}^2$ with $m(\Gamma) = 0$ satisfying condition (C), we consider the stability of the alternative Jensen functional equations on restricted domain Γ of Lebesque measure zero when $X = \mathbb{R}$.

Definition 3.1. A subset K of a topological space E is said to be of the first category if K is a countable union of nowhere dense subsets of E, and otherwise it is said to be of the second category.

Theorem 3.2. (Baire category theorem) Every nonempty open subset of a compact Hausdorff space or a complete metric space is of the second category.

For the proof of the following Lemma 3.3 we refer the reader to [10, Lemma 2.3].

Lemma 3.3. Let B be a subset of $\mathbb{R}$ such that $B^c := \mathbb{R} \setminus B$ is of the first category. Then, for any countable subsets $U \subset \mathbb{R},\ V \subset \mathbb{R} \setminus \{0\}$ and $M > 0$, there exists $t \geq M$ such that

$$U + tV = \{u + tv : u \in U, v \in V\} \subset B. \tag{3.1}$$

From now on we identify $\mathbb{R}^2$ with $\mathbb{C}$.

Lemma 3.4. Let $P = \{(p_j + a_j t, q_j + b_j t) : j = 1, 2, \ldots, r\}$, where $p_j, q_j, a_j, b_j \in \mathbb{R}$ with $a_j^2 + b_j^2 \neq 0$ for all $j = 1, 2, \ldots, r$ and $t \in \mathbb{R}$. Then there exists $\theta \in [0, 2\pi)$, such that $e^{i\theta} P := \{(p_j' + a_j' t, q_j' + b_j' t) : j = 1, 2, \ldots, r\}$ satisfies $a_j' b_j' \neq 0$ for all $j = 1, 2, \ldots, r$.

Proof. The coefficient a_j' and b_j' are given by

$$a_j' = a_j \cos\theta - b_j \sin\theta, \quad b_j' = a_j \sin\theta + b_j \cos\theta$$

for all $j = 1, 2, \ldots, r$. Now, the equation

$$\Pi_{j=1}^r (a_j \cos\theta - b_j \sin\theta)(a_j \sin\theta + b_j \cos\theta) = 0$$

has only a finite number of zeros in $[0, 2\pi)$. Thus, we can choose a $\theta \in [0, 2\pi)$ such that $\Pi_{j=1}^r a_j' b_j' \neq 0$. This completes the proof. ■

Theorem 3.5. Let $X = \mathbb{R}$ and $B \subset \mathbb{R}$. Assume that $B^c := \mathbb{R} \setminus B$ is of the first category. Then there is a rotation Γ of $B^2 := B \times B$ such that for any $d \geq 0$ the set $\Gamma_d := \Gamma \cap \{(x, y) \in \mathbb{R}^2 : |x| + |y| \geq d\}$ satisfies the condition (C).

Proof. Let $P = \{(p_j + a_j t, q_j + b_j t) : j = 1, 2, \ldots, r\}$. Then by Lemma 2.4 we can choose a $\theta \in [0, 2\pi)$ such that $e^{-i\theta} P := \{(p_j' + a_j' t, q_j' + b_j' t) : j = 1, 2, \ldots, r\}$ satisfies $a_j' b_j' \neq 0$ for all $j = 1, 2, \ldots, r$. Now, we prove that $\Gamma_d := (e^{i\theta} B^2) \cap \{(x, y) \in \mathbb{R}^2 : |x| + |y| \geq d\}$ satisfies (C). It suffices to show that for given $p_j, q_j, a_j, b_j \in \mathbb{R}$ with $a_j^2 + b_j^2 \neq 0$ for all $j = 1, 2, \ldots, r$, there exists $t \in \mathbb{R}$ such that

$$e^{i\theta} P = \{(p_j' + a_j' t, q_j' + b_j' t) : j = 1, 2, \ldots, r\} \subset B \times B,\ P \subset \{(x, y) : |x| + |y| \geq d\}. \tag{3.2}$$

Let $U = \{p_j', q_j' : j = 1, 2, \ldots, r\},\ V = \{a_j', b_j' : j = 1, 2, \ldots, r\}$. Then we have

$$\{u, v : (u, v) \in e^{i\theta} P\} \subset U + tV. \tag{3.3}$$

Now, by Lemma 2.3, for given $p_j, q_j, a_j, b_j \in \mathbb{R}$ with $a_j^2 + b_j^2 \neq 0$ for all $j = 1, 2, \ldots, r$, there exists $t \geq \max\limits_{1 \leq j \leq r} (|a_j| + |b_j|)^{-1} (|p_j| + |q_j| + d)$ such that

$$U + tV \subset B. \tag{3.4}$$

From (3.2) and (3.3) we have

$$e^{i\theta}P \subset B \times B. \tag{3.5}$$

From the choice of t, using the triangle inequality we have $|u|+|v| \geq d$ for all $(u, v) \in P$. Thus, Γ_d satisfies (C). This completes the proof. ■

Remark 3.6. Similarly, appropriate rotation of $2n$-product B^{2n} of B satisfies the conditions (C) which has $2n$-dimensional Lebesgue measure 0.

Remark 3.7. The set $\mathbb{R}$ of real numbers can be partitioned as

$$\mathbb{R} = B \cup (\mathbb{R} \setminus B)$$

where B is of Lebesgue measure zero and $\mathbb{R} \setminus B$ is of the first category [23, Theorem 1.6]. Thus, in view of Theorem 1.8 we can find a subset $\Gamma_d \subset \{(x, y) \in \mathbb{R}^2 : |x| + |y| \geq d\}$ of Lebesgue measure zero satisfying (C).

Now, we obtain the following results.

Theorem 3.8. Suppose that $f : \mathbb{R} \to Y$ satisfies the inequality

$$\|f(x+y) + f(x-y) + 2f(-x)\| \leq \epsilon$$

for all $(x, y) \in \Gamma_d$. Then there exists a unique additive mapping $A : \mathbb{R} \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \leq 2\epsilon$$

for all $x \in \mathbb{R}$.

Theorem 3.9. Suppose that $f : \mathbb{R} \to Y$ satisfies the inequality

$$\|f(x+y) - f(x-y) + 2f(-y)\| \leq \epsilon$$

for all $(x, y) \in \Gamma_d$. Then there exists a unique additive mapping $A : \mathbb{R} \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{3}{2}\epsilon$$

for all $x \in \mathbb{R}$.

Theorem 3.10. Suppose that $f : \mathbb{R} \to Y$ satisfies the inequality

$$\left\|2f\left(-\frac{x+y}{2}\right) + f(x) + f(y)\right\| \leq \epsilon$$

for all $(x, y) \in \Gamma_d$. Then there exists a unique additive mapping $A : \mathbb{R} \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \leq 2\epsilon$$

for all $x \in \mathbb{R}$.

Theorem 3.11. Suppose that $f : \mathbb{R} \to Y$ satisfies the inequality

$$\left\| 2f\left(-\frac{x-y}{2}\right) + f(x) - f(y) \right\| \le \epsilon$$

for all $(x, y) \in \Gamma_d$. Then there exists a unique additive mapping $A : \mathbb{R} \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \le 2\epsilon$$

for all $x \in \mathbb{R}$.

4. Asymptotic behaviors of the equations

We investigate an asymptotic behavior of $f : \mathbb{R} \to Y$ satisfying (1.9) $\sim$ (1.12) as $|x| + |y| \to \infty$ only for $(x, y) \in \Gamma \subset \mathbb{R}^2$ with $m(\Gamma) = 0$.

As a consequence of Theorem 3.8 we have the following.

Theorem 4.1. Suppose that $f : \mathbb{R} \to Y$ satisfies the condition (1.9). Then there exists a unique additive mapping $A : \mathbb{R} \to Y$ such that

$$f(x) = A(x) + f(0)$$

for all $x \in \mathbb{R}$.

Proof. The condition (1.9) implies that for each $n \in \mathbb{N}$, there exists $d_n > 0$ such that

$$\|f(x+y) + f(x-y) + 2f(-x)\| \le \frac{1}{n} \tag{4.6}$$

for all $(x, y) \in \Gamma_{d_n}$. By Theorem 3.8, there exists a unique additive mapping $A_n : \mathbb{R} \to Y$ such that

$$\|f(x) - A_n(x) - f(0)\| \le \frac{2}{n} \tag{4.7}$$

for all $x \in \mathbb{R}$.

Replacing n by positive integers m, k in (4.2) and using the triangle inequality with the results we have

$$\|A_m(x) - A_k(x)\| \le \frac{2}{m} + \frac{2}{k} \le 4 \tag{4.8}$$

for all $x \in \mathbb{R}$. From the additivity of A_m, A_k, it follows that $A_m = A_k$ for all $m, k \in \mathbb{N}$. Letting $n \to \infty$ in (4.2) we obtain the result. This completes the proof. ■

Similarly, using Theorem 3.9, Theorem 3.10 and Theorem 3.11 we have the following.

Theorem 4.2. Suppose that $f : \mathbb{R} \to Y$ satisfies the condition (1.10). Then f is an unique additive mapping.

Theorem 4.3. Suppose that $f : \mathbb{R} \to Y$ satisfies the condition (1.11). Then $f(x) - f(0)$ is an unique additive mapping.

Theorem 4.4. Suppose that $f : \mathbb{R} \to Y$ satisfies the condition (1.12). Then $f(x) - f(0)$ is an unique additive mapping.

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