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Slant Riemannian maps from Kenmotsu manifolds into Riemannian manifolds

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Abstract

In this paper we obtain some results on slant Riemannian maps from Kenmotsu manifolds into Riemannian manifolds by taking vertical structure vector field. Further, we study necessary and sufficient conditions for slant Riemannian maps to be harmonic and totally geodesic. At last we investigate some decomposition theorems and provide some examples to show the existence of such maps admitting vertical structure vector field.

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1. Introduction

Let (M, g_M) and (N, g_N) be two Riemannian manifolds, where $\dim M = m$ and $dimN = n$. Let f be a differentiable map from a Riemannian manifold M into N. Then according to the conditions on the map f, we call it a harmonic map [4] and a totally geodesic map [1] etc.

On other hand submersions have been studied widely in differential geometry. Riemannian submersions between Riemannian manifolds were studied by O'Neill [11] and Gray [6]. Such submersions between Riemannian manifolds equipped with an additional structure of almost complex type was firstly studied by Watson [20]. There are several kinds of Riemannian submersions like:

Almost Hermitian submersion [19] and slant submersions from almost Hermitian manifolds [17] etc. As we know, Riemannian submersions are related with physics and have their applications in the Yang-Mills theory ([3], [20]), Supergravity and superstring theories ([7], [9]) and Kaluza-Klein theory [8] etc.

A. Fischer [5] introduced a Riemannian map between Riemannian manifolds which generalizes and unifies the notions of an isometric immersion, a Riemannian submersion and an isometry. Let $f : (M, g_M) \to (N, g_N)$ be a differentiable map between Riemannian manifolds such that $0 < rank\ f_* < \min(m, n)$. If we denote the *kernal* space of f_* by $kerf_*$ and the orthogonal complementary space of $kerf_*$ by $(kerf_*)^\perp$ in TM, then the TM has the following orthogonal decomposition:

$$TM = \ker f_* \oplus (kerf_*)^\perp \tag{1.1}$$

where $\ker f_* = D \oplus \{\xi\}$, D is a distribution in $\ker f_*$ and ξ is orthogonal vector field to D in $\ker f_*$.

Also, if we denote the *range* of f_* by $rangef_*$ and for a point $q \in M$ the orthogonal complementary space of $rangef_{*f(q)}$ by $(rangef_{*f(q)})^\perp$ in $T_{f(q)}N$. Then the tangent space $T_{f(q)}N$ has the following orthogonal decomposition:

$$T_{f(q)}N = (rangef_{*f(q)}) \oplus (rangef_{*f(q)})^\perp.$$

A differentiable map $f : (M, g_M) \to (N, g_N)$ is called a Riemannian map at $q \in M$ if the horizontal restriction $f^h_{*q} : (\ker f_{*q})^\perp \to (rangef_{*f(q)})$ is linear isometry between the inner product space $((\ker f_{*q})^\perp, g_M(q)|_{(\ker f_{*q})^\perp})$ and $(rangef_{*f(q)}, g_N(f(q))|_{(rangef_{*q})})$. Fischer [5] define that a Riemannian map which is as isometric as it can be. On the other hand, a differentiable map $f : (M, g_M) \to (N, g_N)$ between Riemannian manifolds (M, g_M) and (N, g_N) is called a Riemannian map if it satisfies the equation:

$$g_N(f_*X, f_*Y) = g_M(X, Y), \quad \text{for } X, Y \in \Gamma(\ker f_*)^\perp. \tag{1.2}$$

It follows that isometric immersions and Riemannian submersions are particular cases of Riemannian maps with $\ker f_* = \{0\}$ and $(rangef_*)^\perp = 0$ respectively. There are alots of papers on this topic. Sahin defined the notion of slant Riemannian maps from almost Hermitian manifolds into Riemannian manifolds [15]. In this paper we have studied slant Riemannian maps from Kenmotsu manifolds into Riemannian manifolds.

2. Preliminaries

An $n-$dimensional differentiable manifold M is said to have an almost contact structure if there exist, a tensor field ϕ of type $(1, 1)$, a vector field ξ, a $1-$form η and the Riemannian metric g on M such that:

$$\phi^2 X = -X + \eta(X)\xi, \phi\xi = 0, \eta \circ \phi = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

and

$$g(X, \xi) = \eta(X), \qquad \text{for any } X, Y \text{ on } M. \tag{2.4}$$

An almost contact structure (ϕ, ξ, η) is said to be normal if the almost complex structure J on the product manifold $M \times R$ is given by

$$J\left(X, F\frac{d}{dt}\right) = \left(\phi X - F\xi, \eta(X)\frac{d}{dt}\right),$$

where $J^2 = -I$ and F is the differentiable function on $M \times R$ has no torsion i.e., J is integrable.

The condition for normality is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on M, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Finally, the fundamental 2−form Φ is given by $\Phi(X, Y) = g(X, \phi Y)$.

An almost contact metric structure (ϕ, ξ, η, g) is said to be Kenmotsu manifold [2] if

$$\begin{aligned} (\nabla_X \phi)Y &= g(\phi X, Y)\xi - \eta(Y)\phi X, \\ \nabla_X \xi &= X - \eta(X)\xi, \end{aligned} \tag{2.5}$$

for any X, Y on M, where ∇ denotes the Levi-Civita connection of the metric g on M.

Suppose $f : (M, g_M) \to (N, g_N)$ is differentiable map between Riemannian manifolds. Then the differential f_* of f can be viewed as a section of the bundle $Hom(TM, f^{-1}TN) \to M$, where $f^{-1}TN$ is the pullback bundle which has fibres $(f^{-1}TN)_q = T_{f(q)}N$, $q \in M$. $Hom(TM, f^{-1}TN)$ has a connection ∇ induced from the Riemannian connection ∇^M and the pullback connection. Then the second fundamental form of f is given by

$$(\nabla f_*)(X, Y) = \nabla^f_X f_* Y - f_*(\nabla^M Y), \tag{2.6}$$

for any $X, Y \in \Gamma TM$. The second fundamental form is symmetric [1].

Bi-harmonic Riemannian maps and the second fundamental form $(\nabla f_*)(U, V)$, for all $U, V \in (\Gamma(\ker f_*)^{\perp})$ of a Riemannian map has components in $range f_*$ [18]. Now we have the following:

Lemma 2.1. Let $f : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds. Then

$$g_M((\nabla f_*)(U, V), f_*(W)) = 0, \qquad \text{for all } U, V, W \in (\Gamma(\ker f_*)^{\perp}). \tag{2.7}$$

As a result of above Lemma, we get

$$(\nabla f_*)(U, V) \in (\Gamma(range f_*)^{\perp}) \qquad \text{for all } U, V \in (\Gamma(\ker f_*)^{\perp}). \tag{2.8}$$

For the tension field of a Riemannian map between Riemannian manifolds, we have the following:

Lemma 2.2. ([16]) Let f be Riemannian map from a Riemannian manifold (M, g_M) to Riemannian manifold (N, g_N). Then the tension field τ of f is

$$\tau = -m_1 f_*(H_1) + m_2 H_2, \tag{2.9}$$

where $m_1 = \dim(\ker f_*)$, $m_2 = rank\ f$, H_1 and H_2 are the mean curvature vector fields of the distribution $\ker f_*$ and $range\ f_*$, respectively.

Let $f : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds. The geometry of Riemannian submersions was defined by O'Neill tensors $\mathcal{T}$ and $\mathcal{A}$ [13] as

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \tag{2.10}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.11}$$

for vector fields E, F on M, where ∇ is the Riemannian connection of metric g. For any $E \in \Gamma TM$, $\mathcal{A}_E$ and $\mathcal{T}_E$ are skew-symmetric operators on $(\Gamma TM, g_M)$ reversing the vertical and the horizontal distributions. It is also seen that $\mathcal{A}$ is horizontal $\mathcal{A} = \mathcal{A}_{\mathcal{H}E}$ and $\mathcal{T}$ is vertical $\mathcal{T} = \mathcal{T}_{\mathcal{V}E}$ the tensor field $\mathcal{T}$ satisfies:

$$\mathcal{T}_U W = \mathcal{T}_W U\ , \qquad \text{for } U,\ W \in \Gamma(\ker F_*). \tag{2.12}$$

On the other hand from equations (2.10) and (2.11), we have

$$\nabla_X Y = \mathcal{T}_X Y + \widehat{\nabla}_X Y, \tag{2.13}$$

$$\nabla_X Z = \mathcal{H}\nabla_X Z + \mathcal{T}_X Z, \tag{2.14}$$

$$\nabla_Z X = \mathcal{A}_Z X + \mathcal{V}\nabla_Z X, \tag{2.15}$$

$$\nabla_Z W = \mathcal{H}\nabla_Z W + \mathcal{A}_Z W, \tag{2.16}$$

for any $X, Y \in (\ker f_*)$ and $Z, W \in \Gamma(\ker f_*)^\perp$.

We denote by ∇^N both the Riemannian connection on (N, g_N) and pullback connection along f. Then according to [12], for any vector field U on M and any section Z of $(\Gamma(range\ f_*)^\perp)$, where $(\Gamma(range\ f_*)^\perp)$ is the subbundle of $f^{-1}(TN)$ with fiber $(f_*(T_qM))^\perp$ orthogonal complement of $f_*(T_qM)$ for g_N over q, we have $\nabla_U^{f\perp} V$ which is the orthogonal projection of $\nabla_U^N Z$ on $(f_*(T_qM))^\perp$. In 1987, Nore showed that $\nabla^{f\perp}$ is a linear connection on $(f_*(T_qM))^\perp$ such that $\nabla^{f\perp} g = 0$. Now define S_Z as

$$\nabla^N_{f_*U} Z = -S_Z f_* U + \nabla_U^{f\perp} Z, \tag{2.17}$$

where $S_Z f_* U$ is the tangential component of $\nabla^N_{f_*U} Z$. It is easy to see that $S_Z f_* U$ is bilinear in Z and f_*U, $S_Z f_* U$ at q depends only on Z_q and $f_{*q}U_q$, we obtain

$$g_N(S_Z f_* U, f_* V) = g_N(Z, (\nabla f_*)(U, V)), \tag{2.18}$$

for any $U, V \in (\ker f_*)^\perp$ and $Z \in \Gamma(range\ f_*)^\perp$. Since (∇f_*) is symmetric it follows that S_Z is symmetric linear transformation of $range\ f_*$.

3. Slant Riemannian maps

Let f be a Riemannian map from an almost contact metric manifold $(M, \phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). If for any non-zero vector $X \in \Gamma \ker f_* - \{\xi\}$, the angle $\theta(X)$ between ϕX and the space $\ker f_*$ is constant, i.e., it is independent of the choice of the point $q \in M$ and choice of the tangent vector X in $\ker f_* - \{\xi\}$, then we say that f is a slant Riemannian map. In this definition, the angle θ is called the slant angle of the slant Riemannian map and $\theta \neq 0, \dfrac{\pi}{2}$.

Let f be a Riemannian map from an Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then for $X \in \Gamma(\ker f_*)$, we have

$$X = PX + \eta(X)\xi, \tag{3.1}$$

where $PX \in \ker f_* - \{\xi\}$.

For $X \in \Gamma(\ker f_*)$, we have

$$\phi X = \psi X + \omega X, \tag{3.2}$$

where ψX and ωX are vertical and horizontal components of ϕX, respectively.

Also for $V \in \Gamma(\ker f_*)^\perp$, we have

$$\phi V = BV + CV, \tag{3.3}$$

where BV and CV are vertical and horizontal components of ϕV, respectively.

Let f be slant Riemannian map from a Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ to Riemannian manifold (N, g_N) such that $\xi \in \Gamma(\ker f_*)$. Then, $\omega(\ker f_*)$ is a subspace of $(\ker f_*)^\perp$. Thus it follows that $(\ker f_*)_q \oplus \omega(\ker f_*)_q$ is invariant with respect to ϕ. Then for every $q \in M$, there exists an invariant subspace μ_q of $(\ker f_*)^\perp$ such that

$$T_qM = (\ker f_*)_q \oplus \omega(\ker f_*)_q \oplus \mu.$$

For any $X, Y \in \Gamma(\ker f_*)$, defines

$$(\nabla_X \psi)Y = \widehat{\nabla}_X \psi Y - \psi \widehat{\nabla}_X Y, \tag{3.4}$$

$$(\nabla_X \omega)Y = \mathcal{H}\nabla_X \omega Y - \omega \widehat{\nabla}_X Y. \tag{3.5}$$

Lemma 3.1. Let $(M, \phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. Let f be slant Riemannian map from Kenmotsu manifold to Riemannian manifold, then we have

$$(\nabla_X \psi)Y = B\mathcal{T}_X Y - \mathcal{T}_X \omega Y + g(\psi X, Y)\xi - \eta(Y)\psi X, \tag{3.6}$$

$$(\nabla_X \omega)Y = C\mathcal{T}_X Y - \mathcal{T}_X \psi Y - \eta(Y)\omega X, \tag{3.7}$$

where ∇ is the Levi-Civita connection on M, for any $X, Y \in \Gamma(\ker f_*)$.

Theorem 3.2. Let f be a Riemannian map from Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then f is a slant Riemannian map if and only if there exists a constant $\lambda \in [-1, 0]$ suct that

$$\psi^2 X = \lambda(X - \eta(X)\xi), \tag{3.8}$$

for any $X \in \Gamma(\ker f_*)$. If f is a slant Riemannian map, then $\lambda = -\cos^2\theta$.

Remark 3.3. By using the theorem above, it is easy to see that

$$g_M(\psi X, \psi Y) = \cos^2\theta(g_M(X, Y) - \eta(X)\eta(Y)), \tag{3.9}$$

$$g_M(\omega X, \omega Y) = \sin^2\theta(g_M(X, Y) - \eta(X)\eta(Y)), \tag{3.10}$$

for any $X, Y \in \Gamma(\ker f_*)$, where $\theta \neq 0, \dfrac{\pi}{2}$.

Also by using equation (3.9) we can conclude that

$$\{e_1, \sec\theta\psi e_1, e_2, \sec\theta\psi e_2, ..., e_n, \sec\theta\psi e_n, \xi\},$$

is an orthonormal frame for $\Gamma(\ker f_*)$. On the other hand by using equation (3.10) one can easily see that

$$\{\csc\theta\omega e_1, \csc\theta\omega e_1, ..., \csc\theta\omega e_n\},$$

is an orthonormal frame for $\Gamma(\omega(\ker f_*))$. As in slant immersions, we call the frame

$$\{e_1, \sec\theta\psi e_1, e_2, \sec\theta\psi e_2, ..., e_n, \sec\theta\psi e_n, \xi, \csc\theta\omega e_1, \csc\theta\omega e_1, ..., \csc\theta\omega e_n\},$$

an adapted frame for slant Riemannian maps.

We note that since the distribution $\ker f_*$ is integrable it follows that $\mathcal{T}_X Y = \mathcal{T}_Y X$ for $X, Y \in \Gamma(\ker f_*)$. Then the following lemma:

Theorem 3.4. Let f be a Riemannian map from Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). If ω is parallel with respect to ∇ on $\Gamma(\ker f_*)$, then

$$\mathcal{T}_{\psi X}\psi X = -\cos^2\theta(\mathcal{T}_X X - \eta(X)\mathcal{T}_X\xi), \tag{3.11}$$

for $X \in \Gamma(\ker f_*)$.

Theorem 3.5. Let F be a Riemannian map from Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then f is harmonic if and only if

$$\mathcal{T}_{\psi e_i}\psi e_i = -\cos^2\theta\mathcal{T}_{e_i} e_i, \tag{3.12}$$

$$trace|_{\omega(\ker F_*)}{}^* f_*(S_{E_j} f_*(.)) \in \Gamma(\mu), \tag{3.13}$$

and

$$trace|_{\mu}{}^* f_*(S_{E_j} f_*(.)) \in \Gamma\omega(\ker f_*), \tag{3.14}$$

where $\{e_1, \sec\theta\psi e_1, e_2, \sec\theta\psi e_2, ..., e_r, \sec\theta\psi e_r, \xi\}$ is an orthonormal frame for $\Gamma(\ker f_*)$ and $\{E_k\}$ is an orthonormal frame of $\Gamma((rangef_*)^\perp)$.

Proof. We take a canonical orthonormal frame $\{e_1, \sec\theta\psi e_1, e_2, \sec\theta\psi e_2, \dots, e_r, \sec\theta\psi e_r, \xi, \csc\theta\omega e_1, \csc\theta\omega e_2, \dots, \csc\theta\omega e_{2r}, \bar{e}_1, \dots, \bar{e}_n\}$ such that $\{e_1, \sec\theta\psi e_1, e_2, \sec\theta\psi e_2, \dots, e_r, \sec\theta\psi e_r, \xi\}$ is an orthonormal basis of $\ker f_*$, $\{\csc\theta\omega e_1, \csc\theta\omega e_2, \dots, \csc\theta\omega e_{2r}\}$ is an orthonormal basis of $\omega(\ker f_*)$ and $\{\bar{e}_1, \dots, \bar{e}_s\}$ of μ, where θ is the slant angle. Then f is harmonic if and only if

$$\sum_{i=1}^{r}(\nabla f_*)(e_i, e_i)+\sec^2\theta\sum_{i=1}^{r}(\nabla f_*)(\psi e_i, \psi e_i)+(\nabla f_*)(\xi, \xi)+\csc^2\theta\sum_{i=1}^{2r}(\nabla f_*)(\omega e_i, \omega e_i)$$

$$+\sum_{j=1}^{s}(\nabla f_*)(\bar{e}_j, \bar{e}_j) = 0.$$

By using equations (2.6) and (2.13), we get

$$\begin{aligned} &\sum_{i=1}^{r}(\nabla f_*)(e_i, e_i) + \sec^2\theta\sum_{i=1}^{r}(\nabla f_*)(\psi e_i, \psi e_i) + (\nabla f_*)(\xi, \xi) \\ = \;& -f_*(\mathcal{T}_{e_i}e_i + \sec^2\theta\mathcal{T}_{\psi e_i}\psi e_i). \end{aligned} \tag{3.15}$$

On the other hand from Lemma (1), we know

$$\csc^2\theta\sum_{i=1}^{2r}(\nabla f_*)(\omega e_i, \omega e_i)$$

$$+\sum_{j=1}^{s}(\nabla f_*)(\bar{e}_j, \bar{e}_j) \in \Gamma((rangef_*)^\perp).$$

Then we can write

$$\begin{aligned} &\csc^2\theta\sum_{i=1}^{2r}(\nabla f_*)(\omega e_i, \omega e_i) + \sum_{j=1}^{s}(\nabla f_*)(\bar{e}_j, \bar{e}_j) \\ = \;& \csc^2\theta\sum_{i=1}^{2r}\sum_{k=1}^{p}g_2((\nabla f_*)(\omega e_i, \omega e_i), E_k)E_k + \sum_{i=1}^{s}\sum_{k=1}^{p}g_2((\nabla f_*)(\bar{e}_j, \bar{e}_j), E_k)E_k \end{aligned}$$

where E_k is an orthonormal frame of $\Gamma((rangef_*)^\perp)$. Then using equation (2.18), we have

$$\csc^2\theta\sum_{i=1}^{2r}(\nabla f_*)(\omega e_i,\omega e_i)+\sum_{j=1}^{s}(\nabla f_*)(\bar{e}_j,\bar{e}_j)$$
$$=\csc^2\theta\sum_{i=1}^{2r}\sum_{k=1}^{p}g_2(S_{E_k}f_*(\omega e_i),\,f_*(\omega e_i))E_k$$
$$+\sum_{i=1}^{s}\sum_{k=1}^{p}g_2((\nabla f_*)(S_{E_k}f_*(\bar{e}_j),\,f_*(\bar{e}_j))E_k.$$

■

We now obtain necessary and sufficient conditions for a slant Riemannian map f to be totally geodesic. A differentiable map between Riemannian manifolds (M, g_M) and (N, g_N) is called a totally geodesic map [1] if $(\nabla f_*)(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$.

Theorem 3.6. Let f be a Riemannian map from Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then f is totally geodesic if and only if

$$g_M(\mathcal{T}_X\omega Y, BU) = -g_N((\nabla f_*)(X, \omega\psi Y), f_*(U)) + g_N((\nabla f_*)(X, \omega Y), f_*(CU)),$$
$$g_M(\mathcal{A}_U\omega X, BV) = g_N((\nabla^F_U f_*(\omega\psi X), f_*(V)) - g_N(\nabla^F_U f_*(\omega X), f_*(CV)),$$

and

$$\nabla^f_U f_*(V) + f_*(C(\mathcal{A}_U BV + \mathcal{H}\nabla^M_U CV) + (\mathcal{V}\nabla^M_U BV + \mathcal{A}_U CV) \in \Gamma(range f_*),$$

for $X, Y \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^\perp$, where ∇^M is the Live-Civita connection of M.

Proof. For the decomposition of the total manifold of a slant Riemannian map, it follows that f is totally geodesic if and only if $g_N((\nabla f_*)(X, Y), f_*(U)) = 0$, $g_N((\nabla f_*)(U, X), f_*(V)) = 0$ and $(\nabla f_*)(U, V) = 0$ for $X, Y \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^\perp$. First, since f is a Riemannian map, from (2.6) we obtain

$$g_N((\nabla f_*)(X, Y), f_*(U)) = g_M(\nabla^M_X Y, U).$$

Since M is a Kenmotsu manifold, using equations (3.2) and (3.3), we get

$$g_N((\nabla f_*)(X, Y), f_*(U)) = -\cos^2\theta g_M(\nabla^M_X Y, U) + g_M(\nabla^M_X\omega\psi Y, U)$$
$$-g_M(\nabla^M_X\omega Y, BU) - g_M(\nabla^M_X\omega Y, CU).$$

Taking account that f is a Riemannian map using equations (2.6) and (2.14) we have

$$g_N((\nabla f_*)(X, Y), f_*(U)) = \sec^2\theta\{-g_M(\mathcal{T}_X\omega Y, BU) - g_M((\nabla f_*)(X, \omega\psi Y), f_*(U))$$
$$+g_N((\nabla f_*)(X, \omega Y), f_*(CU))\}.$$

In similar way, we also have

$$g_N((\nabla f_*)(U,X), f_*(V)) = \sec^2\theta\{-g_M(\mathcal{A}_U\omega X, BV) + g_N((\nabla_U^f f_*(\omega\psi X), f_*(V)) - g_N(\nabla_U^f f_*(\omega X), f_*(CV))\}.$$

On the other hand, using equations (2.6) and (2.5) we derive

$$(\nabla f_*)(U,V) = \nabla_U^f f_*V + f_*(\phi(\nabla_U^M \phi V - (\nabla_U^M \phi)V)),$$

for $U, V \in \Gamma(\ker f_*)^\perp$. Then using equations (2.3), (3.2), (3.3) and (2.13) – (2.16) we obtain

$$\begin{aligned}(\nabla f_*)(U,V) &= \nabla_U^f f_*V + f_*(B\mathcal{A}_U BV + C\mathcal{A}_U BV + \psi\mathcal{V}\nabla_U^M BV \\ &+\omega\mathcal{V}\nabla_U^M BV + B\mathcal{H}\nabla_U^M CV + C\mathcal{H}\nabla_U^M CV \\ &+\psi\mathcal{A}_U CV + \omega\mathcal{A}_U CV).\end{aligned}$$

Since

$$B\mathcal{A}_U BV + \psi\mathcal{V}\nabla_U^M BV + B\mathcal{H}\nabla_U^M CV + \psi\mathcal{A}_U CV \in \Gamma(\ker f_*)^\perp,$$

we have

$$\begin{aligned}(\nabla f_*)(U,V) &= \nabla_U^f f_*V + f_*(C\mathcal{A}_U BV + \omega\mathcal{V}\nabla_U^M BV \\ &+C\mathcal{H}\nabla_U^M CV + \omega\mathcal{A}_U CV).\end{aligned}$$

■

A decomposition theorem Via slant Riemannian maps

In this section we find necessary and sufficient conditions for the total manifold of a slant Riemannian map to be a locally product Riemannian manifold. Let g_1 be a Riemannian metric tensor on the manifold $M_1 = B \times F_1$ and assume that the canonical foliations D_1 and D_2 intersect perpendicularly everywhere. Then from de Rham's theorem [14], we know that g_1 is the metric tensor of a usual product Riemannian manifold if and only if D_1 and D_2 are totally geodesic foliations.

Theorem 3.7. Let f be a slant Riemannian map from Kenmotsu manifold $(M, \phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then M is a locally product Riemannian manifold if and only if

$$g_M(\mathcal{T}_X\omega Y, BU) = -g_N((\nabla f_*)(X, \omega\phi Y), f_*(U)) + g_N((\nabla f_*)(X, \omega Y), f_*(CU)),$$

and

$$g_N((\nabla f_*)(U, BV), f_*(\omega X)) = g_N(f_*(V), \nabla_U^f f_*(\omega\psi X)) - g_N(f_*(CV), \nabla_U^F f_*(\omega X)),$$

for $X, Y \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^\perp$.

Proof. For $X, Y \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^\perp$, from equations (2.3), (3.2), (3.3) and theorem (1), we get

$$\begin{aligned} g_M(\nabla^M_U V, X) &= -\cos^2\theta g_1(V, \nabla^M_U X) + g_1(V, \nabla^M_U \omega\psi X) \\ &\quad -g_1(BV, \nabla^M_U \omega X) - g_1(CV, \nabla^M_U \omega X). \end{aligned}$$

Taking into account that f is a Riemannian map using equation (2.6) we get

$$\begin{aligned} g_M(\nabla^M_U V, X) &= \sec^2\theta\{-g_N(f_*(V), (\nabla f_*)(U, \omega\psi X)) + g_N(f_*(V), \nabla^F_U f_*(\omega\psi X)) \\ &\quad -g_N((\nabla f_*)(U, BV), f_*(\omega X)) + g_N((\nabla f_*)(U, \omega X), f_*(CV)) \\ &\quad -g_N(f_*(CV), \nabla^f_U f_*(\omega X))\}. \end{aligned}$$

Then using Lemma (1) implies that

$$\begin{aligned} g_M(\nabla^M_U V, X) &= \sec^2\theta\{g_N(f_*(V), \nabla^f_U f_*(\omega\psi X)) - g_N((\nabla f_*)(U, BV), f_*(\omega X)) \\ &\quad -g_N(f_*(CV), \nabla^F_U f_*(\omega X))\}. \end{aligned}$$

■

4. Example

Note that given an Euclidean space R^{2m+1} with coordinates $(x_1, \ldots, x_{2m}, x_{2m+1})$ we can canonically choose an almost contact metric structure (ϕ, ξ, η, g) on R^{2m+1} as follows:

$$\begin{aligned} &\phi\left(a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + \cdots + a_{2m-1}\frac{\partial}{\partial x_{2m-1}} + a_{2m}\frac{\partial}{\partial x_{2m}} + a_{2m+1}\frac{\partial}{\partial x_{2m+1}}\right) \\ = &\left(-a_2\frac{\partial}{\partial x_1} + a_1\frac{\partial}{\partial x_2} + \cdots - a_{2m}\frac{\partial}{\partial x_{2m-1}} + a_{2m-1}\frac{\partial}{\partial x_{2m}}\right) \end{aligned}$$

where $\xi = \dfrac{\partial}{\partial x_{2m+1}}$ and $a_1, a_2, \ldots, a_{2m}, a_{2m+1}$ are $C^\infty-$real valued functions in R. Let $\eta = dx_{2m+1}$, g is Euclidean metric and $\left\{\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \ldots, \dfrac{\partial}{\partial x_{2m}}, \dfrac{\partial}{\partial x_{2m+1}}\right\}$ is orthonormal basis of vector fields on R^{2m+1}.

Example 4.1. ([10]) We consider the 5−dimensional manifold $M^5 = \{(x_1, \ldots, x_5) \in R^5 : x_5 > 0\}$, where $(x_1, \ldots, x_5)$ are the standard coordinate in R^5, the vector field $E_1 = x_5\dfrac{\partial}{\partial x_1}$, $E_2 = x_5\dfrac{\partial}{\partial x_2}$, $E_3 = x_5\dfrac{\partial}{\partial x_3}$, $E_4 = -x_5\dfrac{\partial}{\partial x_4}$, $E_5 = -x_5\dfrac{\partial}{\partial x_5}$ are linearly independent at each point of M. Let g_5 be the Riemannian metric on M^5 defined by $g_5(E_i, E_i) = 1$ and $g_5(E_i, E_j) = 0$, $i \neq j$ for all $i, j = 1, 2, \ldots, 5$. Let $\eta(.) =$

$g_5(E_i, .)$. Let ϕ be the $(1, 1)$ tensor field defined by $\phi E_1 = -E_2$, $\phi E_2 = E_1, \phi E_3 = -E_4, \phi E_4 = E_3, \phi E_5 = 0$. Then the using linearty of ϕ and g_5, we have $\phi^2 = -I + \eta \otimes E_5, \eta(E_5) = 1$ and $g_5(\phi X, \phi Y) = g_5(X, Y) - \eta(X)\eta(Y)$, for any $X, Y \in \Gamma(TM^5)$. Thus (ϕ, ξ, η, g_5) is an almost contact metric structure. Now, using Koszul formula, we have, for any $X, Y \in \Gamma(TM^5)$,

$$(\nabla_X \phi)Y = g_5(\phi X, Y)\xi - \eta(Y)\phi X,$$

thus $(M^5, \phi, \xi, \eta, g_5)$ is a Kenmotsu manifold.

Example 4.2. Let M^5 is a Kenmotsu manifold as in example (1) and R^3 is a Riemannian manifold. The Riemannian metric tensor field g_{R^3} is given by $g_{R^3} = \begin{bmatrix} \frac{1}{x_5^2} & 0 & 0 \\ 0 & \frac{1}{x_5^2} & 0 \\ 0 & 0 & \frac{1}{x_5^2} \end{bmatrix}$ on R^3.

Let $f : R^5 \to R^3$ be a Riemannian map defined by

$$f(x_1, \ldots, x_5) = \left(\frac{x_2 + x_3}{\sqrt{2}}, 0, x_4 \right)$$

Then, we have

$$\ker f_* =< E_1, \frac{1}{\sqrt{2}}(E_2 - E_3), E_5 > \text{ and } (\ker f_*)^\perp =< \frac{1}{\sqrt{2}}(E_2 + E_3), E_4 >$$

Thus, f is a slant Riemannian map with $\lambda = \frac{\pi}{4}$.

Example 4.3. Let M^5 is a Kenmotsu manifold as in example (1) and R^4 is a Riemannian manifold. The Riemannian metric tensor field g_{R^4} is given by $g_{R^4} = \begin{bmatrix} \frac{1}{x_5^2} & 0 & 0 & 0 \\ 0 & \frac{1}{x_5^2} & 0 & 0 \\ 0 & 0 & \frac{1}{x_5^2} & 0 \\ 0 & 0 & 0 & \frac{1}{x_5^2} \end{bmatrix}$ on R^4.

Let $f : R^5 \to R^4$ be a Riemannian map defined by

$$f(x_1, \ldots, x_5) = \left(\frac{x_1 - x_3}{\sqrt{2}}, 0, x_2, 0 \right)$$

Then, we have

$$\ker f_* =< \frac{1}{\sqrt{2}}(E_1 + E_3), E_4, E_5 > \text{ and } (\ker f_*)^{\perp} =< \frac{1}{\sqrt{2}}(E_1 - E_3), E_2 >$$

Thus, f is a slant Riemannian map with $\lambda = \frac{\pi}{4}$.

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