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An asymptotic analysis of compressible twophase flow equations near initial time[1]

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Abstract
We asymptotically analyze compressible twophase flow near the initial time. The formal inner limit asymptotic expansions are derived for the solutions of compressible equations. Specifically, under the singular limit process, we find the inner terms and fast variables in the exterior domain and within the mixing zone.

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1. Introduction

We asymptotically analyze compressible twophase flow [2, 6, 7, 13, 14] near the initial time. The nondimensionalized equations of compressible isentropic ideal twophase flow are a nonlinear hyperbolic system in the volume fraction β_k^λ, velocity v_k^λ, density ρ_k^λ, and pressure p_k^λ of fluid k, defined by the equations

$$\frac{\partial \beta_k^\lambda}{\partial t} + v^{*\lambda}\frac{\partial \beta_k^\lambda}{\partial z} = 0\,, \tag{1.1}$$

$$\beta_k^\lambda\left(\frac{\partial \rho_k^\lambda}{\partial t} + v_k^\lambda\frac{\partial \rho_k^\lambda}{\partial z}\right) + \beta_k^\lambda\rho_k^\lambda\frac{\partial v_k^\lambda}{\partial z} + \rho_k^\lambda(v_k^\lambda - v^{*\lambda})\frac{\partial \beta_k^\lambda}{\partial z} = 0, \tag{1.2}$$

$$\beta_k^\lambda\rho_k^\lambda\left(\frac{\partial v_k^\lambda}{\partial t} + v_k^\lambda\frac{\partial v_k^\lambda}{\partial z}\right) + \lambda^2\beta_k^\lambda\frac{\partial p_k^\lambda}{\partial z} + \lambda^2(p_k^\lambda - p^{*\lambda})\frac{\partial \beta_k^\lambda}{\partial z} = \beta_k^\lambda\rho_k^\lambda g(t), \tag{1.3}$$

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depending on a large dimensionless parameter λ. Here $p_k^\lambda = p_k(\rho_k^\lambda)$ and an equation of state $p_k(\rho_k) = A_k \rho_k^{\gamma_k}$, $\gamma_k > 1$ is given with $\partial p_k/\partial \rho_k(\rho_k) > 0$ for $\rho_k > 0$ and the entropy A_k assumed to be constant within each fluid but $A_1 \neq A_2$. The fluids are distinguished by a subscript k, *i.e.*, $k = 1$ for the light fluid and $k = 2$ for the heavy fluids, and $g = g(t) > 0$ is the gravity. The parameter λ is the reciprocal of the Mach number, $M = |v_m|(\gamma p(\rho_m)/\rho)^{-1/2}$, the ratio of fluid speed to sound speed, where ρ_m is the mean density and $|v_m|$ is a typical mean fluid velocity which is the ratio of time units to space units. In fact, $\lambda = M^{-1}(\gamma A)^{-1/2}$.

The interfacial quantities v^* and p^* of twophase flow have been proposed [6, 7, 12, 13] by closure relations

$$q^* = \mu_1^q q_2 + \mu_2^q q_1, \quad q = v, p, \tag{1.4}$$

$$\mu_k^q(\beta_k, d_k^q) = \frac{\beta_k}{\beta_k + d_k^q \beta_{k'}}. \tag{1.5}$$

Here the primed index k' denotes the fluid complementary to fluid k, i.e., $k' = 3 - k$. We note that $\mu_1^q + \mu_2^q = 1$ and $\mu_k^q \geq 0$ and that $\mu_k^q/\beta_k$is continuous on $0 \leq \beta_k \leq 1$ and for all t. The μ_k^q thus depends on a single parameter d_k^q. [13, 14] proposed a closure for the constitutive law $d_k^q(t)$ and it was compared in a validation study based on simulation data [1, 18].

Under appropriate conditions, one expects that the compressible twophase flow solutions β_k^λ, v_k^λ, p_k^λ converge to the incompressible solutions as $\lambda \to \infty$. In contrast with the incompressible limit problem of single phase flow, that of the compressible twophase flow equations requires advanced techniques in asymptotics of time-singular and layer-type. The incompressible limit of the single phase compressible Euler or Navier-Stokes equations has been studied in higher space dimensions from various points of view [3, 8, 15, 16]. On the other hand, [4, 12] discussed the outer limit process of the compressible twophase flow equations describing the fluid motions away from the initial time. The slow variables in the outer limit asymptotic expansions have a slow scale of motion and they have been determined through second order in closed form. Information supplied from the weakly compressible theory plays a role in resolving underdetermination of incompressible pressures. This aspect of two-phase flow in the incompressible limit appears to be unique.

As a background flow of the incompressible limit, $\lambda \to \infty$, the incompressible flow in the volume fraction β_k^∞, velocity v_k^∞, and scalar pressure p_k^∞, defined by the equations

$$\frac{\partial \beta_k^\infty}{\partial t} + v^{*\infty}\frac{\partial \beta_k^\infty}{\partial z} = 0, \tag{1.6}$$

$$\beta_k^\infty \frac{\partial v_k^\infty}{\partial z} + (v_k^\infty - v^{*\infty})\frac{\partial \beta_k^\infty}{\partial z} = 0, \tag{1.7}$$

$$\beta_k^\infty \rho_k^\infty \left(\frac{\partial v_k^\infty}{\partial t} + v_k^\infty \frac{\partial v_k^\infty}{\partial z}\right) + \beta_k^\infty \frac{\partial p_k^\infty}{\partial z} + (p_k^\infty - p^{*\infty})\frac{\partial \beta_k^\infty}{\partial z} = \beta_k^\infty \rho_k^\infty g(t) \tag{1.8}$$

are considered, where ρ_k^∞ is the constant density of phase k. We assume that all state variables are piecewise C^1 functions with discontinuous derivatives at the mixing zone edges $z = Z_k^\infty(t)$ of incompressible flow. Analytic solutions of the incompressible problem (1.6)-(1.8) have been obtained in closed form [5, 6, 7].

In this paper, we are concerned with the limiting motion of the solution $U_k^\lambda \equiv (\beta_k^\lambda, v_k^\lambda, \rho_k^\lambda, p_k^\lambda)^{tr}$ of the compressible equations (1.1)-(1.3) near the initial time as $\lambda \to \infty$. Specifically under the singular limit process, we derive inner limit asymptotic expansions for the solutions of the compressible equations. Each order of asymptotic expansions for the compressible solutions describing the incompressible limit process has an independent existence, defined as proportional to a derivative of the compressible solution with respect to λ evaluated at the value $\lambda = 0$ of the expansion parameter. The inner terms, uniformly valid in space, in closed form through first order in the expansion of volume fraction, density and pressure, and zero-th order in the expansion of velocity have been evaluated in [10] and are summarized in Sec. 3.1. The main result of this paper is the evaluation of the inner terms in higher order of λ^{-1} through second order in the expansion of volume fraction, density and pressure, and first order in the expansion of velocity. They are determined in the exterior and the mixing zone, respectively in Sec. 3.2. The fast variables are defined as the inner expansion terms minus the common terms resulting from the outer expansion evaluated in the inner limit to each order. These variables oscillate on the fast time scale and are described by linear wave equations. In Sec. 3.2, they are also found in the exterior and the mixing zone. It represents that fast scale acoustical oscillations can first appear in the second order of the asymptotic expansion of β_k^λ, ρ_k^λ, p_k^λ and in the first order of the expansion v_k^λ. In Sec. 2 we present the constitutive asymptotic assumptions and conditions which are required to derive formal asymptotic expansions for compressible solutions. In higher order in λ^{-1}, we are concerned here, there exist four transition-layers having new fast transitional variables. The fast transition-layer expansions will be defined later and furthermore, the uniformly valid inner limit terms will be determined by matching the inner limit in the exterior and the incompressible mixing zone and the fast transition-layers expansions through higher order in future studies. For simplicity, we suppress superscript λ's of compressible variables from now on.

2. Asymptotic Conditions

We specify boundary conditions for compressible and incompressible flow,

$$v_1(z^{+\infty}) = 0, \quad p_2(z^{-\infty}) = const, \tag{2.9}$$

where $z = z^{+\infty}(z = z^{-\infty})$ denotes the position of the upper (lower) wall of a finite but large domain $\mathcal{D}$.

We introduce the asymptotic expansions of compressible solutions in the form

$$\begin{aligned}\beta_k &= \beta_k^{(0,s)} + \lambda^{-1}\beta_k^{(1,s)} + \lambda^{-2}\left(\beta_k^{(2,s)} + \beta_k^{(2,f)}\right) + O(\lambda^{-3}),\\ v_k &= v_k^{(0,s)} + \lambda^{-1}\left(v_k^{(1,s)} + v_k^{(1,f)}\right) + O(\lambda^{-2}),\\ \rho_k &= \rho_k^{(0,s)} + \lambda^{-1}\rho_k^{(1,s)} + \lambda^{-2}\left(\rho_k^{(2,s)} + \rho_k^{(2,f)}\right) + O(\lambda^{-3}).\end{aligned} \tag{2.10}$$

The equation of state gives the expansion

$$\begin{aligned}p_k &= p_k^{(0,s)} + \lambda^{-1}p_k^{(1,s)} + \lambda^{-2}\left(p_k^{(2,s)} + p_k^{(2,f)}\right) + O(\lambda^{-3})\\ &= p_k(\rho_k^{(0,s)}) + \lambda^{-1}c_k^2(\rho_k^{(0,s)})\rho_k^{(1,s)}\\ &+ \lambda^{-2}\left(\frac{1}{2}\frac{\partial^2 p_k}{\partial \rho_k^2}(\rho_k^{(0,s)})\rho_k^{(1,s)2} + c_k^2(\rho_k^{(0,s)})\rho_k^{(2,s)} + c_k^2(\rho_k^{(0,s)})\rho_k^{(2,f)}\right) + O(\lambda^{-3}),\end{aligned} \tag{2.11}$$

where $c_k^2(\rho) = \dfrac{\partial p_k}{\partial \rho_k}(\rho)$. The compressible solutions are assumed to have the initial conditions

$$\begin{aligned}\beta_k(z,0) &= \beta_k^{\infty}(z,0) + \lambda^{-1}\beta_k^{(1,s)}(z,0) + \lambda^{-2}\beta_k^{(2,s)}(z,0),\\ v_k(z,0) &= v_k^{\infty}(z,0) + \lambda^{-1}\left[v_k^{(1,s)}(z,0) + v_k^{(1)}(z)\right],\\ \rho_k(z,0) &= \rho_k^{\infty}(z,0) + \lambda^{-2}\left[c_k^{-2}(\rho_k^{\infty})\rho_k^{\infty}(z,0) + \rho_k^{(2)}(z)\right],\\ p_k(z,0) &= p_k(\rho_k^{\infty}) + \lambda^{-2}\left[p_k^{\infty}(z,0) + p_k^{(2)}(z)\right],\end{aligned} \tag{2.12}$$

where $v_k^{(1)}(z)$, $\rho_k^{(2)}(z)$, and $p_k^{(2)}(z)$ belong to C^1 on $(-1)^k z \le (-1)^k Z_k(0)$ and $|||v_k^{(1)}(z)||| = O(1)$, $|||\rho_k^{(2)}(z)||| = O(1)$ and $|||p_k^{(2)}(z)||| = O(1)$. The variables

$$U_k^{(m,s)} \equiv \left(\beta_k^{(m,s)}, v_k^{(m,s)}, \rho_k^{(m,s)}, p_k^{(m,s)}\right)^{tr},$$

$m = 0, 1, 2$, have a slow scale of motion and they have been determined in closed form through second order [12].

We denote $Z_k = Z_k(t)$ as the position of the mixing zone edge k, defined as the location of vanishing β_k and $V_k = dZ_k/dt$ as the velocity of the edge k. At edge k, the following boundary data holds $v_k = V_k(t)$ at $z = Z_k(t)$. The two phase flow model depends on the motions of the mixing zone edges Z_k and closure for the interfacial averages with the constitutive law d_k^q, $q = v, p$. The mixing zone edges Z_k are not well characterized for compressible flows. Thus the velocities or trajectories of the edges of the mixing zone must be provided as data. Here we asymptotically assume the velocities or trajectories of the edges of the mixing zone with a specific limit term. A uniformly valid asymptotic expansion for the compressible mixing zone edge is assumed as the following

$$Z_k(t) = Z_k^{(0,s)}(t) + \lambda^{-1} Z_k^{(1,s)}(t) + \lambda^{-2}\left(Z_k^{(2,s)}(t) + Z_k^{(2,f)}(t,\lambda)\right) + O(\lambda^{-3}). \quad (2.13)$$

Here $\sum_{j=0}^{m} \lambda^{-j} Z_k^{(j,s)}$, $m = 0, 1, 2$, denotes the location of vanishing $\beta_k^{(m,s)}$. Thus Z_k and each of the expansion coefficients $Z_k^{(m,s)}$ and $Z_k^{(2,f)}$ are input to the model equations. We assume that the compressible edge moves faster than the incompressible edge with no initial perturbation. A similar assumption is applied to any finite number of terms in the expansion (2.12). We assume that the zero-th order term in the expansion (2.12) equals to the incompressible edge trajectory $Z_k^{\infty}(t)$. Thus we require

$$\begin{gathered} Z_k^{(0,s)}(t) = Z_k^{\infty}(t),\ Z_k(0) = Z_k^{\infty}(0), \\ (-1)^k Z_k^{(m,t)}(t) \geq 0,\ m = 1, 2,\ t = s, f. \end{gathered} \quad (2.14)$$

The variable $Z_k^{(2,f)}$ is oscillatory on the fast time scale $\tau \equiv \lambda t$ while the terms $Z_k^{(m,s)}$, $m = 0, 1, 2$, are the slow variables with a slow scale of motion. We assume that the fast variable $Z_k^{(2,f)}$ decays exponentially in τ away from the initial curve $t = 0$. The formally uniformly valid expansion (2.13) leads to the inner and outer expansion under the corresponding limit process. The reduced expansion is assumed in the derivation of inner and outer expansions for compressible solutions in Sec. 3 and [10, 12]. Following the inner limit process $\lambda \to \infty$ with τ fixed $\neq \infty$, (2.13) leads to the inner limit expansion, valid near $t = 0$:

$$\begin{aligned} Z_k(\tau) &= \widehat{Z}_k^{(0)}(\tau) + \lambda^{-1}\widehat{Z}_k^{(1)}(\tau) + \lambda^{-2}\widehat{Z}_k^{(2)}(\tau) + O(\lambda^{-3}) \\ &= Z_k^{(0,s)}(0) + \lambda^{-1}\tau \dot{Z}_k^{(0,s)}(0) \\ &+ \lambda^{-2}\left(\frac{\tau^2}{2}\ddot{Z}_k^{(0,s)}(0) + \tau \dot{Z}_k^{(1,s)}(0) + Z_k^{(2,s)}(0) + Z_k^{(2,f)}(\tau)\right) + O(\lambda^{-3}). \end{aligned} \quad (2.15)$$

Refer to [9, 10]. The initial conditions associated with (2.14) are

$$\widehat{Z}_k^{(0)}(0) = Z_k^{\infty}(0),\ \widehat{Z}_k^{(m)}(0) = 0,\ m \geq 1. \quad (2.16)$$

Notice that the fast variable $Z_k^{(2,f)}$ consist of the inner term $\widehat{Z}_k^{(2)}$ minus common terms to order λ^{-2}. Under the outer limit process, $\lambda \to \infty$ with t fixed $\neq 0$, (2.13) leads to the outer expansion

$$Z_k(t) = Z_k^{(0,s)}(t) + \lambda^{-1} Z_k^{(1,s)}(t) + \lambda^{-2} Z_k^{(2,s)}(t) + O(\lambda^{-3}) \quad (2.17)$$

valid away from the initial curve $t = 0$.

Since the edge velocity of the compressible flow satisfies $V_k = \dot{Z}_k = v_k(Z_k, t)$, it must have an asymptotic expansion associated with the expansion (2.12) in the form

$$\begin{aligned} V_k(t) =& V_k^{(0,s)}(t) + \lambda^{-1}\left(V_k^{(1,s)}(t) + V_k^{(1,f)}(t,\lambda)\right) \\ &+ \lambda^{-2}\left(V_k^{(2,s)}(t) + V_k^{(2,f)}(t,\lambda)\right) + O(\lambda^{-3}). \end{aligned} \tag{2.18}$$

The formal asymptotic expansion (2.18) leads to the inner limit expansion

$$\begin{aligned} V_k(\tau) &= \widehat{V}_k^{(0)}(\tau) + \lambda^{-1}\widehat{V}_k^{(1)}(\tau) + O(\lambda^{-2}) \\ &= V_k^{(0,s)}(0) + \lambda^{-1}\left(\tau \dot{V}_k^{(0,s)}(0) + V_k^{(1,s)}(0) + V_k^{(1,f)}(\tau)\right) + O(\lambda^{-2}). \end{aligned} \tag{2.19}$$

which is valid near the initial curve $t = 0$ under the limit process $\lambda \to \infty$ with τ fixed $\neq \infty$, and to the outer limit expansion

$$V_k(t) = V_k^{(0,s)}(t) + \lambda^{-1}V_k^{(1,s)}(t) + \lambda^{-2}V_k^{(2,s)}(t) + O(\lambda^{-3}) \tag{2.20}$$

which is valid away from the initial curve $t = 0$ under the limit process $\lambda \to \infty$ with t fixed $\neq 0$.

The compressible constitutive laws $d_k^v(t)$ and $d_k^p(t)$ are asymptotically assumed with a specific limit term as follows

$$d_k^q(t,\lambda) = d_k^{q(0,s)}(t) + \lambda^{-1}\left(d_k^{q(1,s)}(t) + d_k^{q(1,f)}(t,\lambda)\right) + O(\lambda^{-2}),\ q = v, p, \tag{2.21}$$

where $d_k^{q(m,s)}(t)$, $d_k^{q(m,f)}(\tau) \in C([0,\infty))$, and we assume $d_k^{q(0,s)}(t) = d_k^{q\infty}(t)$. Similarly, we obtain [9] that the expansion (2.21) leads to the inner limit asymptotic expansion

$$\begin{aligned} d_k^q(\tau) &= \widehat{d}_k^{q(0)}(\tau) + \lambda^{-1}\widehat{d}_k^{q(1)}(\tau) + (\lambda^{-2}) \\ &= d_k^{q(0,s)}(0) + \lambda^{-1}\left(\tau\frac{dd_k^q(0,s)}{dt}(0) + d_k^{q(1,s)}(0) + d_k^{q(1,f)}(\tau)\right) + O(\lambda^{-2}), \end{aligned} \tag{2.22}$$

and to the outer limit expansion

$$d_k^q(t) = d_k^{q(0,s)}(t) + \lambda^{-1}d_k^{q(1,s)}(t) + \lambda^{-2}d_k^{q(2,s)}(t) + O(\lambda^{-3}),\ q = v, p. \tag{2.23}$$

In [10] the first order inner expansion was defined by five regions, $\widehat{\mathcal{E}}_k^{(1)} \cup \widehat{\mathcal{T}}_k^{(1)} \cup \widehat{\mathcal{M}} \cup \widehat{\mathcal{T}}_{k'}^{(1)} \cup \widehat{\mathcal{E}}_{k'}^{(1)}$, including two transition-layers through $z = \overline{\widehat{Z}_i^{(1)}}$, $i = k, k'$. In higher order in λ^{-1}, we are concerned here, we have two additional fast transition-layers extending out to $z = \overline{\widehat{Z}_i^{(2)}}$, $i = k, k'$, so the second order expansion is uniquely defined by seven regions $\widehat{\mathcal{E}}_k^{(2)} \cup \widehat{\mathfrak{T}}_k^{(2)} \cup \widehat{\mathfrak{T}}_k^{(1)} \cup \widehat{\mathcal{M}} \cup \widehat{\mathfrak{T}}_{k'}^{(1)} \cup \widehat{\mathfrak{T}}_{k'}^{(2)} \cup \widehat{\mathcal{E}}_{k'}^{(2)}$. The regions are defined by

$$\widehat{\mathcal{E}}_{i'}^{(n)} = \left\{(z,t) : (-1)^i \overline{\widehat{Z}_i^{(n)}} \le (-1)^i z\right\}, \tag{2.24}$$

$$\widehat{\mathfrak{T}}_{i'}^{(n)} = \left\{(z,t) : (-1)^i \overline{\widehat{Z}_i^{(n-1)}} \le (-1)^i z < (-1)^i \overline{\widehat{Z}_i^{(n)}}\right\},\ n = 1, 2, \tag{2.25}$$

$$\widehat{\mathcal{M}} = \left\{(z,t) : \widehat{Z}_1^{(0)} < z < \widehat{Z}_2^{(0)}\right\}, \tag{2.26}$$

where

$$\overline{\widehat{Z}_k^{(m)}}(t) \equiv \sum_{j=0}^{m} \lambda^{-j} \widehat{Z}_k^{(j)}, m = 0, 1, 2. \tag{2.27}$$

denotes the position of boundaries of the fast transition-layers. The fast transitional layer expansions are derived uniformly in the transition-layers $\widehat{\mathfrak{T}}_i^{(n)}$, $n = 1, 2$, $i = k, k'$. Each order term of λ^{-1} in the expansions satisfy simple differential equations and they are solved in closed form [11].

3. Inner Limit Expansions

The motion of the fast variables is understandable by making the change of variables to the fast time scale $\tau \equiv \lambda t$. Then the compressible equations (1.1)-(1.3) reduce to

$$\lambda \frac{\partial \beta_k}{\partial \tau} \quad + \quad v^* \frac{\partial \beta_k}{\partial z} = 0\,, \tag{3.28}$$

$$\beta_k \left(\lambda \frac{\partial \rho_k}{\partial \tau} + v_k \frac{\partial \rho_k}{\partial z}\right) \quad + \quad \beta_k \rho_k \frac{\partial v_k}{\partial z} + \rho_k (v_k - v^*) \frac{\partial \beta_k}{\partial z} = 0\,, \tag{3.29}$$

$$\beta_k \rho_k \left(\lambda \frac{\partial v_k}{\partial \tau} + v_k \frac{\partial v_k}{\partial z}\right) \quad + \quad \lambda^2 \beta_k \frac{\partial p_k}{\partial z} + \lambda^2 (p_k - p^*) \frac{\partial \beta_k}{\partial z} = \beta_k \rho_k g(t)\,, \tag{3.30}$$

for $U_k(z,\tau)$. Assuming the initial data (2.12) for the compressible solutions and the inner limit asymptotic expansions (2.15), (2.19) and (2.22) for Z_k, V_k and d_k^q, $q = v, p$, we want to derive the inner limit asymptotic expansions, valid near $t = 0$, of the solutions of the compressible equations (3.28)-(3.30), uniformly valid in z.

We introduce inner limit asymptotic expansions associated with the inner limit $\lambda \to \infty$ with τ fixed $\neq \infty$:

$$U_k(z,\tau) = \widehat{U}_k^{(0)}(z,\tau) + \lambda^{-1} \widehat{U}_k^{(1)}(z,\tau) + \lambda^{-2} \widehat{U}_k^{(2)}(z,\tau) + O(\lambda^{-}3)\,, \tag{3.31}$$

where $\widehat{U}_k^{(m)}(z,\tau) \equiv \left(\widehat{\beta}_k^{(m)}, \widehat{v}_k^{(m)}, \widehat{\rho}_k^{(m)}, \widehat{p}_k^{(m)}\right)^{tr}$, $m = 0, 1, 2, \cdots$. The equation of state gives the expansion relations

$$\widehat{p}_k^{(0)} = p_k(\widehat{\rho}_k^{(0)})\ , \quad \widehat{p}_k^{(1)} = c_k^2(\widehat{\rho}_k^{(0)})\widehat{\rho}_k^{(1)}\ , \quad \widehat{p}_k^{(2)} = \frac{1}{2}\frac{\partial^2 p_k}{\partial \rho_k^2}(\widehat{\rho}_k^{(0)})\widehat{\rho}_k^{(1)2} + c_k^2(\widehat{\rho}_k^{(0)})\widehat{\rho}_k^{(2)} \tag{3.32}$$

between the terms in the expansions of ρ_k and p_k. From (3.31) and (3.32), the inner limit expansion for the mixing coefficient $\mu_k^q(\beta_k, d_k^q)$, $q = v, p$ is introduced. Refer to [10].

Our concern is to derive the inner terms in the inner limit asymptotic expansions (3.31) for the solutions of the compressible equations. The equations for the inner limit terms $\widehat{U}_k^{(m)}(z,\tau) \equiv \left(\widehat{\beta}_k^{(m)}, \widehat{v}_k^{(m)}, \widehat{\rho}_k^{(m)}, \widehat{p}_k^{(m)}\right)^{tr}$, $m = 0, 1$, are derived by repeated application of the inner limit expansions (3.31) to the compressible equations (3.28)-(3.30), and by equating terms of the same order of λ. Within a single power of λ, the inner terms are defined as a solution of simple differential equations. We solve the inner term equations in each region defined in (2.25). The inner terms, uniformly valid in space, are determined by matching the inner limit in the exterior and the incompressible mixing zone and the fast transition-layer expansions.

3.1. Inner Limit Terms to First Order

We substitute the inner limit asymptotic expansions (3.31) into the compressible equations (3.28)-(3.30), and equate powers of λ. Since λ is arbitrary, the coefficient of λ^m for each order m must vanish, defining equations for the inner terms to each order. We first solve the inner term $\widehat{\beta}_k^{(0)}, \widehat{\rho}_k^{(0)}, \widehat{p}_k^{(0)}$ equations. They are determined completely by the initial data of incompressible flow in (2.12). Then the inner terms $\widehat{v}_k^{(0)}, \widehat{\beta}_k^{(1)}, \widehat{\rho}_k^{(1)}, \widehat{p}_k^{(1)}$, uniformly valid in z are found by solving a subsystem of hyperbolic differential equations and by matching at the outer and inner edges of regions [10]. Specifically, it has been shown that there are no fast scale acoustical oscillation in the asymptotic expansions of β_k, ρ_k, p_k through first order and in the order zero expansion of v_k. The result is summarized in the following theorem.

Theorem 3.1. Assume (2.9), (2.15), (2.19), (2.22), and the initial data (2.12). Then the compressible solutions have the uniformly valid inner expansions in z to $O(\lambda^{-1})$ for β_k, ρ_k, p_k and to $O(1)$ for v_k of the form

$$\begin{aligned}\beta_k &= \beta_k^{\infty}(z,0) + \lambda^{-1}\widehat{\beta}_k^{(1)} + O(\lambda^{-2}),\\ v_k &= v_k^{\infty}(z,0) + O(\lambda^{-1}),\\ \rho_k &= \rho_k^{\infty} + O(\lambda^{-2}),\\ p_k &= p^{(0)} + O(\lambda^{-2}),\end{aligned} \tag{3.33}$$

where $p^{(0)} = p_k(\rho_k^\infty)$ and the inner term $\widehat{\beta}_k^{(1)}(z,\tau)$ satisfies

$$\widehat{\beta}_k^{(1)}(z,\tau) = \begin{cases} 0 & \text{in } \widehat{\mathcal{E}}_1^{(1)},\ \widehat{\mathcal{E}}_2^{(1)} \\ \lambda\left(z - Z_{k'}^\infty(0)\right)\dfrac{\partial \beta_k^\infty}{\partial z}(Z_{k'}^\infty(0) + (-1)^k 0, 0) & \\ +\tau \dfrac{\partial \beta_k^\infty}{\partial t}(Z_{k'}^\infty(0) + (-1)^k 0, 0) & \text{in } \widehat{\mathcal{T}}_k^{(1)} \\ \beta_k^{(1,s)}(z,0) + \tau \dfrac{\partial \beta_k^\infty}{\partial t}(z,0) & \text{in } \widehat{\mathcal{M}} \\ -\beta_{k'}^{(1,s)}(z,t) & \text{in } \widehat{\mathcal{T}}_{k'}^{(1)} \end{cases}. \tag{3.34}$$

3.2. Higher Order Inner Terms

Using the zero-th order and first order inner terms determined in Theorem 3.1, we find the inner terms $\widehat{v}_k^{(1)}, \widehat{\beta}_k^{(2)}, \widehat{\rho}_k^{(2)}, \widehat{p}_k^{(2)}$ in the expansions (3.31) in the exterior domain and within the mixing zone. Following the derivation of the inner limit expansion, the inner terms $\widehat{v}_k^{(1)}, \widehat{\beta}_k^{(2)}, \widehat{\rho}_k^{(2)}, \widehat{p}_k^{(2)}$ are defined as a solution of differential equations. Specifically the fast variables defined as the inner expansion terms minus the common terms satisfy linearized compressible equations. The variables $v_k^{(1,f)}$ and $p_k^{(2,f)}$ solve a hyperbolic IBVP. They are solved by the method of characteristics and a Picard iteration. The remaining equations can thus be viewed as equations for $\beta_k^{(2,f)}$ alone. The solutions are expressed in terms of the initial and the boundary data. The boundary data is provided by matching at the outer and inner edges of regions which exist in order λ^{-m} expansions.

The initial conditions associated with (2.12) are given by

$$\begin{aligned} &\widehat{v}_k^{(1)}(z,0) = v_k^{(1,s)}(z,0) + v_k^{(1)}(z), \quad \widehat{\beta}_k^{(2)}(z,0) = \beta_k^{(2,s)}(z,0), \\ &\widehat{p}_k^{(2)}(z,0) = p_k^\infty(z,0) + p_k^{(2)}(z) \end{aligned} \tag{3.35}$$

in $(-1)^{k'}$¤» $\geq (-1)^{k'} Z_k$, where $v_k^{(1)}(z),\ p_k^{(2)}(z) \in C^1$. We define the fast variables $v_k^{(1,f)}, \beta_k^{(2,f)}, p_k^{(2,f)}$ which are oscillatory part of $\widehat{v}_k^{(1)}, \widehat{\beta}_k^{(2)}$, and $\widehat{p}_k^{(2)}$ as the following

$$\begin{aligned} v_k^{(1,f)}(z,\tau) &\equiv \widehat{v}_k^{(1)} - \left(v_k^{(1,s)}(z,0) + \tau \frac{\partial \beta_k^\infty}{\partial t}(z,0)\right), \\ \beta_k^{(2,f)}(z,\tau) &\equiv \widehat{\beta}_k^{(2)} - \left(\beta_k^{(2,s)}(z,0) + \tau \frac{\partial \beta_k^\infty}{\partial t}(z,0) + \frac{\tau^2}{2}\frac{\partial^2 \beta_k^\infty}{\partial t^2}(z,0)\right), \\ p_k^{(2,f)}(z,\tau) &\equiv \widehat{p}_k^{(2)} - p_k^\infty(z,0). \end{aligned} \tag{3.36}$$

The fast variables $v_k^{(1,f)}$ and $p_k^{(2,f)}$ satisfy coupled PDEs with a linear source term, and coefficients given by solutions of the slow variable expansions. The proof is a combination of the method of characteristics and a Picard iteration. The characteristics for $v_k^{(1,f)}$

and $p_k^{(2,f)}$ are straight lines with known slopes. While the coefficients are known explicitly, the integral along characteristics is no longer elementary, as the equations have two systems of characteristics, each of which carries information to the linear source term. Thus, we solve the equations by a convergence series expansion, but do not have a closed form solution.

3.2.1 Higher Order Inner Terms in the Exterior Domain

In this section, we find the inner terms $\widehat{v}_k^{(1)}$ and $\widehat{p}_k^{(2)}$ in the single phase region $\widehat{\mathcal{E}}_k^{(2)}$. We substitute the asymptotic expansions (3.31) into (3.28)-(3.30) and equate powers of λ. Using (3.33), we isolate the order λ^{-1} terms in the mass equations and the order zero terms in the momentum equation. Since λ is arbitrary, the coefficient of λ must vanish, leading to the equations

$$O(\lambda^{-1}) \quad : \quad \frac{\partial \widehat{\rho}_k^{(2)}}{\partial \tau} + \rho_k^{\infty} \frac{\partial \widehat{v}_k^{(1)}}{\partial z} = 0, \tag{3.37}$$

$$O(\lambda) \quad : \quad \rho_k^{\infty} \widehat{v}_k^{(1)} \frac{\partial \widehat{v}_k^{(1)}}{\partial \tau} + \frac{\partial \widehat{p}_k^{(2)}}{\partial z} = \rho_k^{\infty} g(0), \tag{3.38}$$

for $\widehat{v}_k^{(1)}$, $\widehat{\rho}_k^{(2)}$, $\widehat{p}_k^{(2)}$ in $\widehat{\mathcal{E}}_k^{(2)}$. We note that

$$\widehat{\beta}_k^{(2)} = 0 \quad \text{in } \widehat{\mathcal{E}}_i^{(2)}, \; i = 1, 2. \tag{3.39}$$

Substituting (3.36) into (3.37), (3.38), and multiplying (3.37) by $a_k^2 \equiv dp_k/d\rho_k(\rho_k^{\infty})$ and (3.38) by $1/\rho_k^{\infty}$, the equations reduce to the system of the wave equations,

$$\frac{\partial u_k^{(2,f)}}{\partial \tau} + A_k^0 \frac{\partial u_k^{(2,f)}}{\partial z} = 0 \tag{3.40}$$

for $u_k^{(2,f)} = \left(p_k^{(2,f)}, v_k^{(1,f)}\right)^{tr}$, where A_k^0 is the constant coefficient matrix,

$$A_k^0 = \begin{pmatrix} 0 & \rho_k^{\infty} a_k^2 \\ 1/\rho_k^{\infty} & 0 \end{pmatrix}. \tag{3.41}$$

From (3.35) the initial conditions satisfy

$$u_k^{(2,f)}(z,0) = \left(p_k^{(2)}(z), v_k^{(1)}(z)\right)^{tr} . \tag{3.42}$$

We now discuss the IBVP (3.40)–(3.42) in $\widehat{\mathcal{E}}_k^{(2)}$ using the method of characteristics. The matrix A_k^0 has two distinct eigenvalues $\Lambda_{k,i}^0 \equiv (-1)^i a_k$, $i = 1, 2$. We define Γ_k^0 to be the nonsingular matrix whose column vectors consist of linearly independent eigenvectors of A_k^0,

$$\Gamma_k^0 = \begin{pmatrix} 1/2 & 1/2 \\ -1/(2\rho_k^{\infty} a_k) & 1/(2\rho_k^{\infty} a_k) \end{pmatrix}. \tag{3.43}$$

The characteristic curves $C_{k,i}$, $i = 1, 2$, satisfy

$$C_{k,i} : \frac{dX}{ds} = \Lambda^0_{k,i}. \tag{3.44}$$

Therefore the backward characteristics $C_{k,i}$, $i = 1, 2$, through a point (z, τ) have equations for $0 \leq s \leq \tau$,

$$C_{k,i} : X(s) = \alpha_{k,i}(s, z, \tau) = (-1)^i a_k(s - \tau) + z. \tag{3.45}$$

Introducing a new unknown column vector $U^{(2,f)}_k = \left(U^{(2,f)}_{k,1}, U^{(2,f)}_{k,2}\right)^t \equiv (\Gamma^0_k)^{-1} u^{(2,f)}_k$ and multiplying (3.40) by $(\Gamma^0_k)^{-1}$, we obtain the system

$$\frac{\partial U^{(2,f)}_k}{\partial \tau} + \Lambda^0_k \frac{\partial U^{(2,f)}_k}{\partial z} = 0. \tag{3.46}$$

By the method of characteristics we find the solution

$$\begin{aligned} U^{(2,f)}_{k,k}(z, \tau) &= U^{(2,f)}_{k,k}(\alpha_{k,k}(0, z, \tau), 0) \\ &= \left(p^{(2,f)}_k + (-1)^k \rho^\infty_k a_k v^{(1,f)}_k\right)(z + (-1)^{k'} a_k \tau, 0), \end{aligned} \tag{3.47}$$

$$U^{(2,f)}_{k,k'}(z, \tau) = \begin{cases} U^{(2,f)}_{k,k'}(\alpha_{k,k'}(0, z, \tau), 0) & \text{if } (-1)^{k'} \alpha_{k,k'}(0, z, \tau) \geq \widehat{Z}^{(0)}_{k'} \\ U^{(2,f)}_{k,k'}\left(\overline{\widehat{Z}^{(2)}_{k'}}(\tau^{(2)}_k), \tau^{(2)}_k\right) & \text{if } (-1)^{k'} \alpha_{k,k'}(0, z, \tau) \leq \widehat{Z}^{(0)}_{k'} \end{cases}, \tag{3.48}$$

in $\widehat{\mathcal{E}}^{(2)}_k$, where $\tau^{(2)}_k = \tau^{(2)}_k(z, \tau)$ satisfies $\alpha_{k,k'}(\tau^{(2)}_k, z, \tau) = \overline{\widehat{Z}^{(2)}_{k'}}(\tau^{(2)}_k)$. From (3.47) and (3.48), we note that the solution $U^{(2,f)}_{k,k'}$ depends upon boundary data at the edge $z = \overline{\widehat{Z}^{(2)}_{k'}}(\tau^{(2)}_k)$ to be completely determined in the single phase region $\widehat{\mathcal{E}}^{(2)}_k$ along the outgoing characteristic $\alpha_{k,k'}$ while $U^{(2,f)}_{k,k}$ is determined by initial data along the incoming characteristic $\alpha_{k,k}$.

Furthermore, the solution $U^{(2,f)}_{k,k}$ belongs to C^1 in $\widehat{\mathcal{E}}^{(2)}_k$ under the C^1 initial conditions and it provides boundary data $U^{(2,f)}_{k,k}\left(\overline{\widehat{Z}^{(2)}_{k'}}, \tau\right) \in C^1$ which will be used to solve the related fast variable system in $\widehat{\mathcal{M}}$ in Sec. 4.2.3 once the variables $u^{(2,f)}_k$ are determined in the fast transitional regions $\widehat{\mathcal{T}}^{(2)}_i$, $\widehat{\mathcal{T}}^{(1)}_i$, $i = k, k'$.

In contrast the solution $U^{(2,f)}_{k,k'}$ depends upon boundary data at the edge $z = \overline{\widehat{Z}^{(2)}_{k'}}$ to be completely determined in the single phase region $\widehat{\mathcal{E}}^{(2)}_k$ along the outgoing characteristic $\alpha_{k,k'}$. We assume the boundary data

$$U^{(2,f)}_{k,k'}\left(\overline{\widehat{Z}^{(1)}_{k'}}, \tau\right) = \left(p^{(2,f)}_k + (-1)^k \rho^\infty_k a_k v^{(1,f)}_k\right)\left(\overline{\widehat{Z}^{(2)}_{k'}}, \tau\right) \tag{3.49}$$

which will be found later by matching in [11].

Proposition 3.2. Assume (2.9), (2.15), (2.19), (2.22), and the initial data (2.12). Assume the boundary data (3.49). Then the solution $U^{(2,f)}$ of (3.46) is uniquely determined by (3.47) and (3.48) in the exterior domain $\widehat{\mathcal{E}}_k^{(2)}$. Thus, the inner terms $\widehat{v}_k^{(1)}$ and $p_k^{(2)}$ are also found uniquely in $\widehat{\mathcal{E}}_k^{(2)}$.

3.2.2 Higher Order Inner Terms within the Mixing Zone

In this section we find the inner terms $\widehat{v}_k^{(1)}, \widehat{\beta}_k^{(2)}, \widehat{\rho}_k^{(2)}, \widehat{p}_k^{(2)}$ in the mixing zone $\widehat{\mathcal{M}}$. Using (3.33), we isolate the order λ^{-1} terms in the interface and continuity equation and the order zero terms in the momentum equation, defining the equations

$$O(\lambda^{-1}) : \frac{\partial \widehat{\beta}_k^{(2)}}{\partial \tau} + \widehat{v}^{*(0)} \frac{\partial \widehat{\beta}_k^{(1)}}{\partial z} + \widehat{v}^{*(1)} \frac{\partial \widehat{\beta}_k^{(0)}}{\partial z} = 0, \tag{3.50}$$

$$\begin{aligned} O(\lambda^{-1}) : \widehat{\beta}_k^{(0)} \frac{\partial \widehat{\rho}_k^{(2)}}{\partial \tau} &+ \widehat{\beta}_k^{(0)} \widehat{\rho}_k^{(0)} \frac{\partial \widehat{v}_k^{(1)}}{\partial z} + \widehat{\beta}_k^{(1)} \widehat{\rho}_k^{(0)} \frac{\partial \widehat{v}_k^{(0)}}{\partial z} \\ &+ \widehat{\rho}_k^{(0)} \widehat{\mu}_k^{v(0)} \left(\widehat{v}_k^{(1)} - \widehat{v}_{k'}^{(1)} \right) \frac{\partial \widehat{\beta}_k^{(0)}}{\partial z} + \widehat{\rho}_k^{(0)} \widehat{\mu}_k^{v(0)} \left(\widehat{v}_k^{(0)} - \widehat{v}_{k'}^{(0)} \right) \frac{\partial \widehat{\beta}_k^{(1)}}{\partial z} \\ &+ \widehat{\rho}_k^{(0)} \widehat{\mu}_k^{v(1)} \left(\widehat{v}_k^{(0)} - \widehat{v}_{k'}^{(0)} \right) \frac{\partial \widehat{\beta}_k^{(0)}}{\partial z} = 0, \end{aligned} \tag{3.51}$$

$$\begin{aligned} O(1) : \widehat{\beta}_k^{(0)} \widehat{\rho}_k^{(0)} \frac{\partial \widehat{v}_k^{(1)}}{\partial \tau} &+ \widehat{\beta}_k^{(0)} \widehat{\rho}_k^{(0)} \widehat{v}_k^{(0)} \frac{\partial \widehat{v}_k^{(0)}}{\partial z} + \widehat{\beta}_k^{(0)} \frac{\partial \widehat{p}_k^{(2)}}{\partial z} \\ &+ \widehat{\mu}_k^{p(0)} \left(\widehat{p}_k^{(2)} - \widehat{p}_{k'}^{(2)} \right) \frac{\partial \widehat{\beta}_k^{(0)}}{\partial z} = \widehat{\beta}_k^{(0)} \widehat{\rho}_k^{(0)} g(0) \end{aligned} \tag{3.52}$$

for $\widehat{v}_k^{(1)}, \widehat{\beta}_k^{(2)}, \widehat{\rho}_k^{(2)}, \widehat{p}_k^{(2)}$ in $\widehat{\mathcal{M}}$.

Substituting (3.36) into (3.50)-(3.52) one calculation shows the system

$$\frac{\partial \beta_k^{(2,f)}}{\partial \tau} + v^{*(1,f)} \frac{\partial \widehat{\beta}_k^{(0)}}{\partial z} = 0, \tag{3.53}$$

$$\frac{\partial p_k^{(2,f)}}{\partial \tau} + \rho_k^{\infty} a_k^2 \frac{\partial v_k^{(1,f)}}{\partial z} + \rho_k^{\infty} a_k^2 f_k^v(z) \left(v_k^{(1,f)} - v_{k'}^{(1,f)} \right) + \rho_k^{\infty} a_k^2 g_k^v(z) d_k^{v(1,f)}(\tau) = 0, \tag{3.54}$$

$$\frac{\partial v_k^{(1,f)}}{\partial \tau} + \frac{1}{\rho_k^{\infty}} \frac{\partial p_k^{(2,f)}}{\partial z} + \frac{f_k^p(z)}{\rho_k^{\infty}} \left(p_k^{(2,f)} - p_{k'}^{(2,f)} \right) = 0 \tag{3.55}$$

for $v_k^{(1,f)}$, $\beta_k^{(2,f)}$ and $p_k^{(2,f)}$. Here $a_k^2 = c^2(\rho_k^\infty)$ and for $q = v, p$,

$$f_k^q(z) \equiv \frac{\widehat{\mu}_k^{q(0)}}{\widehat{\beta}_k^{(0)}}\frac{\partial \widehat{\beta}_k^{(0)}}{\partial z} = \left(\frac{\mu_k^{q\infty}}{\beta_k^\infty}\frac{\partial \beta_k^\infty}{\partial z}\right)(z,0) = -d_{k'}^{q\infty}(0) f_{k'}^q(z), \tag{3.56}$$

$$g_k^v(z) \equiv \frac{\partial \widehat{\mu}_k^{v(0)}}{\partial \widehat{d}_k^{v(0)}}\frac{\widehat{v}_k^{(0)} - \widehat{v}_{k'}^{(0)}}{\widehat{\beta}_k^{(0)}}\frac{\partial \widehat{\beta}_k^{(0)}}{\partial z} = -\left[\mu_k^{v\infty}\mu_{k'}^{v\infty} d_{k'}^{v\infty}\frac{v_k^\infty - v_{k'}^\infty}{\beta_k^\infty}\frac{\partial \beta_k^\infty}{\partial z}\right](z,0). \tag{3.57}$$

Observe that $v_k^{(1,f)}$ and $p_k^{(2,f)}$ satisfy a subsystem of hyperbolic PDEs which are linearized compressible equations while $\beta_k^{(2,f)}$ depends on the solution $v_k^{(1,f)}$. The initial condition (2.12) implies the initial data

$$\beta_k^{(2,f)}(z,0) = 0\ ,\ \ v_k^{(1,f)}(z,0) = v^{(1)}(z),\ \ p_k^{(2,f)}(z,0) = p_k^{(2)}(z) \in C^1 \tag{3.58}$$

for the fast variables. In this section, we assume the boundary data

$$\left(\widehat{p}_k^{(2)} + (-1)^k \rho_k^\infty a_k \widehat{v}_k^{(1)}\right)\left(\widehat{Z}_{k'}^{(0)}, \tau\right) = \left(\widehat{p}_k^{(2)} + (-1)^k \rho_k^\infty a_k \widehat{v}_k^{(1)}\right)\left(\overline{\widehat{Z}_{k'}^{(2)}}, \tau\right), \tag{3.59}$$

$$\widehat{v}_k^{(1)}\left(\widehat{Z}_k^{(0)}, \tau\right) = \widehat{v}_k^{(1)}\left(\overline{\widehat{Z}_k^{(2)}}, \tau\right) = \widehat{V}_k^{(1)}. \tag{3.60}$$

These data will be shown later by matching in [11]. We see from (2.19) and (3.36) that the conditions (3.59), (3.60) are equivalent to the boundary data

$$\left(p_k^{(2,f)} + (-1)^k \rho_k^\infty a_k v_k^{(1,f)}\right)\left(\widehat{Z}_{k'}^{(0)}, \tau\right) = \left(p_k^{(2,f)} + (-1)^k \rho_k^\infty a_k v_k^{(1,f)}\right)\left(\overline{\widehat{Z}_{k'}^{(2)}}, \tau\right), \tag{3.61}$$

$$v_k^{(1,f)}\left(\widehat{Z}_k^{(0)}, \tau\right) = v_k^{(1,f)}\left(\overline{\widehat{Z}_k^{(2)}}, \tau\right) = V_k^{(1,f)}. \tag{3.62}$$

Let us first solve the IBVP (3.54), (3.55), (3.58), (3.61), (3.62) for $p_k^{(2,f)}$ and $v_k^{(1,f)}$ in $\widehat{\mathcal{M}}$ by using the method of characteristics and by applying a Picard iteration. We can write (3.54), (3.55) for unknown column vector $u^{(2,f)} = \left(p_1^{(2,f)}, v_1^{(1,f)}, p_2^{(2,f)}, v_2^{(1,f)}\right)^t r$ in the form

$$\frac{\partial u^{(2,f)}}{\partial \tau} + A^0 \frac{\partial u^{(2,f)}}{\partial z} + B^0(z) u^{(2,f)} + G(z,\tau) = 0 \tag{3.63}$$

where A_k^0 was defined in (3.41),

$$A^0 = \begin{pmatrix} A_1^0 & 0 \\ 0 & A_2^0 \end{pmatrix},\ B^0(z) = \begin{pmatrix} B_1^0 & -B_1^0 \\ -B_2^0 & B_2^0 \end{pmatrix},\ B_k^0(z) = \begin{pmatrix} 0 & \rho_k^\infty a_k^2 f_k^v(z) \\ -f_k^p(z)/\rho_k^\infty & 0 \end{pmatrix} \tag{3.64}$$

and the source term is

$$G = \begin{pmatrix} \rho_1^\infty a_1^2 g_1^v d_1^{v(1,f)} \\ 0 \\ \rho_2^\infty a_2^2 g_2^v d_2^{v(1,f)} \\ 0 \end{pmatrix}. \tag{3.65}$$

The constant coefficient matrix A^0 has four distinct eigenvalues $\pm a_1$, $\pm a_2$. The nonsingular matrix Γ^0 whose column vectors consist of linearly independent eigenvector of A^0 has the form

$$\Gamma^0 = \begin{pmatrix} \Gamma_1^0 & 0 \\ 0 & \Gamma_2^0 \end{pmatrix}, \tag{3.66}$$

where Γ_k^0 was given in (3.43) and it satisfies

$$A^0\Gamma^0 = \Gamma^0\Lambda^0, \quad \Lambda^0 = \begin{pmatrix} \Lambda_1^0 & 0 \\ 0 & \Lambda_2^0 \end{pmatrix} \tag{3.67}$$

with the diagonal matrix Λ_k^0 defined in (3.46), whose diagonal entries are eigenvalues of A^0. The characteristics curves $C_{k,i}$, $i = 1, 2$, satisfy (3.44) and therefore, the backward characteristics $C_{k,i}$, $i = 1, 2$ through a point (z, τ) have the equations (3.45). We note that the semilinear hyperbolic system (3.63) has the same characteristics as the system discussed in [10]. We introduce a new unknown vector $U^{(2,f)} \equiv (\Gamma^0)^{-1}u^{(2,f)}$. From (3.63) we find that $U^{(1,f)}$ satisfies hyperbolic IBVP

$$\frac{\partial U^{(2,f)}}{\partial \tau} + \Lambda^0 \frac{\partial U^{(2,f)}}{\partial z} + \overline{B}^{(0)}(z)U^{(2,f)} + (\Gamma^0)^{-1}G = 0 \tag{3.68}$$

with the initial condition

$$U^{(2,f)}(z, 0) = (\Gamma^0)^{-1}u^{(2,f)}(z, 0). \tag{3.69}$$

and the boundary data

$$U_1^{(2,f)}(\widehat{Z}_2^{(0)}, \tau) = U_{1,1}^{(2,f)}(\overline{\widehat{Z}_2^{(2)}}, \tau), \tag{3.70}$$

$$U_2^{(2,f)}(\widehat{Z}_1^{(0)}, \tau) = \left(U_1^{(2,f)} + 2\rho_1^\infty a_1 v_1^{(1,f)}\right)(\widehat{Z}_1^{(0)}, \tau) = U_1^{(2,f)}(\widehat{Z}_1^{(0)}, \tau) + 2\rho_1^\infty a_1 V_1^{(1,f)}, \tag{3.71}$$

$$U_3^{(2,f)}(\widehat{Z}_2^{(0)}, \tau) = \left(U_4^{(2,f)} - 2\rho_2^\infty a_2 v_2^{(0,f)}\right)(\widehat{Z}_2^{(0)}, \tau) = U_4^{(2,f)}(\widehat{Z}_2^{(0)}, \tau) - 2\rho_2^\infty a_2 V_2^{(1,f)}, \tag{3.72}$$

$$U_4^{(2,f)}(\widehat{Z}_1^{(0)}, \tau) = U_{2,2}^{(2,f)}(\overline{\widehat{Z}_1^{(2)}}, \tau) \tag{3.73}$$

in $\widehat{\mathcal{M}}$. Here $\overline{B}^{(0)} = (\Gamma^0)^{-1}B^0\Gamma^0$, $U_j^{(2,f)}$ denotes the $j-$th component of the vector $U^{(2,f)}$ and $U_{k,k}^{(2,f)}$ was given in (3.47).

By the method of characteristics, we find the implicit solution

$$U_{k+l}^{(2,f)}(z,\tau)=\begin{cases} U_{k+l}^{(2,f)}(\alpha_{k,i}(0,z,\tau),0)-\rho_k^{\infty}a_k^2\int_0^{\tau} g_k^v(\alpha_{k,i}(s,z,\tau))d_k^{v(1,f)}(s)ds \\ -\int_0^{\tau}\sum_{j=1}^{4}\overline{B}_{k+l,j}^0(\alpha_{k,i}(s,z,\tau))U_j^{(2,f)}(\alpha_{k,i}(s,z,\tau),s)ds \\ \qquad \text{if } (-1)^i\alpha_{k,i}(0,z,\tau)\geq(-1)^i\widehat{Z}_{i'}^{(0)} \\ U_{k+l}^{(2,f)}\left(\widehat{Z}_{i'}^{(0)}(\overline{\tau}_i^{(k)}),\overline{\tau}_i^{(k)}\right)-\rho_k^{\infty}a_k^2\int_{\overline{\tau}_i^{(k)}}^{\tau} g_k^v(\alpha_{k,i}(s,z,\tau))d_k^{v(1,f)}(s)ds \\ -\int_{\overline{\tau}_i^{(k)}}^{\tau}\sum_{j=1}^{4}\overline{B}_{k+l,j}^0(\alpha_{k,i}(s,z,\tau))U_j^{(2,f)}(\alpha_{k,i}(s,z,\tau),s)ds \\ \qquad \text{if } (-1)^i\alpha_{k,i}(0,z,\tau)\leq(-1)^i\widehat{Z}_{i'}^{(0)} \end{cases}, \tag{3.74}$$

where $l=[k/2],[k/2]+1$, $i=k,k'$ and $i'=3-i$. Here $\overline{B}_{i,j}^0$ denotes the $(i,\ j)$ component of the matrix $\overline{B}^0$ and the time $\overline{\tau}_i^{(k)}$ satisfies $\alpha_{k,i}(\overline{\tau}_i^{(k)},z,\tau)=\widehat{Z}_{i'}^{(0)}(\overline{\tau}_i^{(k)})$. Thus the solution $U_1^{(2,f)}$ and $U_4^{(2,f)}$ is completely determined by the initial data (3.69) and the boundary data (3.70), (3.73). Furthermore, they provide the boundary data $U_1^{(2,f)}(\widehat{Z}_1^{(0)},\tau)$ and $U_4^{(2,f)}(\widehat{Z}_2^{(0)},\tau)$ to be used in the evaluation of $U_2^{(2,f)}$ and $U_3^{(2,f)}$. Since the initial data $v_k^{(1,f)}(z,0)$, $p_k^{(2,f)}(z,0)$ are assumed to be C^1 in $(-1)^k z\leq(-1)^k Z_k(0)$, the boundary data $U_1^{(2,f)}(\widehat{Z}_2^{(0)},\tau)$ and $U_4^{(2,f)}(\widehat{Z}_1^{(0)},\tau)$ belong to $C^1([0,\infty))$ and they satisfy the compatibility conditions.

We write (3.74) symbolically as

$$U^{(2,f)}=\omega^0+SU^{(2,f)}, \tag{3.75}$$

where $\omega^0(z,\tau)$ is defined as the vector solution of the linear system

$$\frac{\partial\omega^0}{\partial\tau}+\Lambda^0\frac{\partial\omega^0}{\partial z}+(\Gamma^0)^{-1}G=0 \tag{3.76}$$

with the same initial and boundary data as the $U^{(2,f)}$ and S is the integral operator taking a vector $U^{(2,f)}$ into a vector $SU^{(2,f)}$. Given sufficient regularity of the data, the mapping $S:C^1\to C^1$ is continuous in C^1, using the maximum norm $|||U^{(2,f)}|||$ in $\widehat{\mathcal{M}}$. To find the explicit solution $U^{(2,f)}$ of the integral equations (3.74), we use the process of a Picard iteration

$$U^{(2,f)[m+1]} = SU^{(2,f)[m]}, m\geq 0, \tag{3.77}$$

$$U^{(2,f)[0]} = \omega^0. \tag{3.78}$$

to show that the series $\sum_{m=0}^{\infty}U^{(2,f)[m]}$ of vectors converges to the solution $U^{(2,f)}$. Refer to [17].

Lemma 3.3. Let $\eta^{[m]}(\tau)$, $m = 0, 1, \cdots$, denote a sequence of nonnegative continuous functions which satisfy the inequalities

$$\eta^{[m+1]}(\tau) \leq c \int_0^{\tau} \eta^{[m]}(s)ds, \; 0 \leq \tau \leq \overline{\tau} \tag{3.79}$$

with a nonnegative constant c. Then it satisfies

$$\eta^{[m]}(\tau) \leq \frac{c^m \tau^m}{m!} \max_{0 \leq s \leq \tau} \eta^{[0]}(s) \tag{3.80}$$

for $0 \leq \tau \leq \overline{\tau}$ and $m = 0, 1, \cdots$. In particular, the sequence $\eta^{[m]}(\tau)$, $0 \leq \tau \leq \overline{\tau}$, is uniformly bounded. If a interval of length τ has magnitude $\overline{\tau} = O(1)$, then the series $\sum_{m=0}^{\infty} \eta^{[m]}(\tau)$ converges in $0 \leq \tau \leq \overline{\tau}$.

Proof. For $m = 0$ the estimate is true. Assume that it is valid up to the index m. Then

$$\eta^{[m+1]}(\tau) \leq c \int_0^{\tau} \frac{c^m \tau^m}{m!} ds \max_{0 \leq s \leq \tau} \eta^{[0]}(s) = \frac{c^{m+1} \tau^{m+1}}{(m+1)!} \max_{0 \leq s \leq \tau} \eta^{[0]}(s) \tag{3.81}$$

and the lemma is proved. ■

If a time interval of length τ has magnitude $\tau = O(\lambda)$, convergence of the series $\sum_{m=0}^{\infty} \eta^{[m]}(\tau)$ is not guaranteed by (3.80). In our case, the integral ranges in S are not greater than $\dfrac{\widehat{Z}_2^{(0)} - \widehat{Z}_1^{(0)}}{a_k} = O(1)$. Thus convergence of (3.77) is obtained as the following

Proposition 3.4. Assume (2.9), (2.15), (2.19), (2.22), and the initial data (2.12). Assume the boundary data (3.59) and (3.60). Suppose that the initial function be $U^{(2,f)}(z, 0)$ belongs to C^1 in $(-1)^k z \leq (-1)^k Z_k(0)$. Then (3.74) has a unique solution $U^{(2,f)} \in C^1$ in $\widehat{\mathcal{M}}$ and the solution $U^{(2,f)}$ satisfies (3.68).

Proof. We set

$$\eta^{[m]}(\tau) = \max \left\{ \left| U_j^{(2,f)[m]}(z, s) \right| : \widehat{Z}_1^{(0)} \leq z \leq \widehat{Z}_2^{(0)}, \; 0 \leq s \leq \tau, \; j = 1, 2, 3, 4 \right\}. \tag{3.82}$$

As an application of the Picard Lemma to (3.82), we obtain

$$\left\| U^{(2,f)[m]}(\tau) \right\| \leq \frac{c^m \overline{\tau}^m}{m!} \left\| U^{(2,f)[0]}(\tau) \right\|, \tag{3.83}$$

$$\left\| S^m \right\| \leq \frac{c^m \overline{\tau}^m}{m!} \tag{3.84}$$

in $0 < \tau < \infty$. Here

$$\overline{\tau} \equiv \max_{k} \left\{ \frac{\widehat{Z}_2^{(0)} - \widehat{Z}_1^{(0)}}{a_k} \right\} = O(1) \tag{3.85}$$

is the maximum integral range in S and a positive constant c is defined as the maximum norm of the matrix $\overline{B}^0$,

$$c \equiv \left\| \overline{B}^0 \right\| = \max_{k=1,2} \left\{ a_k + \frac{\rho_k^\infty a_k^2}{\rho_{k'}^\infty a_{k'}} \right\} \max \left\{ d_k^{v\infty}(0), d_k^{p\infty} \right\} \left\| \frac{\partial \beta_1^\infty}{\partial z}(z, 0) \right\|. \tag{3.86}$$

It follows from (3.84) that $I - S$ is invertible,

$$(I - S)^{-1} = \sum_{m=0}^{\infty} S^m, \tag{3.87}$$

so (3.74) has a unique solution

$$U^{(2,f)} = (I - S)^{-1} \omega^0 = \sum_{m=0}^{\infty} S^m \omega^0 \tag{3.88}$$

in $\widehat{\mathfrak{M}}$. Since a solution of the hyperbolic system (3.68) is a solution of the integral equation (3.75), it follows that the former has at most one solution. We still need to show that every solution of the integral equation (3.75) is a solution of the differential equations (3.68). It is not obvious that the solution $U^{(2,f)}$ of (3.75) has continuous partial derivatives. Since the source term g_k^v in the vector ω^0 and the coefficients f_k^q in the operator S are not differentiable, the derivation of $\dfrac{\partial U^{(2,f)}}{\partial \tau}$ and $\dfrac{\partial U^{(2,f)}}{\partial z}$ from (3.75) must use integration by parts. We show the existence of $\dfrac{\partial U^{(2,f)}}{\partial \tau}$ instead of $\dfrac{\partial U^{(2,f)}}{\partial z}$ because of the simplicity of calculation in this case. From (3.74), we calculate $\tau-$derivative of $U^{(2,f)}$ and we can write it symbolically as

$$\frac{\partial U^{(2,f)}}{\partial \tau} = \frac{\partial \omega^0}{\partial \tau} + S \frac{\partial U^{(2,f)}}{\partial \tau}, \tag{3.89}$$

where S is the same integral operator as in (3.75). Using Lemma 3.3 to the iteration method

$$\left(\frac{\partial U^{(2,f)}}{\partial \tau} \right)^{[m+1]} = S \left(\frac{\partial U^{(2,f)}}{\partial \tau} \right)^{[m]}, \; \left(\frac{\partial U^{(2,f)}}{\partial \tau} \right)^{[0]} = \frac{\partial \omega^0}{\partial \tau}, \; m \geq 0, \tag{3.90}$$

we obtain a unique solution of (3.89),

$$\frac{\partial U^{(2,f)}}{\partial \tau} = (I - S)^{-1} \frac{\partial \omega^0}{\partial \tau} = \sum_{m=0}^{\infty} S^m \frac{\partial \omega^0}{\partial \tau} \tag{3.91}$$

in $0 < \tau < \infty$. Obviously, $SU^{(2,f)}$ can be differentiated with respect to z as well. Thus convergence of the $\sum_{m=0}^{\infty} U^{(2,f)[m]}$ and $\sum_{m=0}^{\infty} U_{\tau}^{(2,f)[m]}$ implies convergence of $\sum_{m=0}^{\infty} U_{z}^{(2,f)[m]}$. In conclusion, Eq. (3.75) has a unique solution in C^1, provided $\omega^0 \in C^1$ and then the solution satisfies the hyperbolic system (3.68). ■

Using the $v_k^{(1,f)}$ determined above, the solution $\beta_k^{(2,f)}$ of Eq. (3.53) is found as follows

Proposition 3.5. Assume (2.9), (2.15), (2.19), (2.22), and the initial data (2.12). Assume the boundary data (3.59) and (3.60). We obtain the fast variable

$$\beta_k^{(2,f)} = \left\{ \sum_{k=1}^{2} \left[\widehat{\mu}_k^{v(0)} \int_0^{\tau} v_{k'}^{(1,f)}(z,s)ds \right] + \frac{\partial \widehat{\mu}_k^{v(0)}}{\partial \widehat{d}_k^{v(0)}} \left(\widehat{v}_k^{(0)} - \widehat{v}_{k'}^{(0)} \right) \right\} \frac{\partial \widehat{\beta}_{k'}^{(0)}}{\partial z}. \quad (3.92)$$

We remark that if $\widehat{\beta}_k^{(0)} = \beta_k^{\infty}(z,0)$ belongs to C^1 in $\widehat{\mathcal{M}}$, the solution $\beta_k^{(2,f)}$ in (3.92) is not differentiable with respect to z. The continuous $z-$derivative of the $\beta_k^{(2,f)}$ in $\widehat{\mathcal{M}}$ requires the initial data $\beta_k^{\infty}(z,0) \in C^2$ in $\widehat{\mathcal{M}}$.

In conclusion, the fast variables found in Propositions 3.1-3.3 determine the inner terms $\widehat{v}_k^{(1)}$, $\widehat{\beta}_k^{(2)}$, $\widehat{\rho}_k^{(2)}$ and $\widehat{p}_k^{(2)}$ by (3.36). Thus fast scale acoustical oscillation can first appear in second order of the asymptotic expansions of β_k, ρ_k and p_k and in first order of the v_k expansion depending on the initial data.

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