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On Weighted (0,2) –Interpolation

[1]Swarnima Bahadur and [2]Sariya Bano

Department of Mathematics and Astronomy, University of Lucknow, Lucknow 226007, India.

Abstract

In this paper, we consider weighted (0,2) -interpolation on the nodes, which are obtained by projecting vertically the zeros of the $(1-x^2)P_n'(x)$, on the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial. We obtain the explicit forms and establish a convergence theorem for that interpolatory polynomial.

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1. INTRODUCTION

P.Turán [8] was first, who initiated the problem of Lacunary interpolation on zeros of $\prod_n(x)=(1-x^2)P_{n-1}'(x)$, where $P_{n-1}(x)$ is the Legendre polynomial of degree $(n-1)$.The problem of (0,2) interpolation on the roots of unity was first studied by O. Kiš [4]. He obtained its regularity, fundamental polynomials and established a convergence theorem for the same

Later on Sharma and his associates [7] considered the convergence of $(0, m_1, m_2, \dots\dots.., m_q)$ interpolation on the unit circle. In 1981, S.D. Riemenschnider and A. Sharma [7] considered the general Lacunary interpolation on the unit circle. In 1996, Siqing Xie [10] considered (0,1,3) interpolation on the nodes, which are vertically projected on the zeros of $(1-x^2)P_n(x)$ onto the unit circle, where $P_n(x)$stands for n^{th} Legendre polynomial, having the zeros $x_k = cos\theta_k$, such that

$1 > x_1 > x_2 > \cdots \ldots\ldots\ldots\ldots\ldots\ldots.. > x_n > -1$. He claimed the regularity, explicit representation and convergence of (0,1,3) –interpolation. In 2004, W.Chen and A. Sharma [3] considered the regularity of $(0,m)$ interpolation on the zeros of $(z^{2n}+1)(z^2-1)$ and of $(z^{2n}+1)(z^n-1)$, which are non- uniformly distributed on the unit circle.

In 2011, S. Bahadur and K.K. Mathur [1] considered the weighted (0,2)*-interpolation on the set of nodes considered by [10] and established a convergence theorem for the same. Later on S. Bahadur and M.Shukla [2] considered (0,2)-interpolation on the nodes, which are obtained by projecting vertically the zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$ on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial, obtained the explicit forms and establish a convergence theorem for the same. This has motivated us to consider weighted (0,2) –interpolation on some set of nodes on unit circle different from above.

In this paper, we consider weighted (0,2) –interpolation on non- uniformly distributed zeros on the unit circle,which are obtained by projecting vertically the zeros of $(1-x^2)P_n'(x)$ onto the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial . We obtain the explicit forms of the interpolatory polynomials and establish a convergence theorem for the same. In section 2 we give some preliminaries and in section 3, we describe the problem and its existence. In section 4, we give the explicit formulae of the interpolatory polynomials. In section 5 and 6, estimation and convergence of interpolatory polynomials are given respectively.

2. PRELIMINARIES:

In this section, we shall give some well-known results, which we shall use.

The differential equation satisfied by $P_n(x)$is

(2.1)$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$

(2.2)$W(z) = \prod_{k=1}^{2n-2}(z-z_k) = K_n P_n'\left(\frac{1+z^2}{2z}\right) z^{n-1}$

(2.3) $R(z) = (z^2-1)\, W(z)$

We shall require the fundamental polynomial of Lagrange interpolation based on the zeros of $R(z)$ and W(z) are respectively given as:

(2.4) $L_k(z) = \frac{R(z)}{R'(z_k)(z-z_k)}$, $\quad k = 0(1)2n-1$

(2.5) $L_{1k}(z) = \frac{W(z)}{W'(z_k)(z-z_k)}$, $\quad k = 1(1)2n-2$

We will also use the following results

(2.6) $$\begin{cases} W'(z_k) = \dfrac{K_n}{2}(z_k^2 - 1)z_k^{n-3}P_n''(x_k), & k = 0(1)n-1 \\ W'(z_{n+k}) = \dfrac{K_n}{2}(z_{n+k}^2 - 1)z_{n+k}^{n-3}P_n''(x_k), & k = 0(1)n-1 \end{cases}$$

(2.7) $$\begin{cases} W''(z_k) = K_n[(n-3)(z_k^2 - 1) - 3]z_k^{n-4}P_n''(x_k) \\ W''(z_{n+k}) = K_n[(n-3)(z_{n+k}^2 - 1) - 3]z_{n+k}^{n-4}P_n''(x_k) \end{cases}$$

(2.8) $\mathrm{R}'(\mathrm{z_k}) = (\mathrm{z_k^2} - 1)\mathrm{W}'(\mathrm{z_k})$

(2.9) $R''(z_k) = 4z_kW'(z_k) + (z_k^2 - 1)W''(z_k)$

We will also use the following well known inequalities

(2.10) $(1 - x^2)|P_n'(x)| \sim n^{\frac{1}{2}}$

(2.11) $(1 - x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2}$

(2.12) $|P_n''(x_k)| \sim k^{\frac{-5}{2}}\ n^4$

(2.13) $J_{ij}(z) = \int_0^z t^{n-1+j}\, W(t)\, dt \qquad , \qquad j = 0,1$

(2.14) $J_{ij}(-1) = (-1)^{n+j}J_{ij}(1)$

For more details one can see [9] .

3. THE PROBLEM AND REGULARITY:

Let $Z_n = \{z_k : k = 0(1)2n-1\}$ satisfying

(3.1) $$Z_n = \left\{ \begin{matrix} z_0 = 1, \quad z_{2n-1} = -1, \\ z_k = cos\theta_k + i\ sin\theta_k, \qquad z_{n+k} = \overline{z_k}\ ,\ k = 1(1)n-1 \end{matrix} \right\}$$

where, $\{x_k = cos\theta_k\ : k = 1(1)n-1\}$ are the zeros of $P_n'(x)$, where $P_n(x)$ stands for n^{th} Legendre polynomial such that

$$1 > x_1 > x_2 > \cdots \ldots \ldots \ldots \ldots \ldots \ldots .. > x_{n-1} > -1.$$

Here we are interested in determine the interpolatory polynomial $Q_n(z)$ of degree $\leq 4n-1$ satisfying the following conditions:

For $k = 0(1)2n-1$,

(3.2) $\begin{cases} Q_n(z_k) & = & \alpha_k \ , \\ [p(z)Q_n(z)]''_{z=z_k} & = & \beta_k \ , \end{cases}$

where α_k and β_k are arbitrary complex constants and weight function $p(z) = \sqrt{(z^2-1)}$. We establish a convergence theorem for the same.

Theorem 3.1: Weighted (0,2)-interpolation is regular on Z_n.

Proof: It is sufficient,if we show the unique solution of (3.2) is $Q_n(z) \equiv 0$, when all data $\alpha_k = \beta_k = 0$.

In this case, we have

$$Q_n(z) = R(z)q(z)$$

where$q(z)$ is polynomial of degree $\leq 2n-1$.

Obviously, $Q_n(z_k) = 0$.

By $\left[(z^2-1)^{1/2}Q_n(z)\right]''_{z=z_k} = 0$, and using (2.3)-(2.8), we obtain

$z_k(z_k^2-1)q'(z_k) + n(z_k^2-1)\,q(z_k) = 0$,k= $0(1)2n-1$

Therefore ,we have

(3.3) $z(z^2-1)q'(z) + n(z^2-1)\,q(z) = (a\,z+b)R(z)$

where a and b are constants. Integrating (3.3), we get

(3.4) $z^n q(z) = a\,J_{11}(z) + bJ_{10}(z) + c$

where,

$$J_{ij}(z) = \int_0^z t^{n-1+j}\,W(t)\,dt\,, \qquad j = 0,1$$

Putting $z = 0$, in (3.4), we have $c = 0$.

Now for $z = 1\ \&\ -1$, we get

$$a\,J_{11}(1) + bJ_{10}(1) = 0$$

$$a\,J_{11}(-1) + bJ_{10}(-1) = 0$$

Using (2.14), we get $a = b = 0$.

Therefore, $Q_n(z) \equiv 0$

Hence the theorem follows.

4. EXPLICIT REPRESENTATION OF INTERPOLATORY POLYNOMIALS:

We shall write $Q_n(z)$ satisfying (3.2) as:

(4.1) $$Q_n(z) = \sum_{k=0}^{2n-1} \alpha_k A_k(z) \quad + \sum_{k=0}^{2n-1} \beta_k B_k(z)$$

where $A_k(z)$ and $B_k(z)$ are unique polynomial, each of degree at most $4n-1$ satisfying the conditions :

For $j,k = 0(1)2n-1$

(4.2) $$\begin{cases} A_k(z_j) & = \quad \delta_{jk} , \\ \left[(z^2-1)^{1/2} A_k(z)\right]''_{z=z_j} & = \quad 0 , \end{cases}$$

(4.3) $$\begin{cases} B_k(z_j) & = \quad 0 , \\ \left[(z^2-1)^{1/2} B_k(z)\right]''_{z=z_j} & = \quad \delta_{jk} , \end{cases}$$

Theorem 4.1: For $k = 0(1)2n-1$, we have

(4.4) $$B_k(z) = b_k z^{-n} R(z) J_k(z)$$

where,

(4.5) $$J_k(z) = \int_0^z t^{n-1} L_k(t)\, dt$$

(4.6) $$b_k = \frac{z_k}{2(z_k^2-1)^{1/2} R'(z_k)}$$

Proof: Obviously $B_k(z_j) = 0$, for each j and k.

Also $\left[(z^2-1)^{1/2} B_k(z)\right]''_{z=z_j} = 0$, for $j \neq k$

For $j = k$, we get (4.6).

This proves the theorem.

Theorem 4.2: For $k = 0(1)2n-1$, we have

(4.7) $$A_k(z) = L_k^2(z) + z^{-n} S_k(z) R(z) + a_k B_k(z) ,$$

where,

(4.8) $S_k(z) = -\frac{1}{R'(z_k)}\int_0^z t^n \frac{[L_k'(t)-L_k'(z_k)L_k(t)]}{(t-z_k)}\, dt$

(4.9) $a_k = 4(z_k^2-1)^{1/2}L_k'(z_k)[z_k + 4(z_k^2-1)L_k'(z_k)]$

Proof: One can check that $A_k(z_j) = \delta_{jk},\ j,k = 0(1)2n-1$

Further from $\left[(z^2-1)^{1/2}A_k(z)\right]''_{z=z_j} = 0$, for j ≠ k ,

we get

(4.10) $S_k'(z_j) = -\frac{z_j^n L_k'(z_j)}{R'(z_k)(z_j-z_k)}$

owing to (4.2).

Therefore, we have

(4.11) $S_k'(z) = -\frac{1}{R'(z_k)} z^n \frac{[L_k'(z)-L_k'(z_k)L_k(z)]}{(z-z_k)}$

integrating (4.11) ,we get(4.8).

From $\left[(z^2-1)^{1/2}A_k(z)\right]''_{z=z_k} = 0$, we get (4.9), which completes the proof of theorem.

5. ESTIMATION OF FUNDAMENTAL POLYNOMIALS :

Lemma 5.1 : Let $L_k(z)$ be given by (2.4).Then

(5.1) $max_{|z|=1} \sum_{k=0}^{2n-1}|L_k(z)| \le c\log n$,

where c is a constant and independent of n and z.

Proof: From maximal principle,we know

$$\lambda_n = max_{|z|=1}\lambda_n(z)$$

$$\lambda_n(z) = \sum_{k=0}^{2n-1}|L_k(z)|$$

Let $z = x+iy$ and $|z| = 1$, then for $0 \le argz \le \pi$ and $k = 1(1)n$

$$L_k(z) = \frac{R(z)}{(z-z_k)R'(z_k)}$$

$$= \frac{(z^2-1)\,W(z)}{(z-z_k)(z_k^2-1)W'(z_k)}$$

Since,

(5.2) $$\begin{cases} z_k = x_k + iy_k \\ |z^2 - 1| = 2\sqrt{1-x^2} \\ |z_k^2 - 1| = 2\sqrt{1-x_k^2} \\ |z - z_k| = \sqrt{2}\sqrt{1 - xx_k - \sqrt{1-x^2}\sqrt{1-x_k^2}} \end{cases}$$

Therefore, we have

$$|L_k(z)| = \frac{\sqrt{1-x^2}|P_n'(x)|}{2\sqrt{2}(1-x_k^2)\sqrt{1-xx_k-\sqrt{1-x^2}\sqrt{1-x_k^2}}|P_n''(x_k)|}$$

$$\leq \frac{\sqrt{1-x^2}|P_n'(x)|(1-xx_k)^{\frac{1}{2}}}{2\sqrt{2}(1-x_k^2)|P_n''(x_k)|(x-x_k)}$$

$$= M_k(x),$$

Also $|L_{n+k}(z)| \leq M_k(x)$

Similarly, for $0 \leq arg z < 2\pi$, $k = 1(1)n$

$|L_k(z)| \leq M_k(z)$ and $|L_{n+k}(z)| \leq M_{n+k}(z)$

(5.3)$\lambda_n \leq 2\sum_{k=0}^{n} M_k(x) + |L_0(z)| + |L_{2n-1}(z)|$

$=2\sum_{|x_k-x|\geq\frac{1}{2}(1-x_k^2)} M_k(x)+2\sum_{|x_k-x|<\frac{1}{2}(1-x_k^2)} M_k(x) + 2$

Using (2.10), we get

(5.4) $\sum_{|x_k-x|\geq\frac{1}{2}(1-x_k^2)} M_k(x) \leq cn^{1/2}\sum_{k=1}^{n}\frac{1}{(1-x_k^2)^{3/2}|P_n''(x_k)|}$

$$\leq c\log n,$$

owing to (2.11) and (2.12).

Similarly,

$$\sum_{|x_k-x|<\frac{1}{2}(1-x_k^2)} M_k(x) \leq c\log n$$

Hence lemma follows from (5.3).

Remark: Let $L_{1k}(z)$ be given by (2.5). Then

$max_{|z|=1} \sum_{k=1}^{2n-2}|L_{1k}(z)| \leq cn^{3/2}\log n$,

where c is a constant and independent of n and z.

Lemma 5.2 : Let $B_k(z)$ be defined in theorem 4.1. Then, we have

(5.5) $\sum_{k=0}^{2n-1}\left|(z^2-1)^{1/2}B_k(z)\right| \le cn^{-2}\log n\ ,\ |z|\le 1$

where c is a constant independent of n and z.

Proof: From (4.4),we have

(5.6) $\sum_{k=0}^{2n-1}\left|(z^2-1)^{1/2}B_k(z)\right|$

$$\le \sum_{k=0}^{2n-1}\left|(z^2-1)^{1/2}R(z)\right| |J_k(z)||b_k|\,|z|^{-n}$$

where,

(5.7) $|b_k| \le \frac{1}{K_n}(1-x_k^2)^{-5/4}|P_n''(x_k)|^{-1}$

(5.8) $\left|(z^2-1)^{1/2}R(z)\right| \quad \le c\,K_n(1-x^2)^{3/4}|P_n'(x)|$

$$|J_k(z)| = \left|\int_0^z t^{n-1}L_k(t)\,dt\right|$$

(5.9) $\qquad \le \int_0^1 t^{n-1}\,|L_k(t)|\,dt$

Using (5.7)-(5.9) in (5.6), we get

$$\sum_{k=0}^{2n-1}\left|(z^2-1)^{1/2}B_k(z)\right| \le \sum_{k=0}^{2n-1}\frac{\left(1-x_k^2\right)^{-5/4}(1-x^2)^{3/4}|P_n'(x)|}{|P_n''(x_k)|}\int_0^1 t^{n-1}|L_k(t)|\,dt$$

Further using (2.10) - (2.12) and lemma (5.1), we get the result.

Lemma 5.3: For $z=e^{i\theta}$ $(0\le\theta<2\pi)$, we have

(5.10) $\sum_{k=0}^{2n-1}|S_k(z)| \le c\,n^{-1/2}\log n,$

where $S_k(z)$ is given by (4.8).

Proof: Differentiating (2.4), we have

(5.11) $\begin{cases} L_k(z) = \frac{R'(z)}{R'(z_k)} - (z-z_k)L_k'(z) \\ L_k'(z) = \frac{R''(z)}{2R'(z_k)} - \frac{1}{2}(z-z_k)L_k''(z) \end{cases}$

Using (2.4), (2.5) and (5.11) in (4.8), we get

(5.12) $$S_k(z) = \frac{1}{(1-z_k^2)\{R'(z_k)\}^2}\int_0^z t^n(1+z_k t)W'(t)\,dt$$

$$+\frac{1}{2R'(z_k)}\int_0^z t^n L_k''(t)\,dt - \frac{L_k'(z_k)}{R'(z_k)}\int_0^z t^n L_k'(t)\,dt$$

$$+\frac{(3n+2)}{(z_k^2-1)R'(z_k)}\int_0^z t^n L_{1k}(t)\,dt$$

$$+\frac{n}{z_k\,R'(z_k)}\int_0^z t^{n-1}[L_k(z)+(z-z_k)L_k'(t)]\,dt$$

$$+\frac{2z_k}{(z_k^2-1)^2R'(z_k)}\int_0^z t^{n+1}L_{1k}(t)\,dt$$

$$+\frac{2n}{(z_k^2-1)R'(z_k)}\int_0^z t^n L_{1k}(t)\,dt$$

$= \quad I_1+I_2+I_3+I_4+I_5+I_6+I_7.$

Using (2.8), (2.11) - (2.12), Lemma 5.1 and Bernstein inequality,we have

$$|I_2|+|I_3| \le cn^{-1/2}\log n$$

Also,we have

$$|I_1|+|I_4|+|I_5|+|I_6|+|I_7| \le \frac{c}{n}$$

Therefore combining all these ,we get (5.10).

Lemma 5.4 : For $z=e^{i\theta}$ $(0\le\theta<2\pi)$,we have

(5.13) $$\sum_{k=0}^{2n-1}\left|(z^2-1)^{1/2}A_k(z)\right| \le c\log n\,,$$

where $A_k(z)$ is given in theorem 4.2 and c is a constant independent of n and z.

Proof: From (4.7) ,we have

(5.14) $$\left|(z^2-1)^{1/2}A_k(z)\right| \le \left|(z^2-1)^{1/2}L_k(z)\right|\,|L_k(z)|$$

$$+\ |z^{-n}|\left|(z^2-1)^{1/2}R(z)\right|\,|S_k(z)|$$

$$+|a_k|\left|(z^2-1)^{1/2}B_k(z)\right|$$

From (4.9), we have

$$|\mathrm{a_k}| \le 4|(z_k{}^2-1)|^{1/2}|L'_k(z_k)|[|z_k|+4|(z_k{}^2-1)||L'_k(z_k)|]$$

(5.15) $\le c\mathrm{n}^2$, owing to (2.11) – (2.12).

Using lemmas 5.1 – 5.3 and (5.15) in (5.14),we get (5.13).

6. CONVERGENCE

In this section, we prove the following:

Theorem 6.1: Let $f(z)$ be continuous for$|z| \le 1$ and analytic for $|z| < 1$. Let the arbitrary β'_k s be such that

(6.1) $|\beta_k| = O\big(n^2\omega(f,n^{-1})\big)$

Then $\{Q_n(z)\}$ defined by

(6.2) $$Q_n(z) = \sum_{k=0}^{2n-1} f(z_k)A_k(z) \quad + \sum_{k=0}^{2n-1} \beta_k B_k(z)$$

satisfies the relation,

(6.3) $$\left|(z^2-1)^{1/2}\{\,Q_n(z)-f(z)\}\right| = O(\omega(f,n^{-1})\log n),$$

where $\omega(f,n^{-1})$ be the modulus of continuity of $f(z)$.

To prove the theorem 6.1, we shall need the following.

Remark 6.1: Let $f(z)$ be continuous for $|z| \le 1$ and analytic for $|z| < 1$. Then there exist a polynomial $F_n(z)$ of degree $\le 4n-1$ satisfying Jackson's inequality.

(6.4) $|f(z)-F_n(z)| \le c\omega(f,n^{-1}),\ z = e^{i\theta}(0 \le \theta < 2\pi)$

And also an inequality due to O.Kiš[4].

(6.5) $\left|F_n^{(m)}(z)\right| \le cn^m\,\omega(f,n^{-1})\,,\ \ m\epsilon I^+.$

Proof: Since $Q_n(z)$ be is uniquely determined polynomial of degree $\le 4n-1$ and the polynomial $F_n(z)$ satisfying (6.4) and (6.5) can be expressed as

$$F_n(z) = \sum_{k=0}^{2n-1} F_n(z_k)A_k(z) \quad + \sum_{k=0}^{2n-1} F_n''(z_k)B_k(z)$$

Then

$$\left|(z^2-1)^{1/2}\{Q_n(z)-f(z)\}\right|$$
$$\le |z^2-1|^{1/2}|Q_n(z)-F_n(z)|$$
$$+\,|z^2-1|^{1/2}|F_n(z)-f(z)|$$
$$\le \sum_{k=0}^{2n-1}|f(z_k)-\ F_n(z_k)|\left|(z^2-1)^{1/2}A_k(z)\right|$$
$$+\sum_{k=0}^{2n-1}\{|\beta_k|+|F_n''(z_k)|\}\left|(z^2-1)^{1/2}B_k(z)\right|$$
$$+\,|z^2-1|^{1/2}|F_n(z)-f(z)|$$

Using (6.1), (6.4), (6.5), Lemma 5.2 and Lemma 5.4, we get (6.3).

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