Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 3 (2017), pp. 1091-1101
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The Classification of Quadratic Rook Polynomials of a Generalized Three Dimensional Board

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Abstract

The theory of rook polynomials provides a way of counting permutations with restricted positions. In this paper, we classify all possible quadratic polynomials that are the rook polynomial for some generalized three dimensional board B. On a three dimensional board, each rook placement prohibits any further rook placements in the union of three intersecting planes formed by the three intersections of the dimensions (i.e.) the plane formed by the rook's row and column, rook's row and tower and the plane formed using the rook's column and tower. Here we prove theorems regarding rook equivalency on the considered three dimensional board which allow us to minimize the number and type of boars that we need to work with for enumerating all possible quadratic rook polynomials.

Keywords: Rook numbers, Three Dimensional Board, Rook polynomial, Generalized Board.

1. INTRODUCTION

Rook theory is the study of permutations described using terminology from the game of chess. In chess, the rook is a piece that can capture any opponent's piece in the same row and column provided there are no other pieces positioned between them. Here, a generalized three dimensional board B is any subset of the sequences of an n x n x n chess board for some positive integer n. Rook theory focuses on the placement of non-attacking rooks in a more general situation. Here we generalize the rook theory to three dimensions. In higher dimensions, rooks attack along hyper planes, which correspond to layers of cells with one fixed coordinate. In two dimensions, when we place a rook on a tile, we are no longer able to place a rook on any tile in the same row or column. In three dimensions, when we place a rook in a cell; we can no longer place another rook in the same wall, slab or layer. Here, in this paper, we assume that the rooks attack along lines instead of a rook attacking hyper planes just as the two dimensional case. We prove theorems regarding rook equivalency that helps us to minimize the number of boards and its types we need to work with to complete the required task. Finally we conclude by enumerating all the possible quadratic rook polynomials of the generalized three dimensional board for the r_1 values ranging from $r_1 = 3$ to $r_1 = 10$.

2. BASIC DEFINITIONS:

Definition: 2.1 We define a board to be a square nxn chessboard for any nϵN . A *generalized board* is any subset of squares of the board .Thus, a generalized board can be any argument of squares that is completely contained inside a board (or) it can be the board itself.

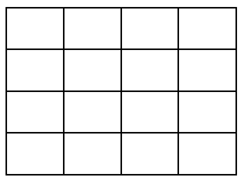


Figure 1(a). Board

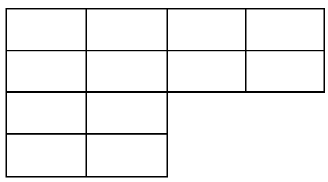


Figure 1(b). Generalized board

Fig 1 (a) is an example of 4x4board.
Fig 1 (b) is the generalized board with 12 squares that is σ contained inside the 4x4 board (Fig (a))

Definition:2.2 A rook polynomial is an arrangement of some numbers of non-attacking rooks on some board .Since rooks attack square in their row and column, a rook placement cannot have more than one rook in a given row (or)column.

A placement of n rooks on an nxn square board can be associated to a permutation σ = $\sigma_{1,}\sigma_{2,\ldots\ldots}\sigma_{n,}$ of 1,2,3,……..n by saying the placement Pσ has a rook on the square (i,j) of the board if and only if σ_i=j.

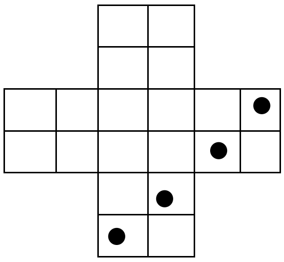


Figure 2: Example of permutation

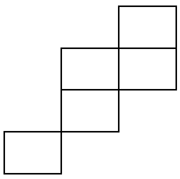
In the above generalized board the maximum number of non- attacking rooks that can be placed is four.

Definition: 2.3 The *rook polynomial* $R_B(x)$ of a board B is generating for the number of arrangements of non attacking rooks.

$$R_B(x) = \sum_{k=o}^{\infty} R_k(B)\, x^k$$

where r_k is the number of ways to place k non attacking rooks on the board. This sum is finite since the board is finite and so there is a maximum number of non-attacking rooks it can hold. Indeed, there cannot be more rooks than the smaller of the number of rows and columns in the board.

Example:



In the above example there are five ways to place one non-attacking rook, seven ways to place two non attacking rooks and three ways to place three non attacking rooks and no way to place four or more non attacking rooks.

Definition: 2.4 The k^{th} *rook number* $r_k(B)$ counts the number of ways to place k non- attacking rooks on a generalized board B. $r_k(B)$ is as r_k when B is clear.

Definition: 2.5 A three dimensional chessboard is simply a pile of two dimensional chessboards stacked one upon another. Considering every two dimensional level of a three dimensional chessboard, we have rows and columns in the traditional sense, each of which is a one dimensional array used to describe position within the board. Define a tower as a one- dimensional array describing position along the added third dimension.

Example:

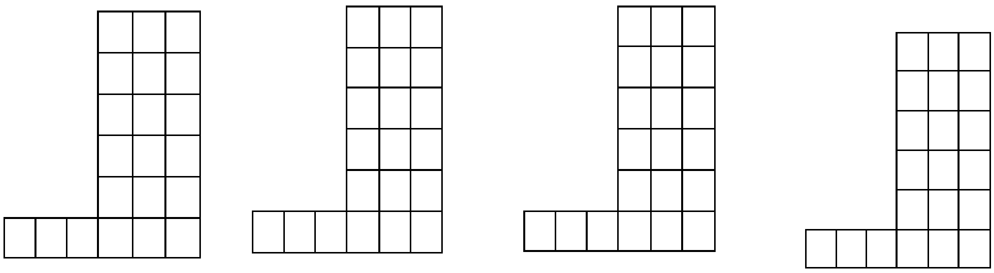


Figure -3

Remark on rook numbers:

1. r_0 is always 1 because there is only one way to place 0 rooks on a generalized three dimensional board.
2. r_1 is always the number of squares of B because a single rook can be placed in any square of B with no other rook to attack it.
3. Since a rook attacks all squares in its row, column and tower, each rook in a rook placement must be in a different row or column or tower.
4. Once we attain r_k=0, we will always have $r_{k+1}, r_{k+2, \ldots\ldots\ldots} = 0$. For example, if we are unable to place 3 non-attacking rooks on a generalized three dimensional board, we cannot place 4 or more non-attacking rooks on the generalized board.
5. If B is contained in an n x n x n board and k>n, then we have r_k=0. For example, in a 3x3x3 board r_4=0 (k=4) as we could not place 4 non-attacking rooks in 3x3x3 board.

Note: r_k could be equal to 0 for smaller values of k as well. For example, the generalized three dimensional board in fig (3) which is completely contained in 6x6x6 board has $r_5 = r_6$ =0 as we could not place 4 or more non-attacking rooks in the board.

3. ROOK EQUIVALENCY OF THREE DIMENSIONAL BOARDS CONTAINED IN TWO ROWS OR COLUMNS:

Lemma: 3.1 Let B be a three dimensional board of darkened squares that decomposes into disjoint sub boards B_1 and B_2. Then

$r_k(B) = r_k(B_1)\, r_k(B_2) + r_{k-1}(B_1)\, r_1(B_2) + \ldots\ldots + r_0(B_1) r_k(B_2)$

Now we define the rook polynomial R(X,B) of the board B as follows:

R(x, B) = $r_0(B) + r_1(B)x + r_2(B)x^2 + \ldots\ldots\ldots + r_n(B)x^n + \ldots..$

Theorem 3.1: Let B be a three dimensional board of darkened squares that decomposes into disjoint sub boards B_1 and B_2. Then

R(X,B) = $R(x, B_1) + R(x, B_2)$.

Proof:

$$\begin{aligned} R(X, B) &= r_0(B) + r_1(B)x + r_2(B)x^2 + \ldots \\ &= 1 + [r_1(B_1)\, r_0(B_2) + r_0(B_1)\, r_1(B_2)]x + \\ &\quad [r_2(B_1)\, r_0(B_2) + r_1(B_1)\, r_1(B_2) + r_0(B_1)\, r_2(B_2)]x^2 + \ldots\ldots. \\ &= [r_0(B_1) + r_1(B_1)x + r_2(B_1)x^2 + \ldots\ldots]x \times [r_0(B_2) + r_1(B_2)x + \\ &\qquad r_2(B_2)x^2 + \ldots\ldots] \\ &= R(x, B_1) + R(x, B_2). \end{aligned}$$

Theorem 3.2: Let B be a non empty three dimensional board such that $r_3(B) = 0$. Then B satisfies at least one of the following criteria:

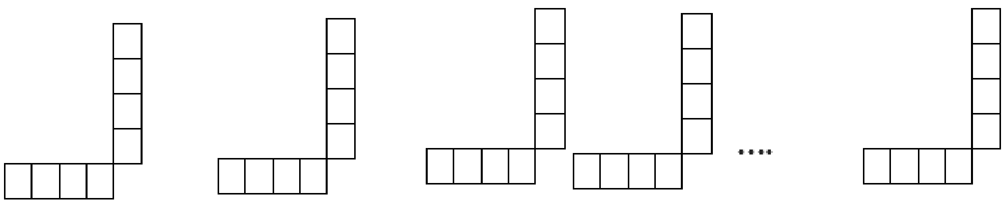
1. B is contained in two or fewer rows;
2. B is contained in two or fewer columns;
3. B is the union of one part of the board contained in one row and another part contained in one column. An example of such a board is shown in figure.

Proof We will prove by contra positive. Hence, we aim to show that if a board does not satisfy any of the properties above, then $r_3 \neq 0$. Assume B does not satisfy any of the listed criteria. So B has at least three rows, at least two columns, and does not have one part of the board contained in one row and one part contained in one column. These conditions force B to have four squares each in a different row and column from the other squares. Non-attacking rooks can be placed on these squares, so $r_3(B) \neq 0$, as desired.

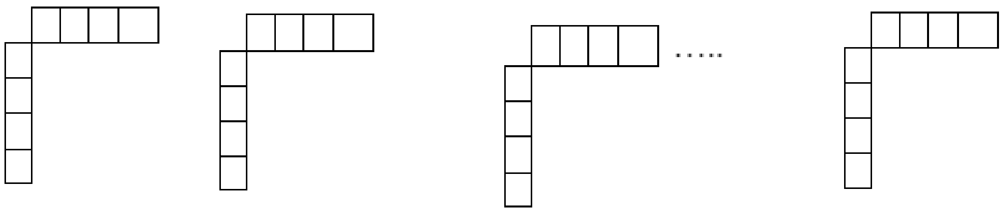
For the purpose of our classification of cubic rook polynomials, we only need to consider boards contained in two rows.

Theorem 3.3: A generalized three dimensional board contained in two columns is rook equivalent to one contained in two rows.

Proof If B is a three dimensional generalized board contained in five rows of an nxn board, Let B' be the three dimensional board obtained by rotating B 90^0 clock wise, as shown in the given figure.



Board B



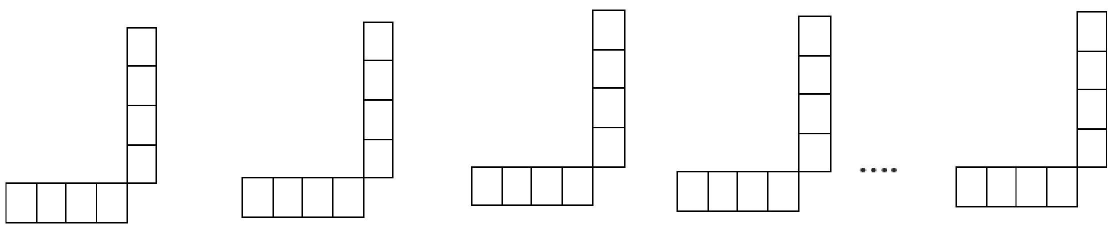
Board B'

It is clear that a B' contained in three rows, B and B' have the same rook numbers. Hence, a board contained in three columns is rook equivalent to a board contained in three rows.

Theorem 3.4: Any generalized three dimensional board with one part contained in one row and another part contained in one column has the same rook polynomial as a board contained within two rows.

Proof

Case 1: Consider the three dimensional board B in which the squares in the column cannot attack any square in the row of any of the towers of B where each tower is a two dimensional board.



Let k be the number of towers (two dimensional boards) of our three dimensional board B.

$r_0(B) = 1$ $r_1(B) = ka + kb = k(a+b)$ where a is the number of squares in the bottom and b is the number of squares in the column.

$r_2(B) = (k-1)b + (k-1)b + \cdots a\ times + \ (k-1)b + (k-1)b + \cdots a\ times + (k-1)b + (k-1)b + \cdots a\ times + \cdots k\ times = \ (k-1)b + (k-1)b + \cdots a\ times$

$$= \ (k-1)ab + (k-1)ab + \cdots k\ times + (k-1)ab$$

$$= \ k(k-1)ab$$

$r_2(B) = k(k-1)ab$

Board B' is contained in two rows with k towers (two dimensional board).

$r_0(B') = 1$ $r_1(B') = \ ka + kb = k(a+b)$

$$r_2(B') = (k-1)b + (k-1)b + \cdots a\ times + \ (k-1)b + (k-1)b + \cdots a\ times \ + \ (k-1)b + (k-1)b + \cdots a\ times + \cdots k\ tim$$

$$= \ (k-1)b + (k-1)b + \cdots a\ times$$

$$= \ (k-1)ab + (k-1)ab + \cdots k\ times + (k-1)ab$$

$$= \ k(k-1)ab$$

$r_2(B') = k(k-1)ab$

Thus $r_2(B) = \ r_2(B') = k(k-1)ab$

Theorem 3.5:

Let B be a generalized three dimensional board whose two dimensional boards contained in two rows and its rook polynomial is quadratic. Given $r_1(B)$, every possible $r_2(B)$ will have the form $r_2(B) = \ k(k-1)\left\{\left[a\left\lfloor\frac{r_1}{k}\right\rfloor - a\right] - i\right\}$ where $0 \le i \le a$ $1 \le a \le \left\lfloor\frac{r_1}{2k}\right\rfloor$

Proof:

Given any generalized three dimensional board B satisfying the given conditions and given the value of r_1, k, we want to find all possible corresponding values of $r_2(B)$ by considering each pair of positive integers a and b of $r_1(B)$ such that $k(a+b) = r_1(B)$

Without loss of generality, let's assume that $a \le b$.

This will imply $ka \le kb$

$$ka + ka \le kb + ka$$

$$2ka \le r_1(B)$$

$$a \le \left\lfloor\frac{r_1}{2k}\right\rfloor$$

$$k(a+b) = \ r_1(B)$$

$$ka + kb = \ r_1(B)$$

$$kb = \ r_1(B) - \ ka$$

$$b = \ \left\lfloor\frac{r_1(B)}{k}\right\rfloor - \ a$$

Here a and b represent the number of squares in the two rows that two dimensional layers of our board B occupy respectively. Then a and b squares can be arranged such that a squares lie consecutively in one row and b squares lie consecutively in the next row. We can rearrange the columns of a generalized board B so that i columns containing squares in both rows lies in the center, columns with empty squares in row 1 lie adjacent on the right, and columns with empty squares in row 2 lie adjacent on the left. Such a rearrangement will not change the rook numbers.
By placing one rook in each row, the possible values of $r_2(B)$ can be found for each value of $0 \leq i \leq a$.

The maximum value of r_2 will be when $i = 0$ (a and b are completely disjoint)

$r_2(B) = k(k-1)ab$ (From Theorem 3.4 Case 1)

The minimum value of r_2 will be when $i = a$ (a and b are completely overlapped)

$r_2(B) = k(k-1)a(b-1)$ (From Theorem 3.4 Case 2)

Every value between $r_2(B) = k(k-1)ab$ and $r_2(B) = k(k-1)a(b-1)$ can be obtained as well.

Beginning with a disjoint pair, shift the row containing a squares one square to the left, which decreases the value of r_2 by one. Continuing the process until a and b are overlapping exhausts all possible values of r_2. Hence, for a given a and b, each choice of i will yield a different possible $r_2(B)$ value.

For any particular arrangement, the value of $r_2(B)$ is equal to,

$$\begin{aligned} r_2(B) &= [k(b-i)a\,(k-1) + i(a-1)(k-1)] \\ &= k(k-1)[\,(b-i)a + i(a-1)] \\ &= k(k-1)[\,ab - ia + ia - i] \\ &= k(k-1)[ab - i] \end{aligned}$$

We know $b = \left\lfloor \frac{r_1(B)}{k} \right\rfloor - a$

Thus $r_2(B) = k(k-1)\left\{a\left[\left\lfloor \frac{r_1}{k} \right\rfloor - a\right] - i\right\}$ where $0 \leq i \leq a$

$$1 \leq a \leq \left\lfloor \frac{r_1}{2k} \right\rfloor$$

Example of finding $r_2(B)$ value when we know $r_1(B) = 12$

Suppose we wanted to classify all quadratic rook polynomials with $r_1(B) = 12$.

Then k can take the values 2, 3, 4 or 6. For other values of k we get non integral values for a and b which cannot be possible.

When $k = 2$

$a + b = 6$

The possible integer pair for a, b are (1,5) (2,4) (3,3) whose corresponding $r_2(B)$ values are given as follows

$r_2(B)$ = 10, 8 for the pair (1,5)

= 16, 18, 20 for the pair (2, 4)

= 12, 14, 16, 18 for the pair (3, 3) by using the formula $r_2(B) = k(k-1)\left\{a\left[\left\lfloor\frac{r_1}{k}\right\rfloor - a\right] - i\right\}$ where $0 \le i \le a$ $1 \le a \le \left\lfloor\frac{r_1}{2k}\right\rfloor$

Thus the rook polynomials when $k = 2$ are $1 + 12x + 8x^2$, $1 + 12x + 10x^2$, $1 + 12x + 12x^2$, $1 + 12x + 14x^2$, $1 + 12x + 16x^2$, $1 + 12x + 18x^2$, $1 + 12x + 20x^2$.

When $k = 3$

$a + b = 4$

The possible integer pair for a, b are (1, 3) (2, 2) whose corresponding $r_2(B)$ values are given as follows

$r_2(B)$ = 12, 18 for the pair (1,3)

= 12, 18, 24 for the pair (2, 2) by using the formula $r_2(B) = k(k-1)\left\{a\left[\left\lfloor\frac{r_1}{k}\right\rfloor - a\right] - i\right\}$ where $0 \le i \le a$ $1 \le a \le \left\lfloor\frac{r_1}{2k}\right\rfloor$

Thus the rook polynomials when $k = 3$ are $1 + 12x + 12x^2$, $1 + 12x + 18x^2$, $1 + 12x + 24x^2$.

When $k = 4$

$a + b = 3$

The possible integer pair for a, b are (1, 2) whose corresponding $r_2(B)$ values are given as follows

$r_2(B)$ = 12, 24 for the pair (1,2) by using the formula $r_2(B) = k(k-1)\left\{a\left[\left\lfloor\frac{r_1}{k}\right\rfloor - a\right] - i\right\}$ where $0 \le i \le a$ $1 \le a \le \left\lfloor\frac{r_1}{2k}\right\rfloor$

Thus the rook polynomials when $k = 4$ are $1 + 12x + 12x^2$, $1 + 12x + 24x^2$.

When $k = 6$

$a + b = 2$

The possible integer pair for a, b are (1, 1) whose corresponding $r_2(B)$ values are given as follows

$r_2(B)$ = 12 for the pair (1,1) by using the formula $r_2(B) = k(k-1)\left\{a\left[\left\lfloor\frac{r_1}{k}\right\rfloor - a\right] - i\right\}$ where $0 \le i \le a \quad 1 \le a \le \left\lfloor\frac{r_1}{2k}\right\rfloor$

Thus the rook polynomials when $k = 6$ are $1 + 12x + 12x^2$.

Hence the possible quadratic rook polynomials of a generalized three dimensional board for $r_1(B) = 12$ are

$1 + 12x + 8x^2$, $1 + 12x + 10x^2$, $1 + 12x + 12x^2$, $1 + 12x + 14x^2$, $1 + 12x + 16x^2$, $1 + 12x + 18x^2$, $1 + 12x + 20x^2$, $1 + 12x + 24x^2$.

4. CLASSIFICATION OF CUBIC ROOK POLYNOMIALS:

The possible quadratic rook polynomials for the generalized three dimensional board under consideration when $r_1(B) = 4$ 5,6,7,8,9,10,11.

For $r_1(B) = 4$

$k = 2$

$1 + 4x + 4x^2$

For $r_1(B) = 5$

No k values possible.

For $r_1(B) = 6$

$k = 2, 3$

$1 + 6x + 2x^2$

$1 + 6x + 4x^2$

$1 + 6x + 6x^2$

$1 + 6x + 12x^2$

For $r_1(B) = 7$

No k values possible.

For $r_1(B) = 8$

$k = 2, 4$

$1 + 8x + 4x^2$

$1 + 8x + 6x^2$

$1 + 8x + 8x^2$

$1 + 8x + 12x^2$

For $r_1(B) = 9$

$k = 3$

$1 + 9x + 6x^2$

$1 + 9x + 12x^2$

For $r_1(B) = 10$

$1 + 10x + 6x^2$

$1 + 10x + 8x^2$

$1 + 10x + 10x^2$

$1 + 10x + 12x^2$

$1 + 10x + 20x^2$

For $r_1(B) = 11$

No k values possible.

CONCLUSION

In this paper, the possible cubic rook polynomials under certain conditions have been enumerated. Basic definitions of rook theory are presented in section 2. In section 3 the idea of rook equivalency is proved by theorems which allow us to simplify the number of bounds that we need to consider to address our given problem. Finally in section 4, we have found bounds for the rook numbers of the cubic rook polynomials for specific boards under certain conditions.

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