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Convergence Theorems of S-iteration Process for Lipschitzian Type Multi-valued Mappings in Banach Spaces

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Abstract

In this paper, we establish the existence theorem and approximating the common fixed point through S-iteration process for two Lipschitzian type multi-valued mappings in the framework of uniformly convex Banach spaces. Our results generalize and unify the corresponding results of Khan and Yildirim [7], Na and Tang [15], Panyanak [16], Sastry and Babu [19], Song and Wang [20], Shahzad and Zegeye [21] and many more results in the sense of wider class of Lipschtzain type multi-valued mapping.

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1. Introduction

Let K be a nonempty subset of a normed space X and $T : K \to K$ be a nonlinear mapping. Then T is said to be:

1. Lipschitzian if for each $n \in \mathbb{N}$, there exists a positive number k_n such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for all } x, y, \in K.$$

2. uniformly k-Lipschitzian if $k_n = k$ for all $n \in \mathbb{N}$,

3. asymptotically nonexpansive [2], if $k_n \geq 1$ and for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} k_n = 1$.

It is easy to see that every nonexpansive mapping T (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$) is asymptotically nonexpansive with sequence $k_n = 1$ and every asymptotically nonexpansive mapping is uniformly k-Lipschitzian with $k = \sup_{n\in\mathbb{N}} k_n$.

It is observe that the class of nearly Lipschitzian mappings is an important generalization of the class of Lipschitzian mappings and was introduced by Sahu in [18] (see [10]).

Let K be a nonempty subset of a Banach space X and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} a_n = 0$. A mappings $T : K \to K$ is said to be nearly Lipschitzian with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n) \quad \text{for all } x, y \in K. \tag{1.1}$$

The infimum of constants k_n in (1.1) is called the nearly Lipschitz constant of T^n and denoted by $\eta(T^n)$.

A nearly Lipschitzian mapping T with the sequence $\{(a_n, \eta(T^n))\}$ is said to be

(i) nearly nonexpansive if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$,

(ii) nearly asymptotically nonexpansive if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \eta(T^n) = 1$,

(iii) nearly uniformly k-Lipschitzian if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$,

(iv) nearly uniformly k-contractive if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

The class of Lipschitz mappings is larger than the classes of nonexpansive mappings and asymptotically nonexpansive mappings. However, the theory of computation of fixed points of non-Lipschitz mappings is equally important and interesting via iterative algorithms and has attracted many researchers in this direction.

On the other hand, the theory of multi-valued nonexpansive mappings is not straightforward than the corresponding theory of single-valued nonexpansive mappings. The study of fixed points for multi-valued contractions and nonexpansive mappings using the

Hausdorff metric was initiated by Markin [13] and Nadler [14]. Later on, an interesting fixed point theory for such maps was developed which has applications in control theory, convex optimization, differential inclusion, and economics (see [3] and references cited therein).

Moreover, the existence of fixed points for multi-valued nonexpansive mappings in uniformly convex Banach spaces was proved by Lim [12]. Different iterative processes have been used to approximate the fixed points of multi-valued nonexpansive mappings (see e.g. Khan and Yildirim [7], Panyanak [16], Sastry and Babu [19], Song and Wang [20], Shahzad and Zegeye [21]).

Recently, Na and Tang [15] introduce an algorithm for solving the fixed point problems for total asymptotically nonexpansive multi-valued mappings in Banach spaces and proved weak and strong convergence theorems.

The purpose of this paper is to establish the existence theorem and approximating the common fixed point through S-iteration process for two Lipschitzian type multi-valued mappings in the framework of uniformly convex Banach spaces. Our results generalize and unify the corresponding results of Kim et al. [9], Kim and Lim [11], Khan and Yildirim [7], Na and Tang [15], Panyanak [16], Sastry and Babu [19], Song and Wang [20], Shahzad and Zegeye [21] and many more results in the sense of wider class of Lipschtzain type multi-valued mapping.

2. Preliminaries

In order to prove our main results, we need the following definitions and lemmas.

Let K be a nonempty closed convex subset of a uniformly convex Banach space X, $\{x_n\}$ be a bounded sequence in X and consider a continuous functional $r_a(\cdot, \{x_n\}) : X \to \mathbb{R}^+$ defined by

$$r_a(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|, \quad x \in X.$$

Then, the infimum of $r_a(\cdot, \{x_n\})$ over K is said to be the *asymptotic radius* of $\{x_n\}$ with respect to K and is denoted by $r_a(K, \{x_n\})$.

A point $z \in K$ is said to be an *asymptotic center* of the sequence $\{x_n\}$ with respect to K if

$$r_a(z, \{x_n\}) = \inf \left\{r_a(x, \{x_n\}) : x \in K\right\},$$

the set of all asymptotic centers of $\{x_n\}$ with respect to K is denoted by $AC(K, \{x_n\})$.

If the asymptotic radius and the asymptotic center are taken with respect to X, then these are simply denoted by $r_a(X, \{x_n\}) = r_a(\{x_n\})$ and $AC(X, \{x_n\}) = AC(\{x_n\})$, respectively.

It is well known that every bounded sequence $\{x_n\}$ in a uniformly convex Banach space X has a unique asymptotic center z with respect to any closed convex subset K of X, that is,

$$AC(K, \{x_n\}) = \{z\}$$

and the inequality

$$\limsup_{n\to\infty} \|x_n - z\| < \limsup_{n\to\infty} \|x_n - x\|,$$

for all $x \neq z$ is known as Opial's condition.

Lemma 2.1. [1] Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $\{x_n\}$ be a bounded sequence in K and $AC(K, \{x_n\}) = \{z\}$. Let $\{y_n\} \subset K$ and $\lim_{m\to\infty} r_a(\{y_m\}) = r_a(K, \{x_n\})$. Then we have

$$\lim_{m\to\infty} y_m = z.$$

Lemma 2.2. [23] Let $\{a_n\}$, $\{\lambda_n\}$ and $\{c_n\}$ be three sequences of nonnegative sequences such that

$$a_{n+1} \leq (1 + \lambda_n)a_n + c_n, \tag{2.2}$$

for all $n \geq n_0$, where n_0 is some positive integer. If $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.3. [17] Let X be a uniformly convex Banach space and $0 < a \leq t_n \leq b < 1$ for all $n \in \mathbb{N}$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that

$$\limsup_{n\to\infty} \|x_n\| \leq r, \quad \limsup_{n\to\infty} \|y_n\| \leq r,$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r,$$

hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

A subset K of a Banach space X is called *proximinal*, if for each $x \in X$, there exists an element $k \in K$ such that

$$||x - k|| = D(x, K) := \inf\{||x - y||, y \in K\}.$$

It is known that weakly compact convex subsets of a Banach space and closed convex subsets of a uniformly convex Banach space are *proximinal*. Let we denote the family of nonempty bounded proximinal subsets of K by $P(K)$.

Let $CB(K)$ denote the class of all nonempty bounded and closed subsets of K. Let H be a Hausdorff metric induced by the metric D of X, that is

$$H(A, B) = \max\left\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\right\},$$

for every $A, B \in CB(X)$.

Let $T : K \to P(K)$ be a multi-valued mapping. An element $x \in X$ such that $x \in Tx$ is called fixed point of T. We denote by $F(T)$ the set of all fixed points of T, that is $F(T) = \{x \in X : x \in Tx\}$.

Definition 2.4. A multi-valued mapping $T : K \to P(K)$ is said to be *Lipschitzian* if for each $n \in \mathbb{N}$, there exists a positive number k_n such that

$$H(T^n x, T^n y)\| \leq k_n \|x - y\| \text{ for all } \ x, y \in K.$$

1. A Lipschitzian mapping T is said to be *multi-valued uniformly k-Lipschitzian* if $k_n = k$ for all $n \in N$,
2. *multi-valued asymptotically nonexpansive*, if $k_n \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} k_n = 1$,
3. *multi-valued nonexpansive*, if $k_n = 1$ for all $n \in \mathbb{N}$.

Now we introduce some wider classes of multi-valued nonlinear mappings which include the classes of Lipschitzian and nearly Lipschitzian mappings.

Definition 2.5. Let K be a nonempty subset of a Banach space X and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} a_n = 0$. A multi-valued mapping $T : K \to P(K)$ is said to be *multi-valued nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that for all $x, y \in K$,

$$H(T^n x, T^n y) \quad \leq \quad k_n(\|x - y\| + a_n). \tag{2.3}$$

The infimum of constants k_n in (2.3) is called the *multi-valued nearly Lipschitz constant* of T^n and denoted by $\eta(T^n)$.

A multi-valued nearly Lipschitzian mapping T with the sequence $\{(a_n, \eta(T^n))\}$ is said to be

(i) *multi-valued nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$,

(ii) *multi-valued nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \eta(T^n) = 1$,

(iii) *multi-valued nearly uniformly k-Lipschitzian* if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$,

(iv) *multi-valued nearly uniformly k-contractive* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

Lemma 2.6. [22] Let $T : K \to P(K)$ be a multi-valued mapping and

$$P_T(x) = \{y \in Tx : \|x - y\| = D(x, Tx)\}.$$

Then the following are equivalent.

(1) $x \in F(T)$;

(2) $P_T(x) = \{x\}$;

(3) $x \in F(P_T)$. Moreover, $F(T) = F(P_T)$.

Definition 2.7. A multi-valued mapping $T : K \to CB(K)$ is called *demiclosed at* $y \in K$ if the sequence $\{x_n\}$ in K is weakly convergent to an element x and the sequence $\{y_n\}(y_n \in Tx_n)$ is strongly convergent to y, then we have $y \in Tx$.

3. Weak Convergence Theorem

In the sequel, let $\Omega = F(S)\bigcap F(T)$ denote the set of all common fixed point of the multi-valued mappings S and T.

First, we introduce and prove the demiclosedness principle for the multi-valued nearly asymptotically nonexpansive mapping.

Proposition 3.1. (Demiclosedness Principle) Let K be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T : K \to CB(K)$ be a multi-valued mapping such that P_T is nearly asymptotically nonexpansive with the sequence $\{(a_n, \eta(T^n))\}$. If $\{x_n\}$ is bounded sequence in K such that $\lim_{n\to\infty} D(x_n, Tx_n) = 0$ and $AC(K, \{x_n\}) = \{z\}$, then z is a fixed point of P_T.

Proof. Let $\{x_n\}$ be a bounded sequence in K. Then there exists a unique asymptotic center p with respect to K, that is $AC(K, \{x_n\}) = \{p\}$. Now we show that p is a common fixed point of T and P_T such that $F(T) = F(P_T) = \{p\}$. Since

$$P_T(x) \;=\; \{y \in Tx : \|y - x\| = D(x, Tx)\},$$

we have

$$\begin{aligned} P_T^2(x) &= \{y \in P_T(x) : \|y - x\| = D(x, P_T(x))\},\\ P_T^3(x) &= \{y \in P_T^2(x) : \|y - x\| = D(x, P_T^2(x))\},\\ &\vdots\\ P_T^n(x) &= \{y \in P_T^{n-1}(x) : \|y - x\| = D(x, P_T^{n-1}(x))\}. \end{aligned} \tag{3.4}$$

So, for any $\nu_n \in P_T^n(x)$, we have $\nu_n \in P_T^{n-1}(x)$, $\nu_n \in P_T^{n-2}(x)$, $\cdots$, $\nu_n \in P_T(x)$, $\nu_n \in Tx$ and so,

$$\lim_{n\to\infty} D(x_n, Tx_n) \le \lim_{n\to\infty} \|x_n - \nu_n\| = 0.$$

Note that, for each $\{x_n\}$, $n \in \mathbb{N}$, there exists a point $u_n \in P_T^n(p)$ such that

$$\begin{aligned} \|x_n - u_n\| &= D(x_n, P_T^n(p)) \\ &\leq d(x_n, P_T^n(x_n)) + D(P_T^n(x_n), P_T^n(p)) \\ &\leq d(x_n, P_T^n(x_n)) + H(P_T^n(x_n), P_T^n(p)) \\ &\leq \|x_n - v_n\| + \eta(T^n)(\|x_n - p\| + a_n). \end{aligned} \tag{3.5}$$

Define a continuous functional $r_a(\cdot, \{x_n\}) : K \to \mathbb{R}^+$ by

$$r_a(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|, \quad x \in K.$$

Then from (3.5), we have

$$\begin{aligned} r_a(\{u_n\}) &= \limsup_{n\to\infty} \|x_n - u_n\| \\ &\leq \limsup_{n\to\infty} \Big[\|x_n - v_n\| + \eta(T^n)(\|x_n - p\| + a_n)\Big] \\ &\leq \limsup_{n\to\infty} \|x_n - v_n\| + \limsup_{n\to\infty} \eta(T^n)\Big(\|x_n - p\| + a_n\Big) \\ &\leq \limsup_{n\to\infty} \|x_n - v_n\| + \eta(T^n)\big(r_a(\{p\}) + a_n\big), \end{aligned}$$

it implies that

$$r_a(\{u_n\}) \leq r_a(\{p\}).$$

Hence, it follows from Lemma 2.1 that $p = u_n \in P_T^n(p)$. So, we have $p \in Tp$, i.e. p is a fixed point of T. From Lemma 2.6, it is also a fixed point of P_T. This completes the proof. ■

Lemma 3.2. Let K be a nonempty closed convex subset of a uniformly convex Banach space X. Let $S, T : K \to P(K)$ be multi-valued mappings with

$$\Omega := F(S) \bigcap F(T) \neq \phi$$

such that P_S and P_T are nearly asymptotically nonexpansive mappings with sequences $\{a_n\}$ and $\{u_n\}$, respectively, such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} u_n < \infty$. Then for the sequence $\{x_n\}$ defined iteratively by

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n v_n \\ x_{n+1} = (1 - \alpha_n)v_n + \alpha_n \mu_n, \end{cases} \tag{3.6}$$

where $v_n \in P_S(x_n)$, $\mu_n \in P_T(y_n)$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists for $p \in \Omega$.

Proof. Let $p \in \Omega$. Then $p \in P_T(p) = \{p\}$ and $p \in P_S(p) = \{p\}$. Using (3.6), we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n)\nu_n + \alpha_n\mu_n - p\| \\ &\le (1-\alpha_n)\|\nu_n - p\| + \alpha_n\|\mu_n - p\| \\ &\le (1-\alpha_n)H(P_S^n(x_n), P_S^n(p)) + \alpha_n H(P_T^n(y_n), P_T^n(p)) \\ &\le (1-\alpha_n)\{(1+u_n)\|x_n - p\| + a_n\} \\ &\quad +\alpha_n\{(1+u_n)\|y_n - p\| + a_n\} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\|y_n - p\| &= \|(1-\beta_n)x_n + \beta_n\nu_n - p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\|\nu_n - p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n H(P_T^n(x_n), P_T^n(p)) \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\{(1+u_n)\|x_n - p\| + a_n\} \\ &\le (1+\beta_n u_n)\|x_n - p\| + a_n\beta_n. \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7), we have

$$\begin{aligned}\|x_{n+1} - p\| &\le (1-\alpha_n)\|x_n - p\| + \alpha_n(1+u_n\beta_n)\|x_n - p\| + \alpha_n a_n\beta_n \\ &= (1+u_n\alpha_n\beta_n)\|x_n - p\| + a_n\alpha_n\beta_n, \\ &\le (1+u_n)\|x_n - p\| + a_n. \end{aligned} \tag{3.9}$$

Since $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1} a_n < \infty$, by Lemma 2.2, we have that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in \Omega$. ■

Lemma 3.3. Let K be a nonempty closed convex subset of a uniformly convex Banach space X. Let S, T, P_T, P_S and $\{x_n\}$ be defined as in Lemma 3.2 with $\Omega \neq \phi$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences such that $0 < a \le \alpha_n \le \beta_n < 1$. Then

$$\lim_{n\to\infty} \|x_n - P_S(x_n)\| = 0$$

and

$$\lim_{n\to\infty} \|x_n - P_T(y_n)\| = 0.$$

Proof. By Lemma 3.2, we have $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in \Omega$. Assume that $\lim_{n\to\infty} \|x_n - p\| = c$, for some $c \ge 0$. For $c = 0$, the result is trivial. Suppose that $c > 0$. From (3.8), we have

$$\limsup_{n\to\infty} \|y_n - p\| \le c. \tag{3.10}$$

Since P_T^n is a nearly asymptotically quasi-nonexpansive mapping, we have

$$\begin{aligned}\|\mu_n - p\| &\leq \|\mu_n - P_T^n(p)\| \\ &\leq H(P_T^n(y_n), P_T^n(p)) \\ &\leq (1+u_n)\|y_n - p\| + a_n.\end{aligned}$$

Using (3.10), we have

$$\limsup_{n\to\infty} \|\mu_n - p\| \leq c. \tag{3.11}$$

Similarly, we have

$$\limsup_{n\to\infty} \|\nu_n - p\| \leq c. \tag{3.12}$$

Moreover, $c = \limsup_{n\to\infty} \|x_{n+1} - p\|$ means that

$$\begin{aligned} c &= \lim_{n\to\infty} \Big[\|(1-\alpha_n)\nu_n + \alpha_n\mu_n - p\|\Big] \\ &\leq \lim_{n\to\infty} \Big[(1-\alpha_n\|\nu_n - p\| + \alpha_n\|\mu_n - p|\Big] \\ &\leq \Big[(1-\alpha_n)\limsup_{n\to\infty}\|\nu_n - p\| + \alpha_n \limsup_{n\to\infty}\|\mu_n - p\|\Big].\end{aligned}$$

Using (3.11) and (3.12), we have

$$c \leq \lim_{n\to\infty}((1-\alpha_n)c + \alpha_n c) = c.$$

Hence by using Lemma 2.3, we have

$$\lim_{n\to\infty} \|\nu_n - \mu_n\| = 0. \tag{3.13}$$

On the other hand,

$$\begin{aligned}\|x_{n+1} - p\| &\leq (1-\alpha_n)\|\nu_n - p\| + \alpha_n\|\mu_n - p\| \\ &\leq \alpha_n\|\nu_n - \mu_n\| + \|\nu_n - p\|\end{aligned}$$

it gives that

$$c \leq \liminf_{n\to\infty} \|\nu_n - p\|. \tag{3.14}$$

Hence from (3.12) and (3.14), we have

$$\lim_{n\to\infty} \|\nu_n - p\| = c. \tag{3.15}$$

Thus,

$$\begin{aligned}\|\nu_n - p\| &\leq \|\nu_n - \mu_n\| + \|\mu_n - p\| \\ &\leq \|\nu_n - \mu_n\| + H(P_T^n(y_n), P_T^n(p)) \\ &\leq \|\nu_n - \mu_n\| + (1 + u_n)\{\|y_n - p\| + a_n\}.\end{aligned}$$

Using (3.15), we have

$$c \leq \limsup_{n\to\infty} \|y_n - p\|. \tag{3.16}$$

Using (3.10) and (3.16), we have

$$\lim_{n\to\infty} \|y_n - p\| = c. \tag{3.17}$$

Applying Lemma 2.3, we obtain

$$\lim_{n\to\infty} \|x_n - \nu_n\| = 0$$

Similarly, we get

$$\lim_{n\to\infty} \|x_n - \mu_n\| = 0.$$

Since

$$D(x, P_S(x)) = \inf_{z\in P_S(x)} ||x - z||,$$

we have

$$D(x, P_S(x_n)) \leq \|x_n - \nu_n\| \to 0 \text{ as } n \to \infty$$

and

$$D(x, P_T(y_n)) \leq \|x_n - \mu_n\| \to 0 \text{ as } n \to \infty.$$

This completes the proof. ■

Theorem 3.4. Let K be a nonempty closed convex subset of a uniformly convex Banach space X with Opial's condition. Let S, T, P_T, P_S and $\{x_n\}$ be as in Lemma 3.2 with $\Omega \neq \phi$. Then the sequence $\{x_n\}$ generated by (3.6) converges weakly to a common fixed point of S and T.

Proof. From Lemma 3.2, there exists the limit $\lim_{n\to\infty} \|x_n - p\|$, for $p \in \Omega$. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in Ω. To prove this, let z_1 and z_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Proposition 3.1, there exists $\nu_n \in Tx_n$ such that $\lim_{n\to\infty} \|\nu_n - x_n\| = 0$. Since $I - P_T$ is demiclosed with respect to zero, by Proposition 3.1, we obtain $z_1 \in F(P_T) = \Omega$. Similarly, $z_1 \in F(P_S) = \Omega$. Again in the same manner, we can prove that $z_2 \in \Omega$.

Next, we prove the uniqueness. Let $z_1 \neq z_2$. Then, by Opial's condition

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{n_i\to\infty} \|x_{n_i} - z_1\| \\
&< \lim_{n_i\to\infty} \|x_{n_i} - z_2\| \\
&= \lim_{n\to\infty} \|x_n - z_2\| \\
&\leq \lim_{n_j\to\infty} \|x_{n_j} - z_2\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - z_1\| \\
&= \lim_{n\to\infty} \|x_n - z_1\|,
\end{aligned}$$

which is a contradiction. Hence $\{x_n\}$ converges weakly to a common fixed point in Ω. This completes the proof. ■

4. Strong convergence theorems

We now give some strong convergence theorems. Our first strong convergence theorem is valid in general real Banach spaces. We then apply this theorem to obtain a result in uniformly convex Banach spaces. We also use the method of direct construction of Cauchy sequence as indicated by Song and Cho [22] (and opposed to [21]) but used also by many other authors including [4, 5, 6].

Theorem 4.1. Let K be a nonempty closed convex subset of a uniformly convex Banach space X. Let S, T, P_T, P_S and $\{x_n\}$ be as in Lemma 3.2 with $\Omega \neq \phi$. Then the sequence $\{x_n\}$ generated by (3.6) converges strongly $p \in \Omega$ if and only if

$$\liminf_{n\to\infty} D(x_n, \Omega) = 0,$$

where

$$D(x_n, \Omega) = \inf\{\|x_n - p\| : p \in \Omega\}.$$

Proof. Necessity is obvious.
Conversely, suppose that $\liminf_{n\to\infty} D(x_n, \Omega) = 0$. From (3.9), we have

$$\|x_{n+1} - p\| \leq (1 + u_n)\|x_n - p\| + a_n, \quad n \in \mathbb{N}.$$

This gives that

$$D(x_{n+1}, \Omega) \leq (1 + u_n)D(x_n, \Omega) + a_n, \quad n \in \mathbb{N}.$$

Since, $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} a_n < \infty$, by the hypothesis $\liminf_{n\to\infty} D(x_n, \Omega) = 0$, we have $\lim_{n\to\infty} D(x_n, \Omega) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. The following arguments similar to those given in [4] (see Lemma 5), we obtain the following inequality

$$||x_{n+m} - p|| \leq L\Big[||x_n - p|| + \sum_{j=n}^{\infty} a_j\Big],$$

for every $p \in \Omega$ and for all $m, n \geq 1$, where $L = e^{\sum_{j=n}^{n+m-1} u_j} > 0$. Since $\sum_{n=1}^{\infty} u_n < \infty$,

$$L^* = e^{\sum_{n=1}^{\infty} u_n} \geq L = e^{\sum_{j=n}^{n+m-1} u_j} > 0.$$

Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} D(x_n, \Omega) = 0$ and $\sum_{n=1}^{\infty} a_n < \infty$, there exists a positive integer n_0 such that for all $n \geq n_0$,

$$D(x_n, \Omega) < \frac{\epsilon}{4L^*} \quad \text{and} \quad \sum_{j=n_0}^{\infty} a_j < \frac{\epsilon}{6L^*}.$$

In particular, $\inf\{\|x_{n_0} - p\| : p \in \Omega\} < \frac{\epsilon}{4L^*}$. Thus there must exist $p^* \in \Omega$ such that

$$\|x_{n_0} - p^*\| < \frac{\epsilon}{3L^*}.$$

Hence for all $n \geq n_0$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\
&\leq 2L^*\Big[\|x_{n_0} - p\| + \sum_{j=n_0}^{\infty} a_j\Big] \\
&< 2L^*\Big(\frac{\epsilon}{3L^*} + \frac{\epsilon}{6L^*}\Big) \\
&= \epsilon.
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset K of a Banach space X and so it must converge strongly to a point q in K. Now, let $\lim_{n\to\infty} x_n = q$. Then

$$\begin{aligned}
D(q, P_T^n q) &\leq \|x_n - q\| + D(x_n, P_T^n(x_n)) + H(P_T^n(x_n), P_T^n q) \\
&\leq \|x_n - q\| + D(x_n, P_T^n(x_n)) + (1 + u_n)(\|x_n - q\| + a_n) \\
&\leq \|x_n - p\| + \|x_n - v_n\| + (1 + u_n)(\|x_n - q\| + a_n) \\
&\to 0, \ \text{as} \ n \to \infty,
\end{aligned}$$

which implies that $D(q, P_T(q)) = 0$. Similarly, we can show that $D(q, P_S(q)) = 0$. Since P_T and P_S are nearly asymptotically nonexpansive mappings, so $F(P_T)$ and $F(P_S)$ are closed. Therefore, we have $q \in \Omega$. This completes the proof. ■

Let $f : [0, \infty] \to [0, \infty)$ be a nondecreasing function with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ and let $D(x, \Omega) = \inf\{||x - y|| : y \in \Omega\}$. Let $S, T : K \to P(K)$ be two multi-valued mappings with $\Omega \neq \phi$. Then the two maps are said to satisfy condition (C) (see [8]) if for all $x \in K$,

$$D(x, Tx) \geq f(D(x, \Omega)) \quad \text{or} \quad D(x, Sx) \geq f(D(x, \Omega)).$$

Recall that, a mapping $T : K \to P(K)$ is *semi-compact,* if any bounded sequence $\{x_n\}$ satisfying $D(x_n, Tx_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

Applying Lemma 3.2, we can easily obtain the following theorems.

Theorem 4.2. Let K be a nonempty closed convex subset of a uniformly convex Banach space X. Let S, T, P_T, P_S and $\{x_n\}$ be as in Lemma 3.2. Suppose that pair of maps P_S, P_T satisfies condition (C). Then the sequence $\{x_n\}$ defined in (3.6) converges strongly to $p \in \Omega$.

Theorem 4.3. Let K be a nonempty closed convex subset of a complete and uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and S, T, P_T, P_S and $\{x_n\}$ be as in Lemma 3.2. Suppose that a pair of maps P_S, P_T satisfies condition (C). Then the sequence $\{x_n\}$ defined in (3.6) converges strongly to $p \in \Omega$.

Theorem 4.4. Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and S, T, P_T, P_S and $\{x_n\}$ be as in Lemma 3.2. Suppose that one of the maps P_S and P_T is semi-compact. Then the sequence $\{x_n\}$ defined in (3.6) converges strongly to $p \in \Omega$.

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