Common Properties, Implicit Function, Rational Contractive Map and Related Common Fixed Point Theorems in Multiplicative Metric Space

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Abstract

The purpose of this paper is to prove common fixed point theorems using rational contraction for weekly compatible mappings along with property (E.A), common limit range properties satisfying implicit functions in the setup of multiplicative metric space.

Keyword: Rational contractive maps, property (E.A), common limit range properties, contractive map, common fixed point, multiplicative metric space.

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1. INTRODUCTION AND PRELIMINARIES

Common fixed point theorems satisfying certain contraction conditions has many applications and has been at the Centre of various research activities also it is the key concepts to solve many problems of calculus. It is well known that set of all positive real numbers is not complete with respect to usual metric but it is observed that set of all real numbers is a complete multiplicative metric space with respect to the multiplicative absolute value function. This problem was solved in 2008, by Bashirov [1] by introducing the new metric space named Multiplicative Metric Space. The theory of convergence in multiplicative metric space and related fixed point theorems in multiplicative metric space was introduced by Özavsar and Cevikel initiated [6]. Let us start with topological definitions of multiplicative metric space. The notations \( \mathbb{R} \) and \( \mathbb{R}_+ \) are used to represent the set of real numbers and the set of all positive real numbers respectively.
Definition 1.1. [6] Let $X$ be a nonempty set. A multiplicative metric is a mapping $d: X \times X \to \mathbb{R}_+$ satisfying the following conditions:

1. $d(x,y) \geq 1 \ \forall \ x,y \in X$ and $d(x,y) = 1$ if and only if $x = y$;
2. $d(x,y) = d(y,x) \ \forall \ x, y \in X$;
3. $d(x,y) \leq d(x,z) + d(z,y) \ \forall \ x,y \in X$ (multiplicative triangle inequality).

Definition 1.2. [6] Let $(X, d)$ be a multiplicative metric space. Then a sequence $\{x_n\}$ in $X$ is said to be

1. a multiplicative convergent to $x$ if for every multiplicative open ball $B_\varepsilon(x) = \{y \mid d(x,y) \leq \varepsilon, \varepsilon > 1\}$, there exists a natural number $N$ such that $n \geq N$, then $x_n \in B_\varepsilon(x)$, i.e. $d(x_n,x) \to 1$ as $n \to \infty$.
2. a multiplicative Cauchy sequence if $\forall \varepsilon > 1$ for all $m, n > N$, that is, $d(x_n,x_m) \to 1$ as $n \to \infty$.
3. we call a multiplicative metric space complete if every multiplicative Cauchy sequence in it is multiplicative convergent to $x \in X$.

The concept of multiplicative contraction was given by Özavsar and Cevikel [6] in 2012, some fixed point theorem were also proved.

Definition 1.3. [6] Let $f$ be a mapping of a multiplicative metric space $(X, d)$ into itself. Then $f$ is said to be a multiplicative contraction if there exist a real constant $\omega \in [0,1)$ such that

$$d(fx,fy) \leq d^\omega (x,y)$$

for all $x,y \in X$.

Definition 1.4. [2] Let $f$ and $g$ be two mappings of a multiplicative metric space $(X, d)$ into itself, then $f$ and $g$ are said to be

1. commutative mapping if $fgx=gfx \forall x \in X$.
2. Weak commutative mapping if $d(fgx, gfx) \leq d(fx, gx) \forall x \in X$.
3. Weakly compatible if $f$ and $g$ commute at coincidence points, that is, $ft=gt$ for somet $\in X$. Implies that $fgt = gft$.
4. E.A property if there exist a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$ for some $t \in X$.
5. CLR_g property (common limit range of $g$ property) if there exist a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = gt$ for some $t \in X$.
6. CLR_f property (common limit range of $f$ property) if there exist a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = ft$ for some $t \in X$. 
2. MAIN RESULTS
Implicit functions was used by Popa [9], it was a good contractive condition in multiplicative metric space. Many authors have been using the concept of multiplicative metric space e.g. [see 5, 7, 10]. To prove the main result we define a suitable class of the implicit function involving three real non-negative arguments as follows:

Let $\Psi$ is family of functions such that $\phi: \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ is continuous and increasing in each coordinate variable and

1. $\phi (t, t_1, l, l) \leq t, t_1$
2. $\phi (1, t, t_1, l) \leq t$
3. $\phi (t, l, l) \leq t$
4. $\phi (l, 1, l) \leq t$
5. $\phi (l, l, 1) \leq t$

for every $t, t_1 \in \mathbb{R}_+ (t, t_1 \geq 1)$. It is obvious that $\phi (l, l, l) = l$. There exist many functions $\phi \in \Psi$.

Now we prove the following theorems for weakly compatible mappings satisfying implicit function in a multiplicative metric space.

**Theorem 2.1.** Let $A, B, \gamma, \theta$ be self mappings of a multiplicative metric space $(X, d)$ such that

(E1) $\gamma X \subset BX$ and $\theta X \subset AX$

(E2) $d(\gamma x, \theta y) \leq \begin{cases} \phi \left( \frac{d(\gamma x, \theta y) + d(\gamma x, y)}{d(\gamma x, y) + d(\gamma x, \theta y)} \right)^{\omega}, & \text{if } \phi \in \Psi; \\
\end{cases}$

(E3) let us suppose that the pairs $(A, \gamma)$ and $(B, \theta)$ are weakly compatible;
(E4) One of the subspaces $AX$ or $BX$ or $\gamma X$ or $\theta X$ is complete

Then $A, B, \gamma, \theta$ have a unique common fixed point.

**Proof.** Let $x_0$ be any arbitrary point of metric space $X$, since $\gamma X \subset BX$, hence $\exists x_1 \in X$ such that $\gamma x_0 = Bx_1 = y_0$, for this $x_1$. $\exists x_2 \in X$ such that $Ax_2 = \theta x_1 = y_1$. Similarly, an inductive sequence $\{ y_n \}$ can be defined in such a way that,

$\gamma x_{2n} = Bx_{2n+1} = y_{2n}$, $Ax_{2n+2} = \theta x_{2n+1} = y_{2n+1}$.

Next, we prove that $\{ y_n \}$ is a multiplicative sequence in $X$. In fact, $\forall n \in \mathbb{N}$, we have,
From (E2), we have

\[ d(y_{2n}, y_{2n+1}) \leq d(y_{2n}, x_{2n+1}) \]

\[ \leq \phi \left( \begin{array}{c} (A_{x_{2n}}, B_{x_{2n+1}}) [d(\theta x_{2n+1}, x_{2n}) + d(A_{x_{2n}}, \theta x_{2n+1})] \\
\frac{d(B_{x_{2n+1}}, A_{x_{2n}}) + d(y_{2n}, B_{x_{2n+1}})}{} \\
\frac{d(\theta x_{2n+1}, x_{2n}) + d(A_{x_{2n}}, \theta x_{2n+1})}{d(A_{x_{2n}}, x_{2n}) + d(\theta x_{2n+1}, x_{2n})} \end{array} \right)^{\omega} \]

\[ \leq \phi \left( \begin{array}{c} d(y_{2n-1}, y_{2n}) [d(y_{2n+1}, y_{2n}) + d(y_{2n-1}, y_{2n+1})] \\
\frac{d(y_{2n}, y_{2n-1}) + d(y_{2n}, y_{2n})}{d(y_{2n+1}, y_{2n}) + d(y_{2n-1}, y_{2n+1})} \\
\frac{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n-1})}{d(y_{2n-1}, y_{2n}) + d(y_{2n+1}, y_{2n})} \end{array} \right)^{\omega} \]

\[ \leq \phi \left( \begin{array}{c} d(y_{2n-1}, y_{2n}) [d(y_{2n+1}, y_{2n}) + d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] \\
\frac{d(y_{2n+1}, y_{2n-1}) + 1}{d(y_{2n+1}, y_{2n}) + d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})} \\
\frac{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})}{d(y_{2n-1}, y_{2n}) + d(y_{2n+1}, y_{2n})} \end{array} \right)^{\omega} \]

\[ \leq \{ \phi d(y_{2n-1}, y_{2n}).d(y_{2n}, y_{2n+1}), 1, d(y_{2n-1}, y_{2n}) \} \]

\[ d(y_{2n}, y_{2n+1}) \leq d^{\omega} (y_{2n-1}, y_{2n}).d^{\omega} (y_{2n}, y_{2n+1}) \]

This implies that,

\[ d(y_{2n}, y_{2n+1}) \leq d^{\omega - \omega}(y_{2n-1}, y_{2n}) \]

On substituting, \( h = \frac{1}{\omega - \omega} \in (0, \frac{1}{2}) \)

\[ d(y_{2n}, y_{2n+1}) \leq h^{\omega - \omega}(y_{2n-1}, y_{2n}) \]

In a similar way we have,

\[ d(y_{2n+1}, y_{2n+2}) = d(Tx_{2n+1}, y_{2n+2}) = d(y_{2n+2}, Tx_{2n+1}) \]
\[\begin{align*}
&\leq \left\{ \phi \left( \frac{d(Ax_{2n+2}, Bx_{2n+1})[d(Tx_{2n+2}, x_{2n+2}) + d(Ax_{2n+2}, Tx_{2n+1})]}{d(Bx_{2n+1}, Ax_{2n+2}) + d(yx_{2n+2}, Bx_{2n+1})},
\frac{d(Tx_{2n+1}, yx_{2n+2})[d(Bx_{2n+1}, Ax_{2n+2}) + d(yx_{2n+2}, Bx_{2n+1})]}{d(Ax_{2n+2}, yx_{2n+2}) + d(Tx_{2n+1}, yx_{2n+2})} \right) \right\}^\omega \\
&\leq \left\{ \phi \left( \frac{d(y_{2n+1}, y_{2n})[d(y_{2n+1}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})]}{d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n})},
\frac{d(y_{2n+1}, y_{2n+2})[d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n})]}{d(y_{2n+1}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})} \right) \right\}^\omega \\
&\leq \{ \phi(1, d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n+1})) \}^\omega \\
\end{align*}\]

\[d(y_{2n+1}, y_{2n+2}) \leq d^\omega(y_{2n}, y_{2n+1}), d^\omega(y_{2n+1}, y_{2n+2})\]

This implies that,

\[d(y_{2n+1}, y_{2n+2}) \leq d_1^\omega(y_{2n}, y_{2n+1})\]

On substituting, \(h = \frac{\omega}{1 - \omega} \in (0, \frac{1}{2})\)

\[d(y_{2n+1}, y_{2n+2}) \leq d^h(y_{2n}, y_{2n+1})\]

Hence

\[d(y_n, y_{n+1}) \leq d^h(y_{n-1}, y_n) \leq d^h(y_{n-2}, y_{n-1}) \leq \cdots \leq d^h(y_0, y_1)\]

For all \(n \geq 2\), let \(m, n \in \mathbb{N}\) such that \(m \geq n\). Using the triangular multiplicative inequality, we obtain

\[d(y_m, y_n) \leq d(y_m, y_{m-1})d(y_{m-1}, y_{m-2}) \cdots d(y_{n+1}, y_n) \leq d^{h^{m-1}}(y_1, y_0)d^{h^{m-2}}(y_1, y_0) \cdots d^h(y_1, y_0) \leq d^{h^n}(y_1, y_0)\]

This implies that \(d(y_m, y_n) \to 1\) as \(n \to \infty\) and \(m \to \infty\), therefore \(\{y_n\}\) is a multiplicative Cauchy sequence in \(X\).

Now, suppose that \(AX\) is complete, there exist \(u \in AX\) such that

\[y_{n+1} = \theta x_{2n+1} = A x_{2n+2} \to u \quad (n \to \infty).\]
Consequently, we can find \( v \in X \) such that \( Av = u \). Further a multiplicative Cauchy sequence \( \{ y_n \} \) has a convergent subsequence \( \{ y_{2n+1} \} \), therefore the sequence \( \{ y_n \} \) converges and hence a subsequence \( \{ y_{2n} \} \) also converges. Thus we have,

\[
y_{2n} = y x_{2n} = B x_{2n+1} \rightarrow u (n \to \infty).
\]

We claim that \( y v = u \). If possible \( y v \neq u \), substituting \( x = v \) and \( y = x_{2n+1} \) in (E2), we have

\[
dx(v, \theta x_{2n+1}) \leq \left\{ \phi \left\{ \frac{d(\theta x_{2n+1} , yv) + d(Av, \theta x_{2n+1})}{d(\theta x_{2n+1} , Av) + d(yv, Bx_{2n+1})} \right\} \right\}_\omega
\]

Taking \( n \to \infty \), on the two sides of the above inequality,

\[
dx(v, u) \leq \left\{ \phi \left\{ \frac{d(u,v) + d(u,u)}{d(u,v) + d(u,u)} \right\} \right\}_\omega
\]

\[
dx(u,v) \leq d^\omega(u,v), \text{ a contradiction, since } \omega \in (0, \frac{1}{2})
\]

This implies that, \( d(v, u) = 1 \) implies \( y v = u \).

Since \( u = y v \in y X \subset BX \), there exist \( w \in X \) such that \( u = Bw \).

Claim that \( \theta w = u \), if possible \( Tw \neq u \). Substituting \( x = v \) and \( y = w \) in (E2), we have

\[
dx(u, \theta w) = d(v, \theta w)
\]

\[
x(v, \theta w) \leq \left\{ \phi \left\{ \frac{d(Av, Bw) [d(\theta w, yv) + d(Av, \theta w)]}{d(Bw, Av) + d(yv, Bw)} \right\} \right\}_\omega
\]

As \( Av = u = y v = Bw \), we have
A contradiction, since $\omega \in (0, \frac{1}{2})$ implies $u = \theta w$.

Hence we get $u = Av = \gamma v$, that is, $v$ is a coincidence point of $A$, $\gamma$, also $u = Bw = \theta w$, that is $w$ is coincidence point of $B$ and $\theta$. Therefore

$Av = \gamma v = Bw = \theta w = u$.

Since the pairs $A$, $\gamma$ and $B$, are weakly compatible, we have

$\gamma u = \gamma (Av) = A(\gamma v) = Au = w_1 \text{ (say)}$

And

$\theta u = \theta (Bw) = B(\theta w) = Bu = w_2 \text{ (say)}$

From (E2), we have

$d(w_1, w_2) = d(\gamma u, v)$

$$d(u, \theta w) \leq d^\omega (u, T w)$$

Using symmetry and above conditions of $w_1$ and $w_2$ we have

$$d(w_1, w_2) \leq d^\omega (w_1, w_2)$$
on the other hand, since $\omega \in (0, \frac{1}{2})$ implies $d(w_1, w_2) = 1$, which implies that $w_1 = w_2$ and hence we have

$$y u = Au = \theta u = Bu.$$ 

Again using (E2) and symmetry of multiplicative metric space we have,

$$d(y v, \theta u) \leq \left\{ \phi \left( \frac{d(A v, B u)[d(\theta u, y v) + d(A v, \theta u)]}{d(B u, A v) + d(y v, B u)}, \frac{d(A v, y v)[d(B u, \theta u) + d(B u, A v)]}{d(A v, y v) + d(\theta u, y v)} \right) \right\} ^{\omega}$$

$$\leq \left\{ \phi \left( \frac{d(v, \theta u)[d(v, y v) + d(\theta u, y v)]}{d(\theta u, v) + d(v, y v)}, \frac{d(y v, \theta u)[d(\theta u, y v) + d(v, \theta u)]}{d(\theta u, y v) + d(v, \theta u)} \right) \right\} ^{\omega}$$

$$\leq \left\{ \phi \{d(y v, \theta u), d(y v, \theta u), 1\} \right\} ^{\omega}$$

$d(y v, \theta u) \leq d(y v, \theta u)$, on the other hand, since $\omega \in (0, \frac{1}{2})$ implies $d(y v, \theta u) = 1$ i.e. $y v = \theta u (u = \theta u)$ and hence we have $u = y u = Au = \theta u = B u$. Therefore $u$ is a common fixed point of $A$, $B$, $\gamma$ and $\theta$. Similarly, we can complete the proof for the different case in which $B X$ or $\theta X$ or $\gamma X$ is complete.

**UNIQUENESS**

Let $p$ and $q$ are two different common fixed points of $A., B, \gamma, \theta$ then using symmetry of multiplicative metric space and using equation (E2)

$$d(p, q) = d(y p, \theta q)$$

$$\leq \left\{ \phi \left( \frac{d(A p, B q)[d(\theta q, y p) + d(A p, \theta q)]}{d(B q, A p) + d(y p, B q)}, \frac{d(B q, A p)[d(\theta q, y p) + d(B q, \theta q)]}{d(A p, y p) + d(\theta q, y p)} \right) \right\} ^{\omega}$$

$$\leq \left\{ \phi \left( \frac{d(p, q)[d(q, p) + d(p, q)]}{d(q, p) + d(p, q)}, \frac{d(p, q)[d(q, p) + d(p, q)]}{d(p, p) + d(q, p)} \right) \right\} ^{\omega}$$

$$\leq \left\{ \phi \{d(p, q), 1\} \right\} ^{\omega}$$

$d(p, q) \leq d(p, q),$
on the other hand, since \( \omega \in (0, \frac{1}{2}) \) implies \( d(p, q) = 1 \) i.e. \( p = q \), which proves the uniqueness.

**Corollary 2.2.** Let \( A, B, \gamma \) be mappings of a multiplicative metric space \((X, d)\) into itself satisfying

\[
(E5) \quad \gamma X \subseteq BX \text{ and } \gamma X \subseteq AX
\]

\[
(E6) \quad d(\gamma x, \gamma y) \leq \omega \left\{ \frac{d(Ax, By)[d(y, \gamma x) + d(Ax, \gamma y)]}{d(By, Ax) + d(y, \gamma x)} \right\} \\
\quad \left\{ \frac{d(By, Ax) + d(y, \gamma x)}{d(Ax, \gamma y) + d(Ax, \gamma y)} \right\}
\]

For all \( x, y \in X \), where \( \omega \in (0, \frac{1}{2}) \) and \( \phi \in \Psi \);

\( (E3) \) let us suppose that the pairs \((A, \gamma)\) and \((B, \gamma)\) are weakly compatible;

\( (E4) \) One of the subspaces \( AX \) or \( BX \) or \( \gamma X \) is complete

Then \( A, B \) and \( \gamma \) have a unique common fixed point.

In theorem 2.1., if we put \( \theta = \gamma \), then we obtain the corollary 2.2.

**Corollary 2.3.** Let \( \gamma \) and \( \theta \) be mappings of a multiplicative metric space \((X, d)\) into itself satisfying

\[
(E7) \quad d(\gamma x, \theta y) \leq \omega \left\{ \frac{d(x, y)[d(\theta y, \gamma x) + d(x, \theta y)]}{d(x, y) + d(\theta y, \gamma x)} \right\} \\
\quad \left\{ \frac{d(\gamma x, \gamma y) + d(x, \theta y)}{d(\gamma x, \gamma y) + d(x, \gamma y)} \right\}
\]

For all \( x, y \in X \), where \( \omega \in (0, \frac{1}{2}) \) and \( \phi \in \Psi \),

\( (E8) \) One of the subspaces \( \gamma X \) or \( \theta X \) is complete.

Then \( \gamma \) and \( \theta \) have a unique common fixed point.

In theorem 2.1., if we put \( A = B = 1 \), then we obtain the corollary 2.3.

**Theorem 2.4.** Let \( A, B, \gamma, \theta \) be mappings of a multiplicative metric space \((X, d)\) into itself satisfying the conditions \((E1), (E2), (E3)\) and the following conditions:

\( (E9) \) one of the subspaces \( AX \) or \( BX \) or \( \gamma X \) or \( \theta X \) is closed subset of \( X \)

\( (E10) \) the pairs \((A, \gamma)\) and \((B, \theta)\) satisfy the E.A. property.

Then \( A, B, \gamma, \theta \) have a unique common fixed point.

**Proof.** Suppose that the pairs \( A, \gamma \) satisfies the E.A property. Then \( \exists \) a sequence \( \{x_n\} \)

In \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} \gamma x_n = z \) for some \( z \in X \). Since \( SX \subseteq BX \), \( \exists \) a sequence \( \{y_n\} \) in \( X \) such that \( \gamma x_n = B y_n \)
Hence
\[
\lim_{n \to \infty} B y_n = z
\]

Now suppose that \(BX\) is closed subset of \(X\), \(\exists\) a point \(u \in X\) such that \(Bu = z\).

We will show that \(\lim \theta y_n = z\), from inequality (E2), we have
\[
d(y x_n, \theta y_n) \leq \phi\left(\left\{\frac{d(\theta y_n, y x_n) + d(y x_n, x_n)}{d(\theta y_n, y x_n) + d(y x_n, x_n)},\frac{d(\theta y_n, y x_n) + d(y x_n, x_n)}{d(\theta y_n, y x_n) + d(y x_n, x_n)}\right\}\right)^\omega
\]

Taking \(n\) approaches to infinity and using the symmetry property of multiplicative metric space, we have
\[
d(z, \lim_{n \to \infty} \theta y_n) \leq \phi\left\{d\left(\lim_{n \to \infty} \theta y_n, z\right), 1, 1\right\}\]

by (iii)

since \(\omega \in (0, \frac{1}{2})\) implies \(d(z, \lim_{n \to \infty} \theta y_n) = 1\)

Thus we have
\[
\lim_{n \to \infty} A x_n = \lim_{n \to \infty} B x_n = \lim_{n \to \infty} y x_n = \lim_{n \to \infty} \theta x_n = z = Bu
\]

(say)

for some \(u\) in \(X\).

substituting \(x = x_n\) and \(y = u\) in (E2), we have
\[
d(y x_n, \theta u) \leq \phi\left(\left\{\frac{d(\theta u, y x_n) + d(y x_n, x_n)}{d(\theta u, y x_n) + d(y x_n, x_n)},\frac{d(\theta u, y x_n) + d(y x_n, x_n)}{d(\theta u, y x_n) + d(y x_n, x_n)}\right\}\right)^\omega
\]

Taking \(n \to \infty\), we have
\[
d(Bu, \theta u)
\]

\[
\leq \phi\left(\left\{\frac{d(Bu, Bu)[d(\theta u, Bu) + d(Bu, \theta u)]}{d(Bu, Bu) + d(Bu, Bu)},\frac{d(Bu, Bu)[d(\theta u, Bu) + d(Bu, \theta u)]}{d(Bu, Bu) + d(Bu, Bu)}\right\}\right)^\omega
\]

\[
d(Bu, \theta u) \leq \phi\{d(Bu, \theta u), 1, 1\}\]
On the other hand, since \( \omega \in (0, \frac{1}{2}) \) it implies that \( d(Bu, \theta u) = 1 \) i.e. \( Bu = \theta u \).

It is given that \( B \) and \( \theta \) are weakly compatible, we have \( B \theta u = \theta Bu \) and then \( BBu = \theta \theta u = B \theta u = \theta Bu \) also

\[ \theta X \subset AX, \exists v \in X \text{ such that } \theta u = Av. \]

Next we claim that \( Av = \gamma v \), if possible \( Av \neq \gamma v \) putting \( x=v \) and \( y=u \), we have

\[
\begin{align*}
    d(\gamma v, \theta u) & \leq \left\{ \frac{d(Av,Bu)[d(\theta u, \gamma v) + d(Av, \theta u)]}{d(Bu,Av) + d(\gamma v,Bu)} \right\}^\omega \\
    d(Av, \gamma v) & \leq \left\{ \frac{d(Av,Av)[d(Av,Av) + d(\gamma v,Av)]}{d(Av,Av) + d(\gamma v,Av)} \right\}^\omega
\end{align*}
\]

\[
    d(\gamma v, Av) \leq \left\{ \frac{d(Av,Av)[d(Av,Av) + d(Av,Av)]}{d(Av,Av) + d(Av,Av)} \right\}^\omega
\]

\[
    [\text{since, } \theta u = Av \text{ and } Bu = Av]
\]

\[
    d(\gamma v, Av) \leq \left\{ \phi[1, d(Av,\gamma v), 1] \right\}^\omega
\]

\[
    d(\gamma v, Av) \leq d^\omega(\gamma v, Av)
\]

Which is a contradiction because the range of \( \omega \), this implies that \( d(\gamma v, Av) = 1 \) i.e. \( \gamma v = Av \).

Hence we have, \( Bu = \theta u = Av = \gamma v \). Since the pair \( A, \gamma \) are weakly compatible, we have \( A \gamma v = \gamma Av \) and then \( \gamma \gamma v = \gamma Av = A \gamma v = AAv \).

Next we claim that \( \gamma Av = Av \), if possible \( \gamma Av \neq Av \) on substituting \( x=Av \) and \( y=u \), we have

\[
\begin{align*}
    d(\gamma Av, Av) & = d(\gamma Av, \theta u) \\
    & \leq \left\{ \frac{d(AAv,Bu)[d(\theta u, \gamma Av) + d(AAv, \theta u)]}{d(Bu,AAv) + d(\gamma Av,Bu)} \right\}^\omega \\
    & \left\{ \frac{d(AAv,Av)[d(AAv,Av) + d(AAv,Av)]}{d(AAv,Av) + d(AAv,Av)} \right\}^\omega
\end{align*}
\]

\[
    [\text{since, } \theta u = Av \text{ and } Bu = Av]
\]
\[
\begin{align*}
&\leq \left\{ \phi \left( \frac{d(SA_v, A_v)[d(A_v, SA_v) + d(SA_v, A_v)]}{d(A_v, SA_v) + d(SA_v, A_v)} \right) + \frac{d(A_v, SAv)[d(A_v, SAv) + d(SAv, A_v)]}{d(A_v, SAv) + d(SAv, A_v)} \right\}^\omega \\
&\leq \left\{ \phi \left( \frac{d(A_v, SAv)[d(A_v, SAv) + d(SAv, A_v)]}{d(A_v, SAv) + d(SAv, A_v)} \right) + \frac{d(SAv, SAv)[d(A_v, SAv) + d(SAv, A_v)]}{d(SAv, SAv) + d(A_v, SAv)} \right\}^\omega \\
&\leq \left\{ \phi \left( d(\alpha Av, Av), d(Av, \gamma Av), 1 \right) \right\}^\omega \\
&d(\gamma Av, Av) \leq d^\omega (\gamma Av, Av)
\end{align*}
\]

This is a contradiction, since \( \omega \in (0, \frac{1}{2}) \), hence

\( Av = \gamma Av \), hence \( \gamma Av = Av = AAv \). Hence \( Av \) is a common fixed point of \( A \) and \( \gamma \).

Similarly, \( BBu = Bu = \theta Bu \), i.e., \( Bu \) is a common fixed point of \( B \) and \( \theta \) as \( Av = Bu \). \( Av \) is a common fixed point of \( A, B, \gamma \) and \( \theta \).

Similarly we can complete the proof for cases in which \( AX \) or \( \theta X \) or \( \gamma X \) is closed subset of \( X \).

**Uniqueness**

Let \( Av \) and \( Pu \) are two distinct common fixed points of \( A, B, \gamma \) and \( \theta \). Using symmetry of multiplicative metric space and \((E2)\), we have

\[
d(Av, Pu) = d(\gamma Av, \theta Pu)
\]

\[
\leq \left\{ \phi \left( \frac{d(AAv, BPu)[d(\theta Pu, \gamma Av) + d(AAv, \theta Pu)]}{d(BPu, AAv) + d(\gamma Av, BPu)} \right) + \frac{d(\theta Pu, \gamma Av)[d(BPu, AAv) + d(\gamma Av, BPu)]}{d(\theta Pu, \gamma Av) + d(AAv, \theta Pu)} \right\}^\omega
\]

\[
\leq \left\{ \phi \left( \frac{d(Av, Pu)[d(Pu, Av) + d(Av, Pu)]}{d(Pu, Av) + d(Av, Pu)} \right) + \frac{d(Pu, Av)[d(Pu, Av) + d(Av, Pu)]}{d(Pu, Av) + d(Av, Pu)} \right\}^\omega
\]

\[
\leq \left\{ \phi \left( d(Av, Pu), d(Av, Pu), 1 \right) \right\}^\omega
\]

\[
d(Av, Pu) \leq d^\omega (Av, Pu)
\]

A contradiction, since \( \omega \in (0, \frac{1}{2}) \), hence \( d(Av, Pu) = 1 \). i.e. \( Av = Pu \).

Now we prove the following theorems for weakly compatible mappings with common limit range property satisfying the implicit function in a multiplicative metric space.
Lemma 2.5. Let \( A, B, \gamma, \theta \) be mappings of a multiplicative metric space \((X, d)\) satisfying the conditions (E1) and (E2) and the following condition:

\( (E11) \) the pairs \((A, \gamma)\) satisfies CLR\(_A\) property or the pair \((B, \theta)\) satisfies CLR\(_B\) property,

Then the pairs \((A, \gamma)\) and \((B, \theta)\) share the common limit in the range of \(A\) property or \(B\) property.

**Proof.** Let us consider that the pair \((A, \gamma)\) satisfies the common limit range of \(A\) property. Then \( \exists \) a sequence \(\{x_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} \gamma x_n = Az
\]

for some \(z \in X\).

Since \(\gamma X \subseteq BX\), so for each \(x_n\) there exists \(y_n\) in \(X\) such that \(\gamma x_n = B y_n\). Then

\[
\lim_{n \to \infty} B y_n = Az, \text{ hence, we have}
\]

\[
\lim_{n \to \infty} y_n = \lim_{n \to \infty} A x_n = \lim_{n \to \infty} B y_n = Az.
\]

Now we claim that \(\lim_{n \to \infty} \theta y_n = Az\). On substituting \(x = x_n\) and \(y = y_n\) in \((E2)\), we have

\[
d(y x_n, \theta y_n) \leq \phi \left( \frac{d(A x_n, B y_n) [d(\theta y_n, \gamma x_n) + d(A x_n, \theta y_n)]}{d(A x_n, B y_n) + d(\theta y_n, \gamma x_n) + d(A x_n, \theta y_n)} \right) \omega
\]

Taking \(n \to \infty\), we have

\[
d(Az, \lim_{n \to \infty} \theta y_n) \leq \phi \left( \frac{d(Az, Az) [d(\lim_{n \to \infty} \theta y_n, Az) + d(Az, \lim_{n \to \infty} \theta y_n)]}{d(Az, Az) + d(\lim_{n \to \infty} \theta y_n, Az)} \right) \omega
\]

\[
d(Az, \lim_{n \to \infty} \theta y_n) \leq \phi \left( d \left( \lim_{n \to \infty} \theta y_n, 1, 1 \right) \right)^\omega
\]

\[
d(Az, \lim_{n \to \infty} \theta y_n) \leq d^\omega (Az, \lim_{n \to \infty} \theta y_n)
\]
This is a contradiction, since \( \omega \in \left( 0, \frac{1}{2} \right) \), hence
\[
d \left( A z, \lim_{n \to \infty} y_n \right) = 1. \quad \text{Therefore, } A z = \lim_{n \to \infty} \theta y_n \text{ and hence } \lim_{n \to \infty} \theta y_n = A z.
\]
i.e. \( \lim_{n \to \infty} \theta y_n = \lim_{n \to \infty} B y_n = \lim_{n \to \infty} \gamma x_n = \lim_{n \to \infty} A x_n = A z \)
Then the pairs \((A, \gamma)\) and \((B, \theta)\) share the common limit range of \(A\) property.
for the other pair \((B, \theta)\) which shares common limit range of \(B\) property. Since the pair \((B, \theta)\) satisfies common limit range of \(B\) property. Then \(\exists\) a sequence \(\{y_n\}\) in \(X\) such that
\[
\lim_{n \to \infty} B y_n = \lim_{n \to \infty} \theta y_n = B z
\]
for some \(z \in X\). Since \(\theta X \subset A X\), so for each \(y_n \in X\) such that \(\theta y_n = A x_n\).
Then \(\lim_{n \to \infty} A x_n = B z\). hence, we have
\[
\lim_{n \to \infty} A x_n = \lim_{n \to \infty} B y_n = \lim_{n \to \infty} \theta y_n = B z.
\]
Now we claim that \(\lim_{n \to \infty} y x_n = B z\). If possible \(\lim_{n \to \infty} y x_n \neq B z\), On substituting \(x = x_n\) and \(y = y_n\) in \((E2)\), we have
\[
\lim_{n \to \infty} y x_n, \theta y_n
\]
\[
\leq \phi \left\{ \frac{d(A x_n, B y_n) [d(\theta y_n, y x_n) + d(A x_n, \theta y_n)]}{d(B y_n, A x_n) + d(y x_n, B y_n)} , \frac{d(\theta y_n, y x_n) [d(B y_n, A x_n) + d(y x_n, B y_n)]}{d(A x_n, y x_n) + d(\theta y_n, y x_n)} \right\} \omega
\]
Taking \(n \to \infty\), we have
\[
d \left( \lim_{n \to \infty} y x_n, B z \right)
\]
\[
\leq \phi \left\{ \frac{d(B z, B z) [d(B z, \lim_{n \to \infty} y x_n) + d(B z, B z)]}{d(B z, B z) + d \left( \lim_{n \to \infty} y x_n, B z \right) , \frac{d \left( B z, \lim_{n \to \infty} y x_n \right) [d(B z, B z) + d \left( \lim_{n \to \infty} y x_n, B z \right)]}{d \left( B z, \lim_{n \to \infty} y x_n \right) + d(B z, B z) , \frac{d \left( B z, \lim_{n \to \infty} y x_n \right) [d(B z, B z) + d(B z, B z)]}{d \left( B z, \lim_{n \to \infty} y x_n \right) + d(B z, \lim_{n \to \infty} y x_n) } \right\} \omega
\]
\[
d \left( \lim_{n \to \infty} y x_n, B z \right) \leq \left\{ \phi \left[ 1, d(B z, y x_{2n}), 1 \right] \right\} \omega
\]
\[
d \left( \lim_{n \to \infty} y x_n, B z \right) \leq d \omega \left( \lim_{n \to \infty} y x_n, B z \right)
\]
This is a contradiction, since \( \omega \in \left( 0, \frac{1}{2} \right) \), hence
Therefore, \( \lim_{n \to \infty} \gamma x_n = Bz \).

Then the pairs \((A, \gamma)\) and \((B, \theta)\) share the common limit range of \(B\) property.

**Theorem 2.6.** Let \(A, B, \gamma, \theta\) be mappings of a multiplicative metric space \((X, d)\) satisfying the conditions \((E1)\) and \((E2)\) and \((E11)\). Then the pairs \((A, \gamma)\) and \((B, \theta)\) have a coincidence point. Moreover, assume that the pairs \((A, \gamma)\) and \((B, \theta)\) are weakly compatible. Then \(A, B, \gamma\) and \(\theta\) have a unique common fixed point.

**Proof.** Using the lemma (2.5), the pairs \((A, \gamma)\) and \((B, \theta)\) share the common limit range of \(A\) property, that is there exist two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that
\[
\lim_{n \to \infty} \theta y_n = \lim_{n \to \infty} B y_n = \lim_{n \to \infty} \gamma x_n = \lim_{n \to \infty} A y_n = \text{Av}
\]
for some \(v \in X\). First, we claim that \(Av = \gamma v\), substituting \(x=v\) and \(y=y_n\) in \((E2)\), we have
\[
d(\gamma v, \theta y_n) \leq \frac{d(Av, By_n)[d(\theta y_n, \gamma v) + d(Av, \theta y_n)]}{\omega} \leq \frac{d(By_n, Av) + d(\gamma v, By_n)}{d(\theta y_n, \gamma v) + d(Av, \theta y_n)} \leq d(Av, \gamma v) + d(\theta y_n, \gamma v)
\]
Taking \(n\) approaches to infinity, we have
\[
d(\gamma v, Av) \leq \frac{d(Av, Av)[d(Av, Av) + d(Av, Av)]}{d(Av, Av) + d(\gamma v, Av)} \leq d(Av, \gamma v) + d(Av, Av)
\]
\[
d(\gamma v, Av) \leq \left\{ \phi \left\{ 1, d(Av, Av) \right\} \right\}^\omega \leq d(\gamma v, Av) \leq d(\gamma v, Av)
\]
On the other hand \( \omega \in \left( 0, \frac{1}{2} \right) \), implies that \(d(\gamma v, Av) = 1\ i.e.,\ \gamma v = Av\).

Since \(\gamma X \subset BX\), \(\exists w \in X\) such that \(Bw = \gamma v\). Now we claim that \(Bw = \theta w\), if possible \(Bw \neq \theta w\). Putting \(x=v\) and \(y=w\), in \((E2)\), we have
\[
d(Bw, \theta w) = d(yv, \theta w)
\]
\[d(\theta w, \gamma^v) \leq \frac{d(Bw, \theta w) + d(\gamma^v, \theta w)}{d(\gamma^v, \theta w) + d(\theta w, \gamma^v)},
\]
\[d(\theta w, \gamma^v) \leq \frac{d(Bw, \theta w) + d(\theta w, \theta w)}{d(\theta w, \theta w) + d(\gamma^v, \theta w)},
\]
\[d(\gamma^v, \theta w) \leq \frac{d(Bw, \gamma^v) + d(\theta w, \gamma^v)}{d(\gamma^v, \theta w) + d(\theta w, \gamma^v)}.
\]

Since \(A\gamma^v = AA^v = \gamma\gamma^v, \theta w = BB^w = \theta \theta w\).

Finally we claim that \(\gamma A\gamma^v = \gamma \gamma^v\). Putting \(x = A^v\) and \(y = w\) in (E2) we have
\[d(\gamma A\gamma^v, A\gamma^v) = d(\gamma A^v, A^v) \leq \{d(\gamma A^v, A^v), d(\gamma A^v, A^v), 1\},
\]
\[d(\gamma A\gamma^v, A\gamma^v) \leq \{d(\gamma A^v, A^v), d(\gamma A^v, A^v), d(\gamma A^v, A^v)\},
\]
\[d(\gamma A^v, A) \leq \{d(\gamma A^v, A), d(\gamma A^v, A), 1\},
\]
\[d(\gamma A^v, A) = d(\gamma A^v, A).
\]
which is a contradiction hence $\gamma$ Av = Av and hence $\gamma$ Av = Av. $\gamma$ v = AA v = $\gamma$ $\gamma$ v, which implies that Av is a common fixed point of A and $\gamma$. Also, one can easily prove that BB w = $\theta$ $\theta$ w = $\theta$ B w = B $\theta$ w, i.e. B w is common fixed point of B and $\theta$. As Av = B w, Av is a common fixed point of A, B, $\gamma$ and $\theta$. The uniqueness follows easily from (E2). This completes the proof.

REFERENCES
