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Some identities on modified degenerate q-Bernoulli polynomials and numbers

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Abstract

Kim introduced some identities on degenerate q-Bernoulli polynomials, which care defined by the p-adic q-integral on $\mathbb{Z}_p$ (see [17]).

And Kim et al. [10] gave symmetric identities for such polynomials under the symmetric group of degree n, which is the degeneration of the result of Kim and

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Kim in [4]. We mention that such symmetric identities could be obtained nicely by using p-adic q-integration on $\mathbb{Z}_p$, which is defined by Kim [14].

In this paper, we consider a new type of modified degenerate q-Bernoulli polynomials and numbers, and give some interesting identities on such polynomials. And we derive some identities of symmetry for those polynomials, by using p-adic q-integral on $\mathbb{Z}_p$, under the symmetry group of degree n.

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1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic noum $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $q \in \mathbb{C}_p$ be an indeterminate such that $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-analogue of the number x is defined by $[x]_q = \frac{1-q^x}{1-q}$.

Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. Then the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$\begin{aligned} I_q(f) &= \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_q(x+p^N\mathbb{Z}_p) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \quad \text{(see [14])}. \end{aligned} \tag{1.1}$$

In [1], L. Carlitz considered the q-analogue of Bernoulli numbers which are given by the recurrence relation

$$\beta_{0,q} = 1, \quad q(q\beta_q+1)^n - \beta_{n,q} = \begin{cases} 1, & \text{if } n=1 \\ 0, & \text{if } n>1, \end{cases} \tag{1.2}$$

with the usual convention of replacing β_q^n by $\beta_{n,q}$. He defined the q-Bernoulli polynomials as

$$\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} \beta_{l,q} [x]_q^{n-l}, \quad \text{(see [1, 14])}. \tag{1.3}$$

In [14], Kim proved that the Carlitz's q-Bernoulli polynomials are represented as the p-adic q-integral on $\mathbb{Z}_p$ which are given by

$$\int_{\mathbb{Z}_p} [x+y]_q^n d\mu_q(y) = \beta_{n,q}(x), \quad (n \geq 0). \tag{1.4}$$

When $x = 0$, $\beta_{n,q} = \beta_{n,q}(0)$ are the Carlitz q-Bernoulli numbers.

In [2], L. Carlitz also introduced the degenerate Bernoulli polynomials which are given by the generating function

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.5}$$

Note that $\lim B_{n,\lambda}^*(x) = B_n^*(x)$, where $B_n(x)$ are ordinary Bernoulli polynomials (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]). When $x = 0$, $B_{n,\lambda}^* = B_{n,\lambda}^*(0)$ are called the degenerate Bernoulli numbers.

In [17], Kim considered degenerate q-Bernoulli polynomials which are given by the generating function

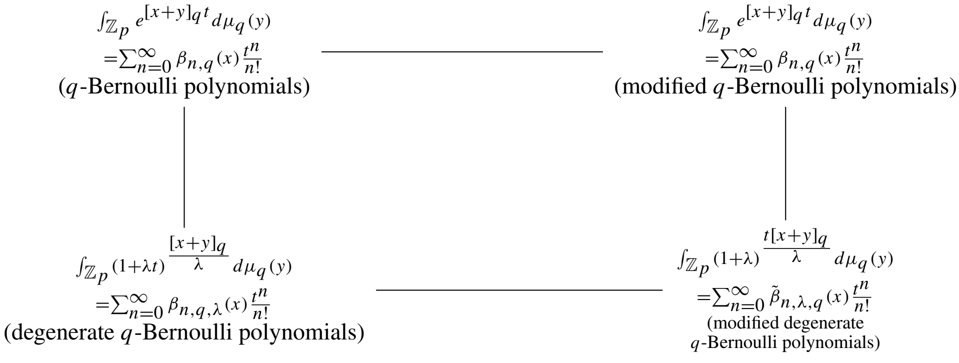
$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda}[x+y]_q} d\mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,\lambda,q}(x)\frac{t^n}{n!}. \tag{1.6}$$

When $x = 0$, $\beta_{n,\lambda,q} = \beta_{n,\lambda,q}(0)$ are called (fully) degenerate q-Bernoulli numbers. Note that $\lim_{\lambda \to 0} \beta_{n,\lambda,q}(x) = \beta_{n,q}(x)$, $(n \geq 0)$.

We note that (1.6) has meaning under the conditions $\lambda, t \in \mathbb{C}_p$ with $0 < |\lambda|_p \leq 1$, $|t|_p < p^{-\frac{1}{p-1}}$.

Since $|\lambda t|_p < p^{-\frac{1}{p-1}}$, $|\log(1+\lambda t)|_p = |\lambda t|_p$ and hence $|\frac{1}{\lambda}\log(1+\lambda t)|_p = |t|_p < p^{-\frac{1}{p-1}}$ and now it makes sense to take the limit as $\lambda \to 0$ (see [17]).

The following diagram shows how q-Bernoulli polynomials become Kim's degenerate q-Bernoulli polynomials and our modified degenerate q-Bernoulli polynomials.



We note that Lee and Jang recently defined and studied the modified degenerate q-Bernoulli polynomials by the generating function:

$$\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[x+y]_q}{\lambda}} d\mu_q(y) = \sum_{n=0}^{\infty} \widetilde{B}_{n,q,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [20])}. \tag{1.7}$$

The generating functions of Stirling numbers are given by

$$(\log(1+t))^n = n!\sum_{l=n}^{\infty} S_1(l,n)\frac{t^l}{l!}, \quad (n \geq 0) \tag{1.8}$$

and

$$(e^t - 1)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}, \quad (n \geq 0), \tag{1.9}$$

where $S_1(n,l)$ are the Stirling numbers of the first kind, and $S_2(l,n)$ are the Striling numbers of the second kind.

We define the modified degenerate q-Bernoulli polynomials by the generating function

$$\int_{\mathbb{Z}_p} (1+\lambda)^{\frac{[x+y]_q}{\lambda}} d\mu_q(y) = \sum_{n=0}^{\infty} \tilde{\beta}_{n,\lambda,q}(x)\frac{t^n}{n!}. \tag{1.10}$$

When $x = 0$, $\tilde{\beta}_{n,\lambda,q}(0) = \tilde{\beta}_{n,\lambda,q}$ are called the modified degenerate q-Bernoulli numbers.

Kim introduced some identities on degenerate q-Bernoulli polynomials, which care defined by the p-adic q-integral on $\mathbb{Z}_p$ (see [17]).

And Kim et al. [10] gave symmetric identities for such polynomials under the symmetric group of degree n, which is the degeneration of the result of Kim and Kim in [4]. We mention that such symmetric identities could be obtained nicely by using p-adic q-integration on $\mathbb{Z}_p$, which is defined by Kim [14].

Recently many authors tried studying on the modified degenerate q-Bernoulli polynomials and numbers in [19, 20].

In this paper, we consider a new type of modified degenerate q-Bernoulli polynomials and numbers, and give some interesting identities on such polynomials. And we derive some identities of symmetry for those polynomials, by using p-adic q-integral on $\mathbb{Z}_p$, under the symmetric group of degree n.

2. Modified degenerate q-Bernoulli polynomials

We define the modified degenerate q-Bernoulli polynomials by the generating function

$$\int_{\mathbb{Z}_p} (1+\lambda)^{\frac{[x+y]_q}{\lambda}} d\mu_q(y) = \sum_{n=0}^{\infty} \tilde{\beta}_{n,\lambda,q}(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 0, \tilde{\beta}_{n,\lambda,q}(0) = \tilde{\beta}_{n,\lambda,q}$ are called the modified degenerate q-Bernoulli numbers. We can verify that

$$\lim_{\lambda\to 0} \tilde{\beta}_{n,\lambda,q}(x) = \beta_{n,q}(x) \tag{2.2}$$

where $\beta_{n,q}(x)$ are the Carlitz q-Bernoulli polynomials (see [1, 14]).

We consider

$$\begin{aligned}\int_{\mathbb{Z}_p} (1+\lambda)^{\frac{[x+y]_q}{\lambda}t} d\mu_q(y) &= \int_{\mathbb{Z}_p} e^{\frac{[x+y]_q}{\lambda}t\log(1+\lambda)} d\mu_q(y) \\ &= \sum_{n=0}^{\infty} \left(\frac{\log(1+\lambda)}{\lambda}\right)^n \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_q(y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\frac{\log(1+\lambda)}{\lambda}\right)^n \beta_{n,q}(x) \frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

From (2.3), we have the following theorem

Theorem 1. *For $n \geq 0$, we have*

$$\tilde{\beta}_{n,\lambda,q}(x) = \left(\frac{\log(1+\lambda)}{\lambda}\right)^n \beta_{n,q}(x).$$

Thus we get a few terms on $\tilde{\beta}_{n,\lambda,q}$ as follows:

$$\begin{aligned}\tilde{\beta}_{0,\lambda,q} &= 1, \\ \tilde{\beta}_{1,\lambda,q} &= -\frac{\log(1+\lambda)}{\lambda} \frac{1}{[2]_q}, \\ \tilde{\beta}_{2,\lambda,q} &= \left(\frac{\log(1+\lambda)}{\lambda}\right)^2 \frac{q}{[2]_q[3]_q}, \ldots\end{aligned}$$

Note that

$$(x)_n = \sum_{l=0}^{n} S_1(n,l) x^l, \quad (n \geq 0)$$

where $S_1(n,l)$ are the Stirling numbers of the first kind.

From the p-adic q-integral representation of the modified degenerate q-Bernoulli polynomials, we consider the following:

$$\begin{aligned}
&\int_{\mathbb{Z}_p} (1+\lambda)^{\frac{[x+y]_q}{\lambda}t} d\mu_q(y) \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{\frac{[x+y]_q}{\lambda}t}{n} \lambda^n d\mu_q(y) \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \sum_{l=0}^{n} S_1(n,l) \left(\frac{[x+y]_q}{\lambda}t\right)^l \frac{\lambda^n}{n!} d\mu_q(y) \qquad (2.4)\\
&= \sum_{l=0}^{\infty} \sum_{n=l}^{\infty} S_1(n,l)\lambda^{n-l} \frac{t^l}{n!} \int_{\mathbb{Z}_p} [x+y]_q^l d\mu_q(y) \\
&= \sum_{l=0}^{\infty} \left(\sum_{n=l}^{\infty} S_1(n,l)\lambda^{n-l} \frac{l!}{n!} \beta_{l,q}(x) \right) \frac{t^l}{l!}.
\end{aligned}$$

Thus we have the following identities.

Theorem 2. *For $n \geq 0$, $\tilde{\beta}_{n,\lambda,q}(x)$ can be written as*

$$\tilde{\beta}_{n,\lambda,q}(x) = \sum_{n=l}^{\infty} S_1(n,l)\lambda^{n-l} \frac{l!}{n!} \beta_{l,q}(x),$$

where $\beta_{l,q}(x)$ are Carlitz's q-Bernoulli polynomials.

Now we observe that

$$[x+y]_q = [x]_q + q^x [y]_q.$$

We can represent (2.1) as follows:

$$\begin{aligned}
&\int_{\mathbb{Z}_p} (1+\lambda)^{\frac{[x+y]_q}{\lambda}t} d\mu_q(y) \\
&= \int_{\mathbb{Z}_p} (1+\lambda)^{\frac{[x]_q}{\lambda}t} (1+\lambda)^{\frac{[y]_q}{\lambda}q^x t} d\mu_q(y) \\
&= (1+\lambda)^{\frac{[x]_q}{\lambda}t} \left(\int_{\mathbb{Z}_p} (1+\lambda)^{\frac{[y]_q}{\lambda}q^x t} d\mu_q(y) \right) \\
&= e^{\frac{[x]_q}{\lambda}t\log(1+\lambda)} \left(\sum_{m=0}^{\infty} \tilde{\beta}_{m,\lambda,q} \frac{q^{mx}t^m}{m!} \right) \\
&= \left(\sum_{l=0}^{\infty} \Big(\frac{\log(1+\lambda)}{\lambda}\Big)^l [x]_q^l \frac{t^l}{l!} \right) \Big(\sum_{m=0}^{\infty} \tilde{\beta}_{m,\lambda,q} \frac{q^{mx}t^m}{m!} \Big) \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \tilde{\beta}_{m,\lambda,q} [x]_q^{n-m} q^{mx} \Big(\frac{\log(1+\lambda)}{\lambda}\Big)^{n-m} \right) \frac{t^n}{n!}
\end{aligned}$$

Thus we have the following identities.

Theorem 2.1. For $n \geq 0$, we have

$$\tilde{\beta}_{n,\lambda,q}(x) = \sum_{m=0}^{n} \binom{n}{m} \tilde{\beta}_{m,\lambda,q} [x]_q^{n-m} q^{mx} \Big(\frac{\log(1+\lambda)}{\lambda} \Big)^{n-m}.$$

We consider the following distribution relation on modified degenerate q-Bernoulli polynomials.

$$\begin{aligned} &\int_{\mathbb{Z}_p} (1+\lambda)^{\frac{[x+y]_q}{\lambda}t} d\mu_q(y) \\ &\qquad = \lim_{N\to} \frac{1}{[dp^N]_q} \sum_{y=0}^{dp^N-1} (1+\lambda)^{\frac{[x+y]_q}{\lambda}t} q^y \\ &\qquad = \lim_{N\to} \frac{1}{[dp^N]_q} \sum_{a=0}^{d-1} \sum_{y=0}^{p^N-1} (1+\lambda)^{\frac{x+a+dy}{\lambda}t} q^{a+dy}. \end{aligned}$$

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{[d]_q [p^N]_{d^q}} \sum_{a=0}^{d-1} \sum_{y=0}^{p^N-1} (1+\lambda)^{\frac{1}{\lambda}[d]_q [\frac{x+a}{d}+y]_{q^d} t} q^a q^{dy} \\ &= \frac{1}{[d]_q} \sum_{a=0}^{d-1} q^a \lim_{N\to\infty} \frac{1}{[p^n]} \sum_{y=0}^{p^N-1} (1+\lambda)^{\frac{1}{\lambda}[d]_q [\frac{x+a}{d}+y]_{q^d} t} q^{dy} \\ &= \frac{1}{[d]_q} \sum_{a=0}^{d-1} q^a \int_{\mathbb{Z}_p} (1+\lambda)^{\frac{1}{\lambda}[d]_q [\frac{x+a}{d}+y]_{q^d} t} d\mu_{q^d}(y) \\ &= \frac{1}{[d]_q} \sum_{a=0}^{d-1} q^a \sum_{n=0}^{\infty} \tilde{\beta}_{n,\lambda,q^d} \Big(\frac{x+a}{a} \Big) \frac{[d]_q^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} [d]_q^{n-1} \sum_{a=0}^{d-1} q^a \tilde{\beta}_{n,\lambda,q^d} \Big(\frac{x+a}{a} \Big) \frac{t^n}{n!}, \end{aligned}$$

where $d \in \mathbb{N}$. Thus we obtain the following:

Theorem 2.2. For $n \geq 0$ and $d \in \mathbb{N}$, we have

$$\tilde{\beta}_{n,\lambda,q}(x) = [d]_q^{n-1} \sum_{a=0}^{d-1} q^a \tilde{\beta}_{n,\lambda,q^d} \Big(\frac{x+a}{d} \Big).$$

For $r \in \mathbb{N}$, we define the modified degenerate q-Bernoulli polynomials of order r as follows:

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda)^{\frac{[x_1+\cdots x_r+x]_q}{\lambda}t} d\mu_q(x_1)\cdots d\mu_q(x_r) = \sum_{n=0}^{\infty} \tilde{\beta}^{(r)}_{n,\lambda,q}(x)(x)\frac{t^n}{n!}. \tag{2.5}$$

When $x=0$, $\tilde{\beta}^{(r)}_{n,\lambda,q}(0) = \tilde{\beta}_{n,\lambda,q}$ are called the modified degenerate q-Bernoulli numbers of order r.

Now we observe that, for $d \in \mathbb{N}$,

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda)^{\frac{[x_1+\cdots x_r+x]_q}{\lambda}t} d\mu_q(x_1)\cdots d\mu_q(x_r) \\
&= \lim_{N\to\infty} \frac{1}{[dp^N]_q^r} \sum_{x_1=0}^{dP^N-1} \cdots \sum_{x_r=0}^{dP^N-1} (1+\lambda)^{\frac{[x_1+\cdots x_r+x]_q}{\lambda}t} q^{x_1+\cdots+x_r} \\
&= \lim_{N\to\infty} \frac{1}{[d]_q^r[p^N]_q^r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} \sum_{x_1=0}^{dP^N-1} \cdots \sum_{x_r=0}^{dP^N-1} \\
&\qquad (1+\lambda)^{\frac{[a_1+\cdots a_r+dx_1+\cdots dx_r]_q}{\lambda}t} q^{a_1+\cdots a_r+x_1+\cdots+x_r} \\
&= \frac{1}{[d]_q^r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} q^{a_1+\cdots a_r} \lim_{N\to\infty} \frac{1}{[p^N]_{q^d}^r} \sum_{x_1=0}^{dP^N-1} \cdots \sum_{x_r=0}^{dP^N-1} \\
&\qquad (1+\lambda)^{\frac{1}{\lambda}[\frac{a_1+\cdots+a_r+x}{d}+x_1+\cdots+x_r]_{q^d}[d]_q t} q^{x_1+\cdots+x_r} \\
&= \frac{1}{[d]_q^r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} q^{a_1+\cdots+a_r} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \\
&\qquad (1+\lambda)^{\frac{1}{\lambda}[\frac{a_1+\cdots+a_r+x}{d}+x_1+\cdots+x_r]_{q^d}[d]_q t} d\mu_{q^d}(x_1)\cdots d\mu_{q^d}(x_r) \\
&= \sum_{n=0}^{\infty} [d]_q^{n-r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} q^{a_1+\cdots a_r} \tilde{\beta}^{(r)}_{n,\lambda,q^d}\left(\frac{a_1+\cdots+a_r+x}{d}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.6}$$

Thus, comparing the coefficients of both sides of (2.6), we have the following theorem.

Theorem 2.3. For $n \geq 0$ and $d \in \mathbb{N}$, we have

$$\tilde{\beta}^{(r)}_{n,\lambda,q}(x) = [d]_q^{n-r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} q^{a_1+\cdots+a_r} \tilde{\beta}^{(r)}_{n,\lambda,q^d}\left(\frac{a_1+\cdots+a_r+x}{d}\right).$$

3. Identities of symmetry for the modified degenerate q-Bernoulli polynomials

We assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda|_p \leq 1, |t|_p < p^{-\frac{1}{p-1}}$. In this section, let $w_1, w_2, \ldots, w_n$ be positive integers. For $N \in \mathbb{N}$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} (1+\lambda)^{\frac{t}{\lambda}[w_1\cdots w_{n-1}y+w_1\cdots w_n x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j]_q} d\mu_{q^{w_1\cdots w_{n-1}}}(y)\\
&\quad= \lim_{N\to\infty}\frac{1}{[w_n p^N]_{q^{w_1\cdots w_{n-1}}}}\\
&\qquad\times\sum_{y=0}^{w_n p^N-1}(1+\lambda)^{\frac{t}{\lambda}[w_1\cdots w_{n-1}y+w_1\cdots w_n x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j]_q} q^{w_1\cdots w_{n-1}y} \qquad (3.1)\\
&\quad= \lim_{N\to\infty}\frac{1}{[w_n p^N]_{q^{w_1\cdots w_{n-1}}}}\sum_{k_n=0}^{w_n-1}\sum_{y=0}^{p^N-1} q^{w_1\cdots w_{n-1}(k_n+w_n y)}\\
&\qquad\times(1+\lambda)^{\frac{t}{\lambda}[w_1\cdots w_{n-1}(k_n+w_n y)+w_1\cdots w_n x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j]_q}.
\end{aligned}$$

Henceforth, we will use the abbreviations

$$\begin{aligned}
W &= w_1 w_2\cdots w_{n-1},\\
W_\sigma &= w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)},\\
YW &= y + w_n x + w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}.
\end{aligned} \qquad (3.2)$$

From (3.1), we note that

$$\begin{aligned}
&\frac{1}{[W]_q}\prod_{u=1}^{n-1}\sum_{k_u=0}^{w_u-1} q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j}\\
&\quad\times(1+\lambda)^{\frac{t}{\lambda}[Wy+Ww_n x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j]_q} d\mu_{q^W}(y)\\
&= \lim_{N\to\infty}\frac{1}{[Ww_n p^N]_q}\prod_{u=1}^{n-1}\prod_{k_u=0}^{w_u-1}\prod_{k_n=0}^{w_n-1}\sum_{y=0}^{p^N-1} q^{W(k_n+w_n y)+\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j w_n}\\
&\quad\times(1+\lambda)^{\frac{t}{\lambda}[W(k_n+w_n y)+Ww_n x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j]_q}.
\end{aligned} \qquad (3.3)$$

It is easy to show that (3.3) is invariant under any permutation in the symmetry group of degree n. Therefore, by (3.3), we obtain the following theorem.

Theorem 3.1. Let $w_1, w_2, \dots, w_n$ be positive integers. Then the following expressions

$$\frac{1}{[W_\sigma]_q}\prod_{u=1}^{n-1}\sum_{k_u=0}^{w_{\sigma(u)}-1} q^{w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j} \times (1+\lambda)^{\frac{t}{\lambda}[W_\sigma y+(w_1+\cdots+w_n)x+w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j]_q} d\mu_{q^{W_\sigma}}(y)$$

are the same for any permutation σ in the symmetry group of order n.

For the meaning of W_σ in the theorem above and W, YW in the equations below, recall the abbreviations given in (3.2).

We can have the following:

$$\Big[Wy + Ww_nx + w_n\sum_{j=1}^{n-1}\big(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\big)k_j\Big]_q = [W]_q\cdot[YW]_{q^W}. \tag{3.4}$$

From (3.4), we note that

$$\begin{aligned}\int_{\mathbb{Z}_p}&(1+\lambda)^{\frac{t}{\lambda}\left[Wy+Ww_nx+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q} d\mu_{q^W}(y)\\ &=\int_{\mathbb{Z}_p}(1+\lambda)^{\frac{t[W]_q}{\lambda}[YW]_{q^W}} d\mu_{q^W}(y)\\ &=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} S_1(n,m)\lambda^{n-m}\frac{m!}{n!}[W]_q^m\tilde{\beta}_{m,\lambda,q^W}\Big(w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\Big)\frac{t^m}{m!}.\end{aligned} \tag{3.5}$$

Therefore, by Theorem 3.1 and (3.5), we obtain the following theorem.

Theorem 3.2. For $m \geq 0$, $w_1, w_2, \dots, w_n \in \mathbb{N}$, the following expressions

$$[W_\sigma]_q^{m-1}\prod_{u=1}^{n-1}\prod_{k_u=0}^{w_{\sigma(u)}-1} q^{\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_{\sigma(i)}\right)k_j w_{\sigma(n)}} \times\sum_{n=m}^{\infty} S_1(n,m)\lambda^{n-m}\frac{m!}{n!}\tilde{\beta}_{m,\lambda,q^{W_\sigma}}\left(w_{\sigma(n)}x + w_{\sigma(n)}\sum_{j=1}^{n-1}\frac{k_j}{w_{\sigma(j)}}\right)$$

are the same for any permutation σ in the symmetry group of order n.

Now we observe that

$$[YW]_{q^W} = \frac{[w_n]_q}{[W]_q}\Big[\sum_{j=1}^{n-1}\big(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\big)k_j\Big]_{q^{w_n}} + q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j}[y+w_nx]_{q^W}. \tag{3.6}$$

By (2.1), we get

$$\begin{aligned}
&\tilde{\beta}_{m,\lambda,q^W}\Big(w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j}\Big) \\
&\qquad = \sum_{m=l}^{\infty} S_1(m,l)\lambda^{m-l} \frac{l!}{m!} \int_{\mathbb{Z}_p} [YW]_{q^W}^l d\mu_{q^W}(y).
\end{aligned} \tag{3.7}$$

From (3.6), we can derive the following equation:

$$\begin{aligned}
&\int_{\mathbb{Z}_p} [YW]_{q^W}^l d\mu_{q^W}(y) \\
&\quad = \sum_{s=0}^{l} \binom{l}{s} \Big(\frac{[w_n]_q}{[W]_q}\Big)^{l-s} \Big[\sum_{j=1}^{n-1} (\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i) k_j\Big]_{q^{w_n}}^{l-s} q^{w_n s \sum_{j=1}^{n-1} \big(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\big) k_j} \\
&\qquad \times \int_{\mathbb{Z}_p} [y + w_n x]_{q^W}^s d\mu_{q^W}(y) \\
&\quad = \sum_{s=0}^{l} \binom{l}{s} \Big(\frac{[w_n]_q}{[W]_q}\Big)^{l-s} \Big[\sum_{j=1}^{n-1} (\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i) k_j\Big]_{q^{w_n}}^{l-s} q^{w_n s \sum_{j=1}^{n-1} \big(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\big) k_j} \\
&\qquad \times \beta_{s,q^W}(w_n x).
\end{aligned} \tag{3.8}$$

By (3.7) and (3.8), we get

$$\begin{aligned}
&\tilde{\beta}_{m,\lambda,q^W}\Big(w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j}\Big) \\
&\quad = \sum_{m=l}^{\infty} \sum_{p=0}^{m} \sum_{s=0}^{p} \binom{p}{s} S_1(m,l)\lambda^{m-l} \frac{l!}{m!} S_1(m,p)\lambda^{m-p} [W]_q^{s-m} [w_n]_q^{p-s} \\
&\qquad \times \Big[\sum_{j=1}^{n-1} (\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i) k_j\Big]_{q^{w_n}}^{p-s} \times q^{w_n s \sum_{j=1}^{n-1} \big(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\big) k_j} \beta_{s,q^W}(w_n x).
\end{aligned} \tag{3.9}$$

From (3.9), we note that

$$[W]_q^{m-1}\prod_{u=1}^{n-1}\sum_{k_u=0}^{w_u-1} q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j}\tilde{\beta}_{m,\lambda,q^W}\Big(w_n x + w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\Big)$$
$$=\prod_{u=1}^{n-1}\sum_{k_u=0}^{w_u-1}\sum_{m=l}^{\infty}\sum_{p=0}^{m}\sum_{s=0}^{p}\binom{p}{s}S_1(m,l)\lambda^{m-l}\frac{l!}{m!}S_1(m,p)\lambda^{m-p}[W]_q^{s-1}[w_n]_q^{p-s}$$
$$\times\Big[\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i)k_j\Big]_{q^{w_n}}^{p=s} q^{(s+1)w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j}\beta_{s,q^W}(w_n x)$$
$$=\sum_{m=l}^{\infty}\sum_{p=0}^{m}\sum_{s=0}^{p}\binom{p}{s}S_1(m,l)\lambda^{m-l}\frac{l!}{m!}S_1(m,p)\lambda^{m-p}[W]_q^{s-1}[w_n]_q^{p-s}\beta_{s,q^W}(w_n x)$$
$$\times\prod_{u=1}^{n-1}\sum_{k_u=0}^{w_u-1} q^{(s+1)w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j}\Big[\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i)k_j\Big]_{q^{w_n}}^{p-s}$$
$$=\sum_{m=l}^{\infty}\sum_{p=0}^{m}\sum_{s=0}^{p}\binom{p}{s}S_1(m,l)\lambda^{m-l}\frac{l!}{m!}S_1(m,p)\lambda^{m-p}[W]_q^{s-1}[w_n]_q^{p-s}\beta_{s,q^W}(w_n x)$$
$$\times K_{n,q^{w_n}}(w_1,\ldots,w_{n-1}\mid p-s,s), \tag{3.10}$$

where

$$K_{n,q^{w_n}}(w_1,\ldots,w_{n-1}\mid p-s,s)=$$
$$\prod_{u=1}^{n-1}\sum_{k_u=0}^{w_u-1} q^{(s+1)w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j}\Big[\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i)k_j\Big]_{q^{w_n}}^{p-s}. \tag{3.11}$$

Therefore, by (3.10) and (3.11), we obtain the following theorem.

Theorem 3.3. Let $m \geq 0$ and $w_1, w_2, \ldots, w_n \in \mathbb{N}$. Then the following expressions

$$\sum_{m=l}^{\infty}\sum_{p=0}^{m}\sum_{s=0}^{p}\binom{p}{s}S_1(l,u)\lambda^{l-u}\frac{l!}{m!}S_1(m,p)\lambda^{m-p}[W_\sigma]_q^{s-1}[w_{\sigma(n)}]_q^{p-s}\beta_{s,q_\sigma^W}(w_{\sigma(n)}x)$$
$$\times K_{n,q^{w_{\sigma(n)}}}(w_{\sigma(1)},\ldots,w_{\sigma(n-1)}\mid p-s,s)$$

are the same for any permutation σ in the symmetry group of order n.

Note that some identities of Bernoulli and Euler polynomials are studied by several authors (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]).

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