Some identities on modified degenerate $q$-Bernoulli polynomials and numbers

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Abstract
Kim introduced some identities on degenerate $q$-Bernoulli polynomials, which care defined by the $p$-adic $q$-integral on $\mathbb{Z}_p$ (see [17]).

And Kim et al. [10] gave symmetric identities for such polynomials under the symmetric group of degree $n$, which is the degeneration of the result of Kim and

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Kim in [4]. We mention that such symmetric identities could be obtained nicely by using $p$-adic $q$-integration on $\mathbb{Z}_p$, which is defined by Kim [14].

In this paper, we consider a new type of modified degenerate $q$-Bernoulli polynomials and numbers, and give some interesting identities on such polynomials. And we derive some identities of symmetry for those polynomials, by using $p$-adic $q$-integral on $\mathbb{Z}_p$, under the symmetry group of degree $n$.

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1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $q \in \mathbb{C}_p$ be an indeterminate such that $|1 - q|_p < p^{-\frac{1}{p-1}}$. The $q$-analogue of the number $x$ is defined by $[x]_q = \frac{1 - qx}{1 - q}$.

Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. Then the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) \mu_q(x + p^N \mathbb{Z}_p),$$

(1.1)

In [1], L. Carlitz considered the $q$-analogue of Bernoulli numbers which are given by the recurrence relation

$$\beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases}$$

(1.2)

with the usual convention of replacing $\beta^n_q$ by $\beta_{n,q}$. He defined the $q$-Bernoulli polynomials as

$$\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{x} \beta_{l,q}[x]^{n-l},$$

(1.3)
Some identities on modified degenerate $q$-Bernoulli polynomials

In [14], Kim proved that the Carlitz’s $q$-Bernoulli polynomials are represented as the $p$-adic $q$-integral on $\mathbb{Z}_p$ which are given by

$$\int_{\mathbb{Z}_p} [x + y]_q^n d\mu_q(y) = \beta_{n,q}(x), \quad (n \geq 0). \quad (1.4)$$

When $x = 0$, $\beta_{n,q} = \beta_{n,q}(0)$ are the Carlitz $q$-Bernoulli numbers.

In [2], L. Carlitz also introduced the degenerate Bernoulli polynomials which are given by the generating function

$$\int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{1}{\lambda}} (x + y)q_y d\mu_q(y) = \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!}. \quad (1.5)$$

Note that $\lim_{\lambda \to 0} B_{n,\lambda}(x) = B_{n}(x)$, where $B_{n}(x)$ are ordinary Bernoulli polynomials (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]). When $x = 0$, $B_{n,\lambda} = B_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

In [17], Kim considered degenerate $q$-Bernoulli polynomials which are given by the generating function

$$\int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{1}{\lambda}} (x + y)q_y d\mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,\lambda,q}(x) \frac{t^n}{n!}. \quad (1.6)$$

When $x = 0$, $\beta_{n,\lambda,q} = \beta_{n,\lambda,q}(0)$ are called (fully) degenerate $q$-Bernoulli numbers.

We note that $\beta_{n,\lambda,q} = \beta_{n,\lambda,q}(0)$ are called (fully) degenerate $q$-Bernoulli numbers.

The following diagram shows how $q$-Bernoulli polynomials become Kim’s degenerate $q$-Bernoulli polynomials and our modified degenerate $q$-Bernoulli polynomials.
We note that Lee and Jang recently defined and studied the modified degenerate $q$-Bernoulli polynomials by the generating function:

$$
\int_{\mathbb{Z}_p} q^{-y} (1 + \lambda) \frac{[x+y]_q}{x} d\mu_q(y) = \sum_{n=0}^{\infty} \tilde{B}_{n,q,\lambda}(x) \frac{t^n}{n!}, \quad \text{(see [20])},
$$

(1.7)

The generating functions of Stirling numbers are given by

$$(\log(1 + t))^n = n! \sum_{l=n}^{\infty} S_1(l, n) \frac{t^l}{l!}, \quad (n \geq 0)$$

(1.8)

and

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!}, \quad (n \geq 0),$$

(1.9)

where $S_1(n, l)$ are the Stirling numbers of the first kind, and $S_2(l, n)$ are the Stirling numbers of the second kind.

We define the modified degenerate $q$-Bernoulli polynomials by the generating function

$$
\int_{\mathbb{Z}_p} (1 + \lambda) \frac{[x+y]_q}{x} d\mu_q(y) = \sum_{n=0}^{\infty} \tilde{\beta}_{n,\lambda,q} (x) \frac{t^n}{n!}.
$$

(2.1)

When $x = 0$, $\tilde{\beta}_{n,\lambda,q}(0) = \tilde{\beta}_{n,\lambda,q}$ are called the modified degenerate $q$-Bernoulli numbers.

Kim introduced some identities on degenerate $q$-Bernoulli polynomials, which care defined by the $p$-adic $q$-integral on $\mathbb{Z}_p$ (see [17]).

And Kim et al. [10] gave symmetric identities for such polynomials under the symmetric group of degree $n$, which is the degeneration of the result of Kim and Kim in [4]. We mention that such symmetric identities could be obtained nicely by using $p$-adic $q$-integration on $\mathbb{Z}_p$, which is defined by Kim [14].

Recently many authors tried studying on the modified degenerate $q$-Bernoulli polynomials and numbers in [19, 20].

In this paper, we consider a new type of modified degenerate $q$-Bernoulli polynomials and numbers, and give some interesting identities on such polynomials. And we derive some identities of symmetry for those polynomials, by using $p$-adic $q$-integral on $\mathbb{Z}_p$, under the symmetric group of degree $n$.

## 2. Modified degenerate $q$-Bernoulli polynomials

We define the modified degenerate $q$-Bernoulli polynomials by the generating function

$$
\int_{\mathbb{Z}_p} (1 + \lambda) \frac{[x+y]_q}{x} d\mu_q(y) = \sum_{n=0}^{\infty} \tilde{\beta}_{n,\lambda,q} (x) \frac{t^n}{n!}.
$$

(2.1)
When \( x = 0 \), \( \tilde{\beta}_{n,\lambda,q}(0) = \tilde{\beta}_{n,\lambda,q} \) are called the modified degenerate \( q \)-Bernoulli numbers. We can verify that

\[
\lim_{\lambda \to 0} \tilde{\beta}_{n,\lambda,q}(x) = \beta_{n,q}(x)
\]

where \( \beta_{n,q}(x) \) are the Carlitz \( q \)-Bernoulli polynomials (see [1, 14]).

We consider

\[
\int_{\mathbb{Z}_p} (1 + \lambda)^{\frac{x+y}{\lambda}} d\mu_q(y) = \int_{\mathbb{Z}_p} e^{\frac{x+y}{\lambda} \log(1+\lambda)} d\mu_q(y) \\
= \sum_{n=0}^\infty \left( \frac{\log(1+\lambda)}{\lambda} \right)^n \int_{\mathbb{Z}_p} [x+y]^n d\mu_q(y) \frac{t^n}{n!} \\
= \sum_{n=0}^\infty \left( \frac{\log(1+\lambda)}{\lambda} \right)^n \beta_{n,q}(x) \frac{t^n}{n!}.
\]

From (2.3), we have the following theorem

**Theorem 1.** For \( n \geq 0 \), we have

\[
\tilde{\beta}_{n,\lambda,q}(x) = \left( \frac{\log(1+\lambda)}{\lambda} \right)^n \beta_{n,q}(x).
\]

Thus we get a few terms on \( \tilde{\beta}_{n,\lambda,q} \) as follows:

\[
\tilde{\beta}_{0,\lambda,q} = 1, \\
\tilde{\beta}_{1,\lambda,q} = -\frac{\log(1+\lambda)}{\lambda} \frac{1}{[2]_q}, \\
\tilde{\beta}_{2,\lambda,q} = \left( \frac{\log(1+\lambda)}{\lambda} \right)^2 \frac{q}{[2]_q[3]_q}, \ldots
\]

Note that

\[
(x)_n = \sum_{l=0}^n S_1(n, l)x^l, \quad (n \geq 0)
\]

where \( S_1(n, l) \) are the Stirling numbers of the first kind.

From the \( p \)-adic \( q \)-integral representation of the modified degenerate \( q \)-Bernoulli polynomials...
polynomials, we consider the following:

\[
\int_{\mathbb{Z}_p} (1 + \lambda \frac{[x+y]_q}{\lambda}) \, d\mu_q(y)
\]

\[
= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \left( \frac{[x+y]_q}{n} \lambda^n \right) d\mu_q(y)
\]

\[
= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \sum_{l=0}^{n} S_1(n, l) \left( \frac{[x+y]_q}{\lambda} \right)^l \lambda^n \, d\mu_q(y)
\]

\[
= \sum_{l=0}^{\infty} \sum_{n=l}^{\infty} \lambda^{n-l} \frac{l!}{n!} \int_{\mathbb{Z}_p} [x+y]_q^l \, d\mu_q(y)
\]

\[
= \sum_{l=0}^{\infty} \left( \sum_{n=l}^{\infty} \lambda^{n-l} \frac{l!}{n!} \beta_{l,q}(x) \right) \frac{l!}{l!}.
\]

Thus we have the following identities.

**Theorem 2.** For \( n \geq 0 \), \( \tilde{\beta}_{n,\lambda,q}(x) \) can be written as

\[
\tilde{\beta}_{n,\lambda,q}(x) = \sum_{n=l}^{\infty} S_1(n, l) \lambda^{n-l} \frac{l!}{n!} \beta_{l,q}(x),
\]

where \( \beta_{l,q}(x) \) are Carlitz’s \( q \)-Bernoulli polynomials.

Now we observe that

\[
[x + y]_q = [x]_q + q^x[y]_q.
\]

We can represent (2.1) as follows:

\[
\int_{\mathbb{Z}_p} (1 + \lambda \frac{[x+y]_q}{\lambda}) d\mu_q(y)
\]

\[
= \int_{\mathbb{Z}_p} (1 + \lambda \frac{[x+y]_q}{\lambda}) (1 + \lambda \frac{[y]_q}{\lambda} q^x) \, d\mu_q(y)
\]

\[
= (1 + \lambda \frac{[y]_q}{\lambda} q^x) \left( \int_{\mathbb{Z}_p} (1 + \lambda \frac{[y]_q}{\lambda} q^x) \, d\mu_q(y) \right)
\]

\[
= e^{\frac{[y]_q}{\lambda} \log(1+\lambda)} \left( \sum_{m=0}^{\infty} \beta_{m,\lambda,q} \frac{q^{mx} x^m}{m!} \right)
\]

\[
= \left( \sum_{l=0}^{\infty} \frac{\log(1+\lambda)}{\lambda} \right) [x]_q^l \frac{l!}{l!} \left( \sum_{m=0}^{\infty} \beta_{m,\lambda,q} \frac{q^{mx} x^m}{m!} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \frac{n}{m} \beta_{m,\lambda,q} [x]_q^{n-m} q^{mx} \left( \frac{\log(1+\lambda)}{\lambda} \right)^{n-m} \right) \frac{t^n}{n!}
\]

(2.4)
Thus we have the following identities.

**Theorem 2.1.** For \( n \geq 0 \), we have

\[
\tilde{\beta}_{n, \lambda, q}(x) = \sum_{m=0}^{n} \binom{n}{m} \tilde{\beta}_{m, \lambda, q}[x]q^{n-m}q^{mx}\left(\frac{\log(1 + \lambda)}{\lambda}\right)^{n-m}.
\]

We consider the following distribution relation on modified degenerate \( q \)-Bernoulli polynomials.

\[
\int_{\mathbb{Z}_p} (1 + \lambda) \frac{x+y}{\lambda} \, dt \mu_q(y)
\]

\[
= \lim_{N \to 1} \frac{1}{[dp^N]_q} \sum_{y=0}^{d-1} (1 + \lambda) \frac{x+y}{\lambda} \, q^y
\]

\[
= \lim_{N \to 1} \frac{1}{[dp^N]_q} \sum_{a=0}^{d-1} \sum_{y=0}^{p^N-1} (1 + \lambda) \frac{x+y+a}{\lambda} \, q^a q^y.
\]

\[
\lim_{N \to \infty} \frac{1}{[d]_q[p^N]_q] \sum_{a=0}^{d-1} \sum_{y=0}^{p^N-1} (1 + \lambda) \frac{x+y+a}{a} \, q^a q^y
\]

\[
= \frac{1}{[d]_q} \sum_{a=0}^{d-1} q^a \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{y=0}^{p^N-1} (1 + \lambda) \frac{x+y+a}{a} \, q^a q^y
\]

\[
= \frac{1}{[d]_q} \sum_{a=0}^{d-1} q^a \int_{\mathbb{Z}_p} (1 + \lambda) \frac{x+y+a}{a} \, q^a q^y \, d\mu_{p^a}(y)
\]

\[
= \frac{1}{[d]_q} \sum_{a=0}^{d-1} q^a \sum_{n=0}^{\infty} \tilde{\beta}_{n, \lambda, q}[x+a] \left(\frac{x+a}{a}\right) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} [d]_q^{n-1} \sum_{a=0}^{d-1} q^a \tilde{\beta}_{n, \lambda, q}[x+a] \left(\frac{x+a}{a}\right) \frac{t^n}{n!}.
\]

where \( d \in \mathbb{N} \). Thus we obtain the following:

**Theorem 2.2.** For \( n \geq 0 \) and \( d \in \mathbb{N} \), we have

\[
\tilde{\beta}_{n, \lambda, q}(x) = [d]_q^{n-1} \sum_{a=0}^{d-1} q^a \tilde{\beta}_{n, \lambda, q}[x+a] \left(\frac{x+a}{d}\right).
\]
For \( r \in \mathbb{N} \), we define the modified degenerate \( q \)-Bernoulli polynomials of order \( r \) as follows:

\[
\int_{Z_p} \cdots \int_{Z_p} (1 + \lambda)^f d\mu_q(x_1) \cdots d\mu_q(x_r) = \sum_{n=0}^{\infty} \tilde{\beta}^{(r)}_{n, \lambda, q}(x) \frac{t^n}{n!}. \tag{2.5}
\]

When \( x = 0 \), \( \tilde{\beta}^{(r)}_{n, \lambda, q}(0) = \tilde{\beta}_{n, \lambda, q} \) are called the modified degenerate \( q \)-Bernoulli numbers of order \( r \).

Now we observe that, for \( d \in \mathbb{N} \),

\[
\int_{Z_p} \cdots \int_{Z_p} (1 + \lambda)^f d\mu_q(x_1) \cdots d\mu_q(x_r)
\]

\[
= \lim_{N \to \infty} \frac{1}{d!} \sum_{x_1=0}^{dN-1} \cdots \sum_{x_r=0}^{dN-1} (1 + \lambda)^f \sum_{a_1=0}^{d} \cdots \sum_{a_r=0}^{d} q^{a_1 + \cdots + a_r} \frac{t^n}{n!}
\]

\[
= \frac{1}{d!} \sum_{a_1=0}^{d} \cdots \sum_{a_r=0}^{d} q^{a_1 + \cdots + a_r} \lim_{N \to \infty} \frac{1}{d!} \sum_{x_1=0}^{dN-1} \cdots \sum_{x_r=0}^{dN-1} (1 + \lambda)^f \sum_{a_1=0}^{d} \cdots \sum_{a_r=0}^{d} q^{a_1 + \cdots + a_r} \frac{t^n}{n!}
\]

\[
= \frac{1}{d!} \sum_{a_1=0}^{d} \cdots \sum_{a_r=0}^{d} q^{a_1 + \cdots + a_r} \int_{Z_p} \cdots \int_{Z_p} (1 + \lambda)^f d\mu_q(x_1) \cdots d\mu_q(x_r)
\]

\[
= \sum_{n=0}^{\infty} [d]^{-r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} q^{a_1 + \cdots + a_r} \tilde{\beta}^{(r)}_{n, \lambda, q}(x) \frac{(a_1 + \cdots + a_r + x)}{d} \frac{t^n}{n!}.
\]

Thus, comparing the coefficients of both sides of (2.6), we have the following theorem.

**Theorem 2.3.** For \( n \geq 0 \) and \( d \in \mathbb{N} \), we have

\[
\tilde{\beta}^{(r)}_{n, \lambda, q}(x) = [d]^{-r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} q^{a_1 + \cdots + a_r} \tilde{\beta}^{(r)}_{n, \lambda, q}(x) \frac{(a_1 + \cdots + a_r + x)}{d} \frac{t^n}{n!}.
\]
3. Identities of symmetry for the modified degenerate \( q \)-Bernoulli polynomials

We assume that \( \lambda, t \in \mathbb{C}_p \) with \( |\lambda|_p \leq 1, |t|_p < p^{-\frac{1}{p-1}} \). In this section, let \( w_1, w_2, \ldots, w_n \) be positive integers. For \( N \in \mathbb{N} \), we have

\[
\int_{\mathbb{Z}_p} (1 + \lambda) \frac{(w_1 \cdots w_{n-1} y + w_1 \cdots w_n x + w_n \sum_{j=1}^{n-1} (\prod_{i=1}^{n-1} w_i) k_j)_q}{w_n p^{N-1}} d\mu_{q^{w_1 \cdots w_{n-1}}} (y) = \lim_{N \to \infty} \frac{1}{w_n p^{N-1}} \sum_{k_n=0}^{w_n-1} \sum_{y=0}^{w_n-1} (1 + \lambda) \frac{(w_1 \cdots w_{n-1} (k_n + w_n y) + w_1 \cdots w_n x + w_n \sum_{j=1}^{n-1} (\prod_{i=1}^{n-1} w_i) k_j)_q}{w_n p^{N-1}} \times (1 + \lambda)
\]

(3.1)

Henceforth, we will use the abbreviations

\[
W = w_1 w_2 \cdots w_{n-1},
\]

\[
W_\sigma = w_\sigma(1) w_\sigma(2) \cdots w_\sigma(n-1),
\]

\[
YW = y + w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j}.
\]

From (3.1), we note that

\[
\frac{1}{[W]_q} \prod_{u=1}^{n-1} \sum_{k_u=0}^{w_u-1} q w_n \sum_{j=1}^{n-1} (\prod_{i=1}^{n-1} x w_i) k_j
\]

\[
\times (1 + \lambda) \frac{(w_1 y + W w_n x + w_n \sum_{j=1}^{n-1} (\prod_{i=1}^{n-1} x w_i) k_j)_q}{d\mu_{q^W} (y)} = \lim_{N \to \infty} \frac{1}{[W w_n p^N]_q} \prod_{u=1}^{n-1} \prod_{k_u=0}^{w_u-1} \sum_{y=0}^{w_n-1} q w_n \sum_{j=1}^{n-1} (\prod_{i=1}^{n-1} x w_i) k_j
\]

(3.3)

It is easy to show that (3.3) is invariant under any permutation in the symmetry group of degree \( n \). Therefore, by (3.3), we obtain the following theorem.
Theorem 3.1. Let \( w_1, w_2, \ldots, w_n \) be positive integers. Then the following expressions

\[
\frac{1}{[W_\sigma]_q} \prod_{u=1}^{n-1} \sum_{k_u=0}^{w_{\sigma(u)}-1} q^{w_{\sigma(u)} \sum_{j=1}^{n-1} \prod_{i \neq j}^{n-1} w_i} k_j
\]

\[
\times (1 + \lambda)
\]

\[
\left[ W \sigma y + (w_1 + \ldots + w_n) x + w_{\sigma(n)} \sum_{j=1}^{n-1} \prod_{i \neq j}^{n-1} w_i \right]_q
\]

\[
\times \left[ y + (w_1 + \ldots + w_n) x + w_{\sigma(n)} \sum_{j=1}^{n-1} \prod_{i \neq j}^{n-1} w_i \right]_q d\mu_q w_{\sigma}(y)
\]

are the same for any permutation \( \sigma \) in the symmetry group of order \( n \).

For the meaning of \( W_\sigma \) in the theorem above and \( W, YW \) in the equations below, recall the abbreviations given in (3.2).

We can have the following:

\[
\left[ Wy + Ww_n x + w_n \sum_{j=1}^{n-1} \prod_{i \neq j}^{n-1} w_i \right]_q = [W]_q \cdot [YW]_q w.
\]

(3.4)

From (3.4), we note that

\[
\int_{Z_p} (1 + \lambda) \frac{[W y + Ww_n x + w_n \sum_{j=1}^{n-1} \prod_{i \neq j}^{n-1} w_i]_q}{[YW]_q w} d\mu_q w(y)
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=m} S_1(n, m) \lambda^{n-m} \frac{m!}{n!} [W]_q^m \beta_{m, \lambda, q} w \left( w_n x + w_n \sum_{j=1}^{n-1} \prod_{i \neq j}^{n-1} w_i \right) \frac{m^n}{m!}.
\]

(3.5)

Therefore, by Theorem 3.1 and (3.5), we obtain the following theorem.

Theorem 3.2. For \( m \geq 0, w_1, w_2, \ldots, w_n \in \mathbb{N} \), the following expressions

\[
[W_\sigma]_q^m \prod_{u=1}^{n-1} \prod_{k_u=0}^{w_{\sigma(u)}-1} q^{w_{\sigma(u)} \sum_{j=1}^{n-1} \prod_{i \neq j}^{n-1} w_i} \frac{\sum_{j=1}^{n-1} \prod_{i \neq j}^{n-1} w_i}{k_j} \sum_{n=m}^{\infty} S_1(n, m) \lambda^{n-m} \frac{m!}{n!} \beta_{m, \lambda, q} w_{\sigma(n)} w_{\sigma(n)} \sum_{j=1}^{n-1} \prod_{i \neq j}^{n-1} w_i \frac{k_j}{w_{\sigma(j)}(j)}
\]

are the same for any permutation \( \sigma \) in the symmetry group of order \( n \).

Now we observe that

\[
[YW]_q^w = \frac{[W]_q [\sum_{j=1}^{n-1} \prod_{i \neq j}^{n-1} w_i]_q}{[W]_q} w_n \sum_{j=1}^{n-1} \prod_{i \neq j}^{n-1} w_i k_j + q^{w_n \sum_{j=1}^{n-1} \prod_{i \neq j}^{n-1} w_i} [y + w_n x]_q w.
\]

(3.6)
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By (2.1), we get

$$
\tilde{\beta}_{m, \lambda, q W}(w_n x + w_n \sum_{j=1}^{n-1} k_j w_j)
= \sum_{m=l}^{\infty} S_1(m, l) \lambda^{m-l} \frac{l!}{m!} \int_{\mathbb{Z}_p} [Y^l W]_q^l d\mu_q(y).
$$

(3.7)

From (3.6), we can derive the following equation:

$$
\int_{\mathbb{Z}_p} [Y^l W]_q^l d\mu_q(y)
= \sum_{s=0}^{l} \binom{l}{s} \left[\frac{[w_n]q}{[W]q}\right] \sum_{j=1}^{n-1} \sum_{i=1, i \neq j}^{n-1} (\prod_{i=1}^{n-1} w_i) k_j
\times \int_{\mathbb{Z}_p} [y + w_n x]_q^s d\mu_q(y)
= \sum_{s=0}^{l} \binom{l}{s} \left[\frac{[w_n]q}{[W]q}\right] \sum_{j=1}^{n-1} \sum_{i=1, i \neq j}^{n-1} (\prod_{i=1}^{n-1} w_i) k_j
\times \beta_{s, q W}(w_n x).
$$

(3.8)

By (3.7) and (3.8), we get

$$
\tilde{\beta}_{m, \lambda, q W}(w_n x + w_n \sum_{j=1}^{n-1} k_j w_j)
= \sum_{m=l}^{\infty} \sum_{p=0}^{m} \sum_{s=0}^{p} \binom{p}{s} S_1(m, l) \lambda^{m-l} \frac{l!}{m!} S_1(m, p) \lambda^{m-p} [W]_q^{m-p} [w_n]_q^{p-s}
\times \left[\sum_{j=1}^{n-1} \sum_{i=1, i \neq j}^{n-1} (\prod_{i=1}^{n-1} w_i) k_j\right]_{q^{w_n}}^{p-s} \times q^{w_n} \sum_{j=1}^{n-1} \sum_{i=1, i \neq j}^{n-1} (\prod_{i=1}^{n-1} w_i) k_j
\times \beta_{s, q W}(w_n x).
$$

(3.9)
From (3.9), we note that

\[ [W]^m_q^{-1} \prod_{u=1}^{m-1} \sum_{k_u=0}^{n-1} q^{j-1} (\prod_{i \neq j}^{n-1} w_i) k_j = \prod_{u=1}^{m-1} \sum_{k_u=0}^{n-1} q^{j-1} (\prod_{i \neq j}^{n-1} w_i) k_j \]

\[ = \prod_{u=1}^{m-1} \sum_{k_u=0}^{n-1} q^{j-1} (\prod_{i \neq j}^{n-1} w_i) k_j \]

\[ \times \sum_{m=1}^{n-1} \sum_{p=0}^{m} \left( \sum_{j=1}^{n-1} (\prod_{i \neq j}^{n-1} w_i) k_j \right)^{p-s} \]

\[ \times K_{n,q^{w_n}}(w_1, \ldots, w_{n-1} | p-s, s), \]

where

\[ K_{n,q^{w_n}}(w_1, \ldots, w_{n-1} | p-s, s) = \prod_{u=1}^{n-1} \sum_{k_u=0}^{n-1} q^{j-1} (\prod_{i \neq j}^{n-1} w_i) k_j \]

\[ \times \sum_{m=1}^{n-1} \sum_{p=0}^{m} \left( \sum_{j=1}^{n-1} (\prod_{i \neq j}^{n-1} w_i) k_j \right)^{p-s} \]

(3.11)

Therefore, by (3.10) and (3.11), we obtain the following theorem.

**Theorem 3.3.** Let \( m \geq 0 \) and \( w_1, w_2, \ldots, w_n \in \mathbb{N} \). Then the following expressions

\[ \sum_{m=1}^{\infty} \sum_{p=0}^{m} \sum_{s=0}^{p} \left( \sum_{j=1}^{n-1} (\prod_{i \neq j}^{n-1} w_i) k_j \right)^{p-s} \beta_{s,q^w}(w_{\sigma(n)} x) \]

\[ \times K_{n,q^{w_n}}(w_{\sigma(1)}, \ldots, w_{\sigma(n-1)} | p-s, s) \]

are the same for any permutation \( \sigma \) in the symmetry group of order \( n \).

Note that some identities of Bernoulli and Euler polynomials are studied by several authors (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]).
Some identities on modified degenerate q-Bernoulli polynomials

References


