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Banach contraction principle on Cone Heptagonal Metric Space

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Abstract

In this paper, we prove Banach contraction principle on cone heptagonal metric spaces. We give an example to support the result. Our result extend and improve some recent results of the following: Azam et al., [Banach contraction principle on cone rectangular metric spaces, Applicable Analysis and Discrete Mathematics, 3 (2), 236–241, 2009], Garg and Agarwal, [Banach Contraction Principle on Cone Pentagonal Metric Space, J. Adv. Studies Topol., 3 (1), 12–18, 2012], Garg, [Banach Contraction Principle on Cone Hexagonal Metric Space, Ultra Scientist, 26 (1), 97–103, 2014], and others.

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1. Introduction

The concept of Banach contraction mapping principle was introduced in [1]. Due to its wide applications; the study of existence and uniqueness of fixed point of a mapping has become a subject of great interest. Many authors proved the Banach contraction principle in various generalized metric spaces.

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In 2007, Huang and Zhang [2] introduced the concept of a cone metric space, they replaced the set of real numbers by an ordered Banach space and proved some fixed point theorems for contractive type conditions in cone metric spaces. Later on many authors have proved some fixed point theorems for different contractive types conditions in cone metric spaces (for e.g., [3, 4, 5]).

Azam et al. [6] introduced the notion of cone rectangular metric space and proved Banach contraction mapping principle in a normal cone rectangular metric space setting.

Garg and Agarwal [7] introduced the notion of cone pentagonal metric space and proved Banach contraction mapping principle in a normal cone pentagonal metric space setting.

Recently, Garg and Agarwal [8] introduced the notion of cone hexagonal metric space and proved Banach contraction mapping principle in a normal cone hexagonal metric space setting.

Very recently, Ampadu [9] introduced the notion of cone heptagonal metric space and proved Chatterjea contraction mapping principle in a normal cone hexagonal metric space setting.

Motivated and inspired by the results of [6, 7, 8, 9], it is our purpose in this paper to continue the study of fixed point of mapping in cone heptagonal metric space setting. Our results extend and improve the results of [6, 7, 8], and many others.

2. Preliminaries

We present some definitions and Lemmas, which will be needed in the sequel.

Definition 2.1. [2] Let E be a real Banach space and P subset of E. P is called a cone if and only if:

(1) P is closed, nonempty, and $P \neq \{0\}$;

(2) $a, b \in \mathbb{R},\ a, b \geq 0$ and $x, y \in P \Longrightarrow ax + by \in P$;

(3) $x \in P$ and $-x \in P \Longrightarrow x = 0$.

Given a cone $P \subseteq E$, we defined a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P.

Definition 2.2. [2] A cone P is called normal if there is a number $K^* > 0$ such that for all $x, y \in E$, the inequality

$$0 \leq x \leq y \Longrightarrow \|x\| \leq K^*\|y\|. \tag{2.1}$$

The least positive number K^* satisfying (2.1) is called the normal constant of P.

In this paper, we always suppose that E is a real Banach space and P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 2.3. [2] Let X be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

Then ρ is called a cone metric on X, and (X, ρ) is called a cone metric space.

Remark 2.4. The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, \infty)$ (e.g., see [2]).

Definition 2.5. [6] Let X be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, w) + \rho(w, z) + \rho(z, y)$ for all $x, y, z \in X$ and for all distinct points $w, z \in X - \{x, y\}$ [Rectangular property].

Then ρ is called a cone rectangular metric on X, and (X, ρ) is called a cone rectangular metric space.

Remark 2.6. Every cone metric space is cone rectangular metric space. The converse is not necessarily true (e.g., see [6]).

Definition 2.7. [7] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for $x, y \in X$;

(3) $d(x, y) \leq d(x, z) + d(z, w) + d(w, u) + d(u, y)$ for all $x, y, z, w, u \in X$ and for all distinct points $z, w, u \in X - \{x, y\}$ [Pentagonal property].

Then d is called a cone pentagonal metric on X, and (X, d) is called a cone pentagonal metric space.

Remark 2.8. Every cone rectangular metric space and so cone metric space is cone pentagonal metric space. The converse is not necessarily true (e.g., see [7]).

Definition 2.9. [8] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for $x, y \in X$;

(3) $d(x, y) \leq d(x, z) + d(z, w) + d(w, u) + d(u, v) + d(v, y)$ for all $x, y, z, w, u, v \in X$ and for all distinct points $z, w, u, v \in X - \{x, y\}$ [Hexagonal property].

Then d is called a cone hexagonal metric on X, and (X, d) is called a cone hexagonal metric space.

Remark 2.10. Every cone pentagonal metric space and so cone rectangular metric space is cone hexagonal metric space. The converse is not true (e.g., see [8]).

Definition 2.11. [9] Let X be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, w) + \rho(w, u) + \rho(u, v) + \rho(v, t) + \rho(t, y)$ for all $x, y, z, w, u, v, t \in X$ and for all distinct points $z, w, u, v, t \in X - \{x, y\}$ [Heptagonal property].

Then ρ is called a cone heptagonal metric on X, and (X, ρ) is called a cone heptagonal metric space.

Remark 2.12. Every cone hexagonal metric space, cone pentagonal metric space and so cone rectangular metric space is cone heptagonal metric space. The converse is not true (e.g., see [9]).

Lemma 2.13. [9] Let (X, ρ) be a cone heptagonal metric space and P be a normal cone with normal constant K^*. Let $\{x_n\}$ be a sequence in (X, ρ) and $x \in X$, then $\{x_n\}$ converges to x if and only if $\|\rho(x_n, x)\| \to 0$ as $n \to \infty$.

Lemma 2.14. [9] Let (X, ρ) be a cone heptagonal metric space and P be a normal cone with normal constant K^*. Let $\{x_n\}$ be a sequence in (X, ρ) and $x \in X$, then $\{x_n\}$ is called Cauchy sequence if and only if $\|\rho(x_n, x_{n+m})\| \to 0$ as $n \to \infty$.

Definition 2.15. [9] If every Cauchy sequence is convergent in (X, ρ), then (X, ρ) is called a complete cone heptagonal metric space.

Lemma 2.16. [9] Let (X, d) be a complete cone heptagonal metric space. Let $\{x_n\}$ be a Cauchy sequence in X and suppose that there is natural number N such that:

1. $x_n \neq x_m$ for all $n, m > N$;
2. x_n, x are distinct points in X for all $n > N$;

3. x_n, y are distinct points in X for all $n > N$;
4. $x_n \to x$ and $x_n \to y$ as $n \to \infty$.

Then $x = y$.

3. Main Results

In this section, we derive the main result of our work, which is an extension of Banach contraction principle in cone heptagonal metric space and we give an example to illustrate the result.

Theorem 3.1. Let (X, ρ) be a complete cone heptagonal metric space, P be a normal cone with normal constant K^*. Suppose the mapping $S : X \to X$ satisfies the following:

$$\rho(Sx, Sy) \leq K\rho(x, y), \tag{3.2}$$

for all $x, y \in X$, where $0 \leq K < 1$. Then S has a unique fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Sx_n = S^{n+1}x_0, \; n = 0, 1, 2, \ldots.$$

Suppose that x_0 is not a periodic point. In fact, if $x_n = x_0$, for all $n \in \mathbb{N}$, from (3.2), it follows that

$$\begin{aligned} \rho(x_0, Sx_0) &= \rho(x_n, Sx_n) \\ &= \rho(S^n x_0, S^{n+1}x_0) \\ &\leq K\rho(S^{n-1}x_0, S^n x_0) \\ &\leq K^2\rho(S^{n-2}x_0, S^{n-1}x_0) \\ &\;\vdots \\ &\leq K^n\rho(x_0, Sx_0). \end{aligned} \tag{3.3}$$

This shows that

$$(K^n - 1)\rho(x_0, Sx_0) \in P.$$

It further implies that

$$\left(\frac{K^n - 1}{1 - K^n}\right)\rho(x_0, Sx_0) \in P.$$

Hence,

$$-\rho(x_0, Sx_0) \in P \text{ and } \rho(x_0, Sx_0) = 0.$$

Thus $Sx_0 = x_0$, this means x_0 is a fixed point of S.

Now, suppose that $x_m \neq x_n$, for all distinct $m, n \in \mathbb{N}$. By using heptagonal property and (3.3), for all $y \in X$, we have

$$\begin{aligned}\rho(y, S^6 y) &\leq \rho(y, Sy) + \rho(Sy, S^2 y) + \rho(S^2 y, S^3 y) + \rho(S^3 y, S^4 y) \\ &\quad + \rho(S^4 y, S^5 y) + \rho(S^5 y, S^6 y) \\ &\leq \rho(y, Sy) + K\rho(y, Sy) + K^2 \rho(y, Sy) + K^3 \rho(y, Sy) \\ &\quad + K^4 \rho(y, Sy) + K^5 \rho(y, Sy) \\ &\leq \sum_{i=0}^{5} K^i \rho(y, Sy).\end{aligned}$$

Similarly,

$$\begin{aligned}\rho(y, S^{11} y) &\leq \rho(y, Sy) + \rho(Sy, S^2 y) + \rho(S^2 y, S^3 y) + \rho(S^3 y, S^4 y) \\ &\quad + \rho(S^4 y, S^5 y) + \rho(S^5 y, S^6 y) + \rho(S^6 y, S^7 y) + \rho(S^7 y, S^8 y) \\ &\quad + \rho(S^8 y, S^9 y) + \rho(S^9 y, S^{10} y) + \rho(S^{10} y, S^{11} y) \\ &\leq \rho(y, Sy) + K\rho(y, Sy) + K^2 \rho(y, Sy) + K^3 \rho(y, Sy) \\ &\quad + K^4 \rho(y, Sy) + K^5 \rho(y, Sy) + K^6 \rho(y, Sy) + K^7 \rho(y, Sy) + K^8 \rho(y, Sy) \\ &\quad + K^9 \rho(y, Sy) + K^{10} \rho(y, Sy) \\ &\leq \sum_{i=0}^{10} K^i \rho(y, Sy).\end{aligned}$$

Now by induction, we obtain for each $k = 0, 1, 2, 3, \ldots$

$$\rho(y, S^{5k+1} y) \leq \sum_{i=0}^{5k} K^i \rho(y, Sy). \tag{3.4}$$

Also by heptagonal property and (3.3), for all $y \in X$, we have

$$\begin{aligned}\rho(y, S^7 y) &\leq \rho(y, Sy) + \rho(Sy, S^2 y) + \rho(S^2 y, S^3 y) + \rho(S^3 y, S^4 y) \\ &\quad + \rho(S^4 y, S^5 y) + \rho(S^5 y, S^7 y) \\ &\leq \rho(y, Sy) + K\rho(y, Sy) + K^2 \rho(y, Sy) + K^3 \rho(y, Sy) \\ &\quad + K^4 \rho(y, Sy) + K^5 \rho(y, S^2 y) \\ &\leq \sum_{i=0}^{4} K^i \rho(y, Sy) + K^5 \rho(y, S^2 y).\end{aligned}$$

Similarly,

$$\begin{aligned}\rho(y, S^{12}y) &\leq \rho(y, Sy) + \rho(Sy, S^2y) + \rho(S^2y, S^3y) + \rho(S^3y, S^4y)\\ &\quad + \rho(S^4y, S^5y) + \rho(S^5y, S^6y) + \rho(S^6y, S^7y) + \rho(S^7y, S^8y)\\ &\quad + \rho(S^8y, S^9y) + \rho(S^9y, S^{10}y) + \rho(S^{10}y, S^{12}y)\\ &\leq \rho(y, Sy) + K\rho(y, Sy) + K^2\rho(y, Sy) + K^3\rho(y, Sy)\\ &\quad + K^4\rho(y, Sy) + K^5\rho(y, Sy) + K^6\rho(y, Sy) + K^7\rho(y, Sy)\\ &\quad + K^8\rho(y, Sy) + K^9\rho(y, Sy) + K^{10}\rho(y, S^2y)\\ &\leq \sum_{i=0}^{9} K^i\rho(y, Sy) + K^{10}\rho(y, S^2y).\end{aligned}$$

By induction, we obtain for each $k = 1, 2, 3, \dots$

$$\rho(y, S^{5k+2}y) \leq \sum_{i=0}^{5k-1} K^i\rho(y, Sy) + K^{5k}\rho(y, S^2y). \tag{3.5}$$

Again for all $y \in X$, we have

$$\begin{aligned}\rho(y, S^8y) &\leq \rho(y, Sy) + \rho(Sy, S^2y) + \rho(S^2y, S^3y) + \rho(S^3y, S^4y)\\ &\quad + \rho(S^4y, S^5y) + \rho(S^5y, S^8y)\\ &\leq \rho(y, Sy) + K\rho(y, Sy) + K^2\rho(y, Sy) + K^3\rho(y, Sy)\\ &\quad + K^4\rho(y, Sy) + K^5\rho(y, S^3y)\\ &\leq \sum_{i=0}^{4} K^i\rho(y, Sy) + K^5\rho(y, S^3y).\end{aligned}$$

Similarly,

$$\begin{aligned}\rho(y, S^{13}y) &\leq \rho(y, Sy) + \rho(Sy, S^2y) + \rho(S^2y, S^3y) + \rho(S^3y, S^4y)\\ &\quad + \rho(S^4y, S^5y) + \rho(S^5y, S^6y) + \rho(S^6y, S^7y) + \rho(S^7y, S^8y)\\ &\quad + \rho(S^8y, S^9y) + \rho(S^9y, S^{10}y) + \rho(S^{10}y, S^{13}y)\\ &\leq \rho(y, Sy) + K\rho(y, Sy) + K^2\rho(y, Sy) + K^3\rho(y, Sy)\\ &\quad + K^4\rho(y, Sy) + K^5\rho(y, Sy) + K^6\rho(y, Sy) + K^7\rho(y, Sy)\\ &\quad + K^8\rho(y, Sy) + K^9\rho(y, Sy) + K^{10}\rho(y, S^3y)\\ &\leq \sum_{i=0}^{9} K^i\rho(y, Sy) + K^{10}\rho(y, S^3y).\end{aligned}$$

So by induction, we obtain for each $k = 1, 2, 3, \ldots$

$$\rho(y, S^{5k+3}y) \leq \sum_{i=0}^{5k-1} K^i \rho(y, Sy) + K^{5k} \rho(y, S^3 y). \tag{3.6}$$

In fact, for all $y \in X$, we also have

$$\begin{aligned} \rho(y, S^9 y) &\leq \rho(y, Sy) + \rho(Sy, S^2 y) + \rho(S^2 y, S^3 y) + \rho(S^3 y, S^4 y) \\ &\quad + \rho(S^4 y, S^5 y) + \rho(S^5 y, S^9 y) \\ &\leq \rho(y, Sy) + K\rho(y, Sy) + K^2 \rho(y, Sy) + K^3 \rho(y, Sy) \\ &\quad + K^4 \rho(y, Sy) + K^5 \rho(y, S^4 y) \\ &\leq \sum_{i=0}^{4} K^i \rho(y, Sy) + K^5 \rho(y, S^4 y). \end{aligned}$$

Similarly,

$$\begin{aligned} \rho(y, S^{14} y) &\leq \rho(y, Sy) + \rho(Sy, S^2 y) + \rho(S^2 y, S^3 y) + \rho(S^3 y, S^4 y) \\ &\quad + \rho(S^4 y, S^5 y) + \rho(S^5 y, S^6 y) + \rho(S^6 y, S^7 y) + \rho(S^7 y, S^8 y) \\ &\quad + \rho(S^8 y, S^9 y) + \rho(S^9 y, S^{10} y) + \rho(S^{10} y, S^{14} y) \\ &\leq \rho(y, Sy) + K\rho(y, Sy) + K^2 \rho(y, Sy) + K^3 \rho(y, Sy) + K^4 \rho(y, Sy) \\ &\quad + K^5 \rho(y, Sy) + K^6 \rho(y, Sy) + K^7 \rho(y, Sy) + K^8 \rho(y, Sy) \\ &\quad + K^9 \rho(y, Sy) + K^{10} \rho(y, S^4 y) \\ &\leq \sum_{i=0}^{9} K^i \rho(y, Sy) + K^{10} \rho(y, S^4 y). \end{aligned}$$

By induction, we obtain for each $k = 1, 2, 3, \ldots$

$$\rho(y, S^{5k+4}y) \leq \sum_{i=0}^{5k-1} K^i \rho(y, Sy) + K^{5k} \rho(y, S^4 y). \tag{3.7}$$

Again, for all $y \in X$, we also have

$$\begin{aligned} \rho(y, S^{10} y) &\leq \rho(y, Sy) + \rho(Sy, S^2 y) + \rho(S^2 y, S^3 y) + \rho(S^3 y, S^4 y) \\ &\quad + \rho(S^4 y, S^5 y) + \rho(S^5 y, S^{10} y) \\ &\leq \rho(y, Sy) + K\rho(y, Sy) + K^2 \rho(y, Sy) + K^3 \rho(y, Sy) \\ &\quad + K^4 \rho(y, Sy) + K^5 \rho(y, S^5 y) \\ &\leq \sum_{i=0}^{4} K^i \rho(y, Sy) + K^5 \rho(y, S^5 y). \end{aligned}$$

Similarly,

$$\begin{aligned}
\rho(y, S^{15}y) &\leq \rho(y, Sy) + \rho(Sy, S^2y) + \rho(S^2y, S^3y) + \rho(S^3y, S^4y) \\
&\quad + \rho(S^4y, S^5y) + \rho(S^5y, S^6y) + \rho(S^6y, S^7y) + \rho(S^7y, S^8y) \\
&\quad + \rho(S^8y, S^9y) + \rho(S^9y, S^{10}y) + \rho(S^{10}y, S^{15}y) \\
&\leq \rho(y, Sy) + K\rho(y, Sy) + K^2\rho(y, Sy) + K^3\rho(y, Sy) + K^4\rho(y, Sy) \\
&\quad + K^5\rho(y, Sy) + K^6\rho(y, Sy) + K^7\rho(y, Sy) + K^8\rho(y, Sy) \\
&\quad + K^9\rho(y, Sy) + K^{10}\rho(y, S^5y) \\
&\leq \sum_{i=0}^{9} K^i\rho(y, Sy) + K^{10}\rho(y, S^5y).
\end{aligned}$$

By induction, we obtain for each $k = 1, 2, 3, \ldots$

$$\rho(y, S^{5k+5}y) \leq \sum_{i=0}^{5k-1} K^i\rho(y, Sy) + K^{5k}\rho(y, S^5y). \tag{3.8}$$

Using inequalities (3.3) and (3.4), for $k = 0, 1, 2, 3, \ldots,$ we have

$$\begin{aligned}
&\rho(S^nx_0, S^{n+5k+1}x_0) \leq K^n\rho(x_0, S^{5k+1}x_0) \\
&\leq K^n\sum_{i=0}^{5k} K^i\rho(x_0, Sx_0) \\
&\leq K^n\Big[\sum_{i=0}^{5k} K^i\big(\rho(x_0, Sx_0) + \rho(x_0, S^2x_0) + \rho(x_0, S^3x_0) + \rho(x_0, S^4x_0) + \rho(x_0, S^5x_0)\big)\Big] \\
&\leq K^n\Big[\sum_{i=0}^{\infty} K^i\big(\rho(x_0, Sx_0) + \rho(x_0, S^2x_0) + \rho(x_0, S^3x_0) + \rho(x_0, S^4x_0) + \rho(x_0, S^5x_0)\big)\Big] \\
&\leq \frac{K^n}{1-K}\Big[\rho(x_0, Sx_0) + \rho(x_0, S^2x_0) + \rho(x_0, S^3x_0) + \rho(x_0, S^4x_0) + \rho(x_0, S^5x_0)\Big].
\end{aligned} \tag{3.9}$$

Similarly for $k = 1, 2, 3, \ldots,$ inequalities (3.3) and (3.5) implies that

$$\begin{aligned}
&\rho(S^n x_0, S^{n+5k+2}x_0) \le K^n \rho(x_0, S^{5k+2}x_0) \\
&\le K^n\Big[\sum_{i=0}^{5k-1} K^i \rho(x_0, Sx_0) + K^{5k}\rho(x_0, S^2x_0)\Big] \\
&\le K^n\Big[\sum_{i=0}^{5k-1} K^i \big(\rho(x_0, Sx_0) + \rho(x_0, S^2x_0) + \rho(x_0, S^3x_0) + \rho(x_0, S^4x_0) + \rho(x_0, S^5x_0)\big) \\
&\qquad + K^{5k}\big(\rho(x_0, Sx_0) + \rho(x_0, S^2x_0) + \rho(x_0, S^3x_0) + \rho(x_0, S^4x_0) + \rho(x_0, S^5x_0)\big)\Big] \\
&\le K^n\Big[\sum_{i=0}^{5k} K^i \big(\rho(x_0, Sx_0) + \rho(x_0, S^2x_0) + \rho(x_0, S^3x_0) + \rho(x_0, S^4x_0) + \rho(x_0, S^5x_0)\big)\Big] \\
&\le K^n\Big[\sum_{i=0}^{\infty} K^i \big(\rho(x_0, Sx_0) + \rho(x_0, S^2x_0) + \rho(x_0, S^3x_0) + \rho(x_0, S^4x_0) + \rho(x_0, S^5x_0)\big)\Big] \\
&\le \frac{K^n}{1-K}\Big[\rho(x_0, Sx_0) + \rho(x_0, S^2x_0) + \rho(x_0, S^3x_0) + \rho(x_0, S^4x_0) + \rho(x_0, S^5x_0)\Big].
\end{aligned} \tag{3.10}$$

Again for $k = 1, 2, 3, \ldots,$ inequalities (3.3) and (3.6) implies that

$$\begin{aligned}
&\rho(S^n x_0, S^{n+5k+3}x_0) \le K^n \rho(x_0, S^{5k+3}x_0) \\
&\le K^n\Big[\sum_{i=0}^{5k-1} K^i \rho(x_0, Sx_0) + K^{5k}\rho(x_0, S^3x_0)\Big] \\
&\le K^n\Big[\sum_{i=0}^{5k-1} K^i \big(\rho(x_0, Sx_0) + \rho(x_0, S^2x_0) + \rho(x_0, S^3x_0) + \rho(x_0, S^4x_0) + \rho(x_0, S^5x_0)\big) \\
&\qquad + K^{5k}\big(\rho(x_0, Sx_0) + \rho(x_0, S^2x_0) + \rho(x_0, S^3x_0) + \rho(x_0, S^4x_0) + \rho(x_0, S^5x_0)\big)\Big] \\
&\le K^n\Big[\sum_{i=0}^{5k} K^i \big(\rho(x_0, Sx_0) + \rho(x_0, S^2x_0) + \rho(x_0, S^3x_0) + \rho(x_0, S^4x_0) + \rho(x_0, S^5x_0)\big)\Big] \\
&\le K^n\Big[\sum_{i=0}^{\infty} K^i \big(\rho(x_0, Sx_0) + \rho(x_0, S^2x_0) + \rho(x_0, S^3x_0) + \rho(x_0, S^4x_0) + \rho(x_0, S^5x_0)\big)\Big] \\
&\le \frac{K^n}{1-K}\Big[\rho(x_0, Sx_0) + \rho(x_0, S^2x_0) + \rho(x_0, S^3x_0) + \rho(x_0, S^4x_0) + \rho(x_0, S^5x_0)\Big].
\end{aligned} \tag{3.11}$$

Again for $k = 1, 2, 3, \dots,$ inequalities (3.3) and (3.7) implies that

$$
\begin{aligned}
&\rho(S^n x_0, S^{n+5k+4} x_0) \le K^n \rho(x_0, S^{5k+4} x_0) \\
&\le K^n \Big[\sum_{i=0}^{5k-1} K^i \rho(x_0, Sx_0) + K^{5k} \rho(x_0, S^4 x_0) \Big] \\
&\le K^n \Big[\sum_{i=0}^{5k-1} K^i \big(\rho(x_0, Sx_0) + \rho(x_0, S^2 x_0) + \rho(x_0, S^3 x_0) + \rho(x_0, S^4 x_0) + \rho(x_0, S^5 x_0)\big) \\
&\qquad + K^{5k} \big(\rho(x_0, Sx_0) + \rho(x_0, S^2 x_0) + \rho(x_0, S^3 x_0) + \rho(x_0, S^4 x_0) + \rho(x_0, S^5 x_0)\big) \Big] \\
&\le K^n \Big[\sum_{i=0}^{5k} K^i \big(\rho(x_0, Sx_0) + \rho(x_0, S^2 x_0) + \rho(x_0, S^3 x_0) + \rho(x_0, S^4 x_0) + \rho(x_0, S^5 x_0)\big) \Big] \\
&\le K^n \Big[\sum_{i=0}^{\infty} K^i \big(\rho(x_0, Sx_0) + \rho(x_0, S^2 x_0) + \rho(x_0, S^3 x_0) + \rho(x_0, S^4 x_0) + \rho(x_0, S^5 x_0)\big) \Big] \\
&\le \frac{K^n}{1-K} \Big[\rho(x_0, Sx_0) + \rho(x_0, S^2 x_0) + \rho(x_0, S^3 x_0) + \rho(x_0, S^4 x_0) + \rho(x_0, S^5 x_0) \Big].
\end{aligned} \tag{3.12}
$$

Also for $k = 1, 2, 3, \dots,$ inequalities (3.3) and (3.8) implies that

$$
\begin{aligned}
&\rho(S^n x_0, S^{n+5k+5} x_0) \le K^n \rho(x_0, S^{5k+5} x_0) \\
&\le K^n \Big[\sum_{i=0}^{5k-1} K^i \rho(x_0, Sx_0) + K^{5k} \rho(x_0, S^5 x_0) \Big] \\
&\le K^n \Big[\sum_{i=0}^{5k-1} K^i \big(\rho(x_0, Sx_0) + \rho(x_0, S^2 x_0) + \rho(x_0, S^3 x_0) + \rho(x_0, S^4 x_0) + \rho(x_0, S^5 x_0)\big) \\
&\qquad + K^{5k} \big(\rho(x_0, Sx_0) + \rho(x_0, S^2 x_0) + \rho(x_0, S^3 x_0) + \rho(x_0, S^4 x_0) + \rho(x_0, S^5 x_0)\big) \Big] \\
&\le K^n \Big[\sum_{i=0}^{5k} K^i \big(\rho(x_0, Sx_0) + \rho(x_0, S^2 x_0) + \rho(x_0, S^3 x_0) + \rho(x_0, S^4 x_0) + \rho(x_0, S^5 x_0)\big) \Big] \\
&\le K^n \Big[\sum_{i=0}^{\infty} K^i \big(\rho(x_0, Sx_0) + \rho(x_0, S^2 x_0) + \rho(x_0, S^3 x_0) + \rho(x_0, S^4 x_0) + \rho(x_0, S^5 x_0)\big) \Big] \\
&\le \frac{K^n}{1-K} \Big[\rho(x_0, Sx_0) + \rho(x_0, S^2 x_0) + \rho(x_0, S^3 x_0) + \rho(x_0, S^4 x_0) + \rho(x_0, S^5 x_0) \Big].
\end{aligned} \tag{3.13}
$$

Thus, by inequalities (3.9), (3.10), (3.11), (3.12) and (3.13) we have, for each m,

$$\begin{aligned}\rho(S^n x_0, S^{n+m}x_0) \leq \frac{K^n}{1-K}\Big[&\rho(x_0, Sx_0) + \rho(x_0, S^2x_0)\\ &+ \rho(x_0, S^3x_0) + \rho(x_0, S^4x_0) + \rho(x_0, S^5x_0)\Big].\end{aligned}$$

Since P is a normal cone with normal constant K^*, we therefore have

$$\begin{aligned}\|\rho(S^n x_0, S^{n+m}x_0)\| \leq \frac{K^nK^*}{1-K}\Big[&\rho(x_0, Sx_0) + \rho(x_0, S^2x_0) + \rho(x_0, S^3x_0)\\ &+ \rho(x_0, S^4x_0) + \rho(x_0, S^5x_0)\Big].\end{aligned}$$

Hence

$$\lim_{n\to\infty} \|\rho(x_n, x_{n+m})\| = 0.$$

Therefore, by Lemma 2.14, the sequence $\{x_n\}$ is a Cauchy sequence in (X, ρ). Since (X, ρ) is complete, then there exists a point $z \in X$ such that $\lim_{n\to\infty} x_n = z$. By Lemma 2.13, we have

$$\lim_{n\to\infty} \|\rho(S^n x_0, z)\| = 0. \tag{3.14}$$

Now, we will show that z is a fixed point of S, i.e. $Sz = z$. Since $x_n \neq x_m$ for $n \neq m$, by hexagonal property and (3.3), we have

$$\begin{aligned}\rho(Sz, z) &\leq \rho(Sz, S^nx_0) + \rho(S^nx_0, S^{n+1}x_0) + \rho(S^{n+1}x_0, S^{n+2}x_0) + \rho(S^{n+2}x_0, S^{n+3}x_0)\\ &\qquad + \rho(S^{n+3}x_0, S^{n+4}x_0) + \rho(S^{n+4}x_0, z)\\ &\leq K\rho(z, S^{n-1}x_0) + K^n\rho(x_0, Sx_0) + K^{n+1}\rho(x_0, Sx_0) + K^{n+2}\rho(x_0, Sx_0)\\ &\qquad + K^{n+3}\rho(x_0, Sx_0) + \rho(S^{n+4}x_0, z).\end{aligned}$$

Since P is a normal cone with normal constant K^*, we therefore have

$$\begin{aligned}\|\rho(Sz, z)\| \leq K^*\Big[&K\|\rho(z, S^{n-1}x_0)\| + K^n\|\rho(x_0, Sx_0)\|\\ &+ K^{n+1}\|\rho(x_0, Sx_0)\| + K^{n+2}\|\rho(x_0, Sx_0)\|\\ &+ K^{n+3}\|\rho(x_0, Sx_0)\| + \|\rho(S^{n+4}x_0, z)\|\Big].\end{aligned}$$

Thus, by (3.14), we have that

$$\lim_{n\to\infty} \|\rho(Sz, z)\| = 0.$$

Therefore, $Sz = z$. That is, S has a fixed point z in X.

Next we show that the fixed point z is unique. For suppose z' be another fixed point of S such that $Sz' = z'$. Therefore,

$$\rho(z, z') = \rho(Sz, Sz') \leq K\rho(z, z').$$

Hence, $z = z'$. This completes the proof of the theorem. ∎

The following example illustrates the result of Theorem 3.1.

Example 3.2. Let $X = \{r, s, t, u, v, w, k\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$ is a normal cone in E. Define $\rho : X \times X \to E$ as follows:

$$\begin{aligned}
\rho(x, x) &= 0, \ \forall x \in X;\\
\rho(r, s) = \rho(s, r) &= (6, 12);\\
\rho(r, t) = \rho(t, r) = \rho(r, u) = \rho(u, r) &= \rho(r, v) = \rho(v, r) = \rho(r, w)\\
&= \rho(w, r) = \rho(s, t) = \rho(t, s)\\
= \rho(s, u) = \rho(u, s) = \rho(s, v) &= \rho(v, s) = \rho(t, v) = \rho(v, t)\\
&= \rho(s, w) = \rho(w, s) = \rho(t, u)\\
= \rho(u, t) = \rho(t, w) = \rho(w, t) &= \rho(u, v) = \rho(v, u) = \rho(u, w)\\
&= \rho(w, u) = \rho(v, w) = (1, 2);\\
\rho(r, k) = \rho(k, r) = \rho(s, k) &= \rho(k, s) = \rho(t, k) = \rho(k, t)\\
&= \rho(u, k) = \rho(k, u) = \rho(v, k) = \rho(k, v)\\
&= \rho(w, k) = \rho(k, w) = (5, 10).
\end{aligned}$$

Then (X, ρ) is a complete cone heptagonal metric space, but (X, ρ) is not a complete cone hexagonal metric space because it lacks the hexagonal property:

$$\begin{aligned}
(6, 12) = \rho(r, s) &> \rho(r, t) + \rho(t, u) + \rho(u, v) + \rho(v, w) + \rho(w, s)\\
&= (1, 2) + (1, 2) + (1, 2) + (1, 2) + (1, 2)\\
&= (5, 10), \text{ as } (6, 12) - (5, 10) = (1, 2) \in P.
\end{aligned}$$

Now, we define a mapping $S : X \to X$ as follows

$$S(x) = \begin{cases} 6, & \text{if } x \neq k;\\ 1, & \text{if } x = k.\end{cases}$$

Hence, we obtain that

$$\begin{aligned}
\rho(S(r), S(s)) = \rho(S(r), S(t)) = \rho(S(r), S(u)) = \rho(S(r),&\\
S(v)) = \rho(S(r), S(w)) = \rho(S(s), S(t))&\\
= \rho(S(s), S(u)) = \rho(S(s), S(v)) = \rho(S(s), S(w))&\\
= \rho(S(t), S(u)) = \rho(S(t), S(v))&\\
= \rho(S(t), S(w)) = \rho(S(u), S(v))&\\
= \rho(S(u), S(w)) = \rho(S(v), S(w)) = 0.&
\end{aligned}$$

And in all other cases $\rho(S(x), S(y)) = (1, 2)$ and $\rho(x, y) = (5, 10)$, for all $x, y \in X$.

Hence, the conditions of Theorem 3.1 holds for all $x, y \in X = \{r, s, t, u, v, w, k\} = \{1, 2, 3, 4, 5, 6, 7\}$, where $K = \frac{1}{6}$, and $w = 6 \in X$ is the unique fixed point of the mappings S.

Remark 3.3. In the above example, results of [6, 7, 8] are not applicable to obtain the fixed of the mapping S on X, since (X, ρ) is not a cone hexagonal metric space.

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