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Equivalence of convergence for various iterative sequences generated by nonlinear mappings in Banach space

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Abstract

The purpose of this paper is to establish the equivalence between the convergence of the modified single Mann, modified double Ishikawa and modified triple three-step iterative sequences for asymptotically pseudo-contractive mappings in uniformly smooth Banach space.

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1. Introduction and Preliminaries

Let E is a Banach space over the real and E^* is the dual space of E. Let $D(T)$, $F(T)$ denote the domain of T and the set of fixed points of T respectively, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|\|f\|, \|f\| = \|x\|\}, \quad x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality mapping by j. For more details, see [13, 23].

Definition 1.1. Let $T : D(T) \subset E \to E$ be a mapping.

(1) The mapping T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in D(T)$.

(2) The mapping T is said to be *asymptotically nonexpansive* [8] if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in D(T)$ and $n = 1, 2, \ldots$.

(3) The mapping T is said to be *asymptotically pseudo-contractive* [22] if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and for any $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \leq k_n \|x - y\|^2$$

for all $n = 1, 2, \ldots$.

The following result follows from Definition 1.1 immediately.

Remark 1.2.

(1) If $T : D(T) \subset E \to E$ is nonexpansive mapping, then T is an asymptotically nonexpansive mapping with a constant sequence $\{1\}$.

(2) If $T : D(T) \subset E \to E$ is asymptotically nonexpansive mapping, then there exists $j(x - y) \in J(x - y)$ such that

$$\begin{aligned}\langle T^n x - T^n y, J(x - y)\rangle &\leq \|T^n x - T^n y\| \|x - y\| \\ &\leq k_n \|x - y\|^2,\end{aligned}$$

for all $x, y \in D(T)$ and $n = 1, 2, \ldots$. Hence T is an asymptotically pseudo-contractive mapping. But the converse is not true in general.

This can be seen from the following example.

Example 1.3. ([21]) Let $E = \mathbb{R}$ and $D = [0, 1]$ and let the mapping $T : D \to D$ be defined by

$$Tx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$$

for all $x \in D$. It can be proved that T is not Lipschitzian [21], and so it is not asymptotically nonexpansive. Since T is monotonically decreasing and $T \circ T = I$, we have

$$\begin{aligned}&(T^n x - T^n y)(x - y) \\ &= \begin{cases} |x - y|^2, & \text{if } n \text{ is even;} \\ (Tx - Ty)(x - y) \leq 0 \leq |x - y|^2, & \text{if } n \text{ is odd.}\end{cases}\end{aligned}$$

This implies that T is an asymptotically pseudo-contractive mapping with a constant sequence $\{1\}$.

Up to recently, Mann [16] and Ishikawa [9] iterative sequences have been studied variously by other researchers [6], [7], [18], [20].

In 2001, Chang [2] extended the result of Schu in Hilbert space ([22], Theorem 2.3) to a real uniformly smooth Banach space and proved the following theorem.

Theorem 1.4. Let K be a nonempty bounded closed convex subset of a uniformly smooth Banach space E. Let $T : K \to K$ be an asymptotically pseudo-contractive mapping with sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and $F(T) \neq \emptyset$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}$ be a sequence in [0, 1] satisfying the following conditions:

(i) $\alpha_n \to 0$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For any $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0.$$

If there exists a strictly increasing function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that

$$\langle T^n x_n - x^*, j(x_n - x^*)\rangle \leq k_n \|x_n - x^*\|^2 - \psi(\|x_n - x^*\|), \quad \forall\, n \geq 0,$$

where $x^* \in F(T)$ is some fixed point of T in K, then $x_n \to x^*$ as $n \to \infty$.

In 2006, Ofoedu [17] proved the folling results.

Theorem 1.5. Let K be a nonempty closed convex subset of a real Banach space E. Let $T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudo-contractive mapping with sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ such that $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\} \subset [0, 1]$ be such that $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$. For arbitrary $x_0 \in K$, let $\{x_n\}$ be iteratively defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0.$$

Suppose there exists a strictly increasing function $\psi : [0, \infty) \to [0, \infty)$, $\psi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n \|x - x^*\|^2 - \psi(\|x - x^*\|), \quad \forall\, x \in K.$$

Then $\{x_n\}$ is bounded.

Theorem 1.6. Let K be a nonempty closed convex subset of a real Banach space E. Let $T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudo-contractive mapping with sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ such that $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\} \subset [0, 1]$ be such that $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$. For arbitrary $x_0 \in K$, let $\{x_n\}$ be iteratively defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0.$$

Suppose there exists a strictly increasing function $\psi : [0, \infty) \to [0, \infty)$, $\psi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n\|x - x^*\|^2 - \psi(\|x - x^*\|), \quad \forall\, x \in K.$$

Then $\{x_n\}$ converges strongly to $x^* \in F(T)$.

In 2009, Chang *et al.* [5] proved the folling results.

Theorem 1.7. Let K be a nonempty closed convex subset of a real Banach space E, $T_i : K \to K$, $i = 1, 2$ be two uniformly L_i-Lipschitzian asymptotically pseudo-contractive mappings with sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ such that $F(T_1) \cap F(T_2) \neq \emptyset$, where $F(T_i)$ is the set of fixed points of T_i in K and p be a point in $F(T_1) \cap F(T_2)$. Let $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ be two sequencess such that $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$, $\sum_{n=0}^{\infty} \beta_n < \infty$ and $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$. For arbitrary $x_0 \in K$, let $\{x_n\}$ be a sequence iteratively defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T_1^n y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T_2^n x_n, \quad n \geq 0. \end{aligned}$$

Suppose there exists a strictly increasing function $\psi : [0, \infty) \to [0, \infty)$, $\psi(0) = 0$ such that

$$\langle T_i^n x - x^*, j(x - x^*)\rangle \leq k_n\|x - x^*\|^2 - \psi(\|x - x^*\|), \quad \forall\, x \in K, \quad i = 1, 2.$$

Then $\{x_n\}$ converges strongly to $x^* \in F(T_1) \cap F(T_2)$.

These theorems motiviated us to introduce and analyze a class of three-step iterative sequence for triple of asymptotically pseudo-contractive mappings.

Let K be a nonempty convex subset of a uniformly convex Banach space E and $R, S, T : K \to K$ be three mappings.

Definition 1.8. ([27]) Let $x_0 \in K$ be a given point, $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $[0, 1]$ satisfying some certain conditions. Then the sequence $\{x_n\} \subset E$

defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n S^n z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n R^n x_n, \quad \forall\, n \geq 0 \end{aligned} \tag{1}$$

is called *the modified triple three-step iterative sequence*.

Definition 1.9. For given $r_0 \in K$, the sequence $\{r_n\}$ defined by

$$\begin{aligned} r_{n+1} &= (1-\alpha_n)r_n + \alpha_n T^n s_n, \\ s_n &= (1-\beta_n)r_n + \beta_n S^n r_n, \quad \forall\, n \geq 0 \end{aligned} \tag{2}$$

is called *the modified double Ishikawa iterative sequence* (or *modified double two-step iterative sequence*), where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ appeared in (2) is the same as in (1).

Definition 1.10. For given $u_0 \in K$, the sequence $\{u_n\}$ defined by

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T^n u_n, \quad \forall\, n \geq 0 \tag{3}$$

is called *the modified single Mann iterative sequence* (or *modified single one-step iterative sequence*), where the sequences $\{\alpha_n\}$ appeared in (3) is the same as in (1).

Remark 1.11. It is easy to see that

(1) if $R = I$(:identity mapping) and $r_0 = x_0$, then the modified triple three-step iterative sequence (1) is reduced to the modified double Ishikawa iterative sequence (2);

(2) if $R = S = I$ and $u_0 = x_0$, then the modified triple three-step iterative sequence (1) is reduced to the modified single Mann iterative sequence (3);

(3) if $S = I$ and $u_0 = r_0$, then the modified double Ishikawa iterative sequence (2) is reduced to the modified single Mann iterative sequence (3).

The concept of asymptotically nonexpansive mapping was introduced by Goebel and Kirk [8] in 1972, which was closely related to the theory of fixed points of mappings in Banach spaces. An early fundamental result due to Goebel and Kirk [8] showed that if E is an uniformly convex Banach space, D is a nonempty bounded closed convex subset of E and $T : D \to D$ is an asymptotically nonexpansive mapping, then T has a fixed point in D. This result is a generalization of the corresponding results in Browder [1] and Kirk [14].

The concept of asymptotically pseudo-contractive mapping was introduced by Schu [22] in 1991.

The iterative approximation problems for nonexpansive mapping, asymptotically nonexpansive mapping and asymptotically pseudo-contractive mapping were studied extensively by Browder [22], Goebel and Kirk [8], Kim *et al.*, [10, 11, 12, 13, 19], Kirk [14], Liu [15], Rhoades [21], Schu [22], Weng [24], Xu [25, 26] and Xu and Roach [28] in the setting of Hilbert space or Banach space.

The purpose of this paper is to establish the equivalence between the convergence of the modified single Mann, modified double Ishikawa and modified triple three-step iterative sequences for asymptotically pseudo-contractive mappings in uniformly smooth Banach space.

2. Main Results

The following lemmas play an important role for our main results.

Lemma 2.1. ([3]) Let E be a real Banach space and let $J : E \to 2^{E^*}$ be the normalized duality mapping. Then for any $x, y \in E$, we have

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall\, j(x + y) \in J(x + y). \tag{4}$$

Lemma 2.2. ([4]) E is an uniformly smooth Banach space if and only if the normalized duality mapping J is single-valued and uniformly continuous on any bounded subset of E.

In this section, we assume that E is a real uniformly smooth Banach space, from Lemma 2.2, J is a single-valued duality mapping.

From now on, we are going to study the equivalence between the convergence of modified triple three-step, modified double Ishikawa and modified single Mann iterative sequences defined by (1), (2) and (3) for asymptotically pseudo-contractive mappings in uniformly smooth Banach spaces.

Theorem 2.3. Let E be a real uniformly smooth Banach space, K be a nonempty closed convex subset of E. Let $R, S, T : K \to K$ be three asymptotically pseudo-contractive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{k\to\infty} k_n = 1$ and $F(R)\cap F(S)\cap F(T) = \{x \in K : Rx = Sx = Tx = x\} \ne \emptyset$. Let $\{x_n\}$, $\{u_n\}$ be the modified triple three-step iterative sequence, modified single Mann iterative sequence defined by (1), (3), respectively, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in [0, 1] satisfying the following conditions;

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = 0;$

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty.$

If $x_0 = u_0 \in K$, range of R, S, T are bounded and there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ satisfying

$$\langle T^n y_n - T^n u_n, J(y_n - u_n)\rangle \le k_n\|y_n - u_n\|^2 - \phi(\|y_n - u_n\|),$$

for all $n \geq 0$, then the following statements are equivalent:

(1) The modified single Mann iterative sequence $\{u_n\}$ converges to $p \in F(R) \cap F(S) \cap F(T) (\subset F(T))$.

(2) The modified triple three-step iterative sequence $\{x_n\}$ converges to $p \in F(R) \cap F(S) \cap F(T)$.

Proof. From Schu ([22], Theorem 2.3), T has a fixed point in K. Since E is an uniformly smooth Banach space, by Lemma 2.2, we know that the normalized duality mapping J is single valued and uniformly continuous on any bounded subset of E. Since the range of R, S, T are bounded, there exists a constant $M_1 > 0$ such that

$$\sup_{x \in K}\{\|Rx\|, \|Sx\|, \|Tx\|\} \leq M_1.$$

And also, we have

$$\sup_{n \geq 0}\{\|Ty_n\|, \|Sz_n\|, \|Rx_n\|, \|Tu_n\|\} \leq M_1.$$

Putting

$$M = M_1 + \|x_0\|,$$

by induction, we can prove that

$$\begin{aligned} \sup_{n \geq 0}\{&\|T^n y_n\|, \|S^n z_n\|, \|R^n x_n\|, \|T^n u_n\|, \\ &\|x_n\|, \|y_n\|, \|z_n\|, \|u_n\|\} \leq M. \end{aligned} \tag{5}$$

From (1), (3), (4) and R, S, T are asymptotically pseudo-contractive mappings, we have

$$\begin{aligned} &\|x_{n+1} - u_{n+1}\|^2 \\ &= \|(1-\alpha_n)(x_n - u_n) + \alpha_n(T^n y_n - T^n u_n)\|^2 \\ &\leq (1-\alpha_n)^2\|x_n - u_n\|^2 \\ &\quad + 2\alpha_n\langle T^n y_n - T^n u_n, J(x_{n+1} - u_{n+1})\rangle \\ &\leq (1-\alpha_n)^2\|x_n - u_n\|^2 \\ &\quad + 2\alpha_n\langle T^n y_n - T^n u_n, J(y_n - u_n)\rangle \\ &\quad + 2\alpha_n\langle T^n y_n - T^n u_n, J(x_{n+1} - u_{n+1}) - J(y_n - u_n)\rangle \\ &\leq (1-\alpha_n)^2\|x_n - u_n\|^2 \\ &\quad + 2\alpha_n(k_n\|y_n - u_n\|^2 - \phi(\|y_n - u_n\|)) \\ &\quad + 2\alpha_n\langle T^n y_n - T^n u_n, J(x_{n+1} - u_{n+1}) - J(y_n - u_n)\rangle. \end{aligned} \tag{6}$$

Now we consider the second term on the right side of (6). From (1) and (4) we have

$$\begin{aligned}
&\|y_n - u_n\|^2 \\
&= \|(1-\beta_n)(x_n - u_n) + \beta_n(S^n z_n - u_n)\|^2 \\
&\le (1-\beta_n)^2\|x_n - u_n\|^2 + 2\beta_n\langle S^n z_n - u_n, J(y_n - u_n)\rangle \\
&\le (1-\beta_n)^2\|x_n - u_n\|^2 + 2\beta_n\|S^n z_n - u_n\|\|y_n - u_n\| \\
&\le (1-\beta_n)^2\|x_n - u_n\|^2 \\
&\quad + 2\beta_n\|S^n z_n - u_n\|(\|y_n - z_n\| + \|z_n - u_n\|) \\
&\le (1-\beta_n)^2\|x_n - u_n\|^2 \\
&\quad + \beta_n(\|S^n z_n - u_n\|^2 + \|y_n - z_n\|^2) \\
&\quad + \beta_n(\|S^n z_n - u_n\|^2 + \|z_n - u_n\|^2) \\
&= (1-\beta_n)^2\|x_n - u_n\|^2 + 2\beta_n\|S^n z_n - u_n\|^2 \\
&\quad + \beta_n\|y_n - z_n\|^2 + \beta_n\|z_n - u_n\|^2. \qquad (7)
\end{aligned}$$

We consider the fourth term on the right side of (7). From (1) and (4),

$$\begin{aligned}
&\|z_n - u_n\|^2 \\
&= \|(1-\gamma_n)(x_n - u_n) + \gamma_n(R^n x_n - u_n)\|^2 \\
&\le (1-\gamma_n)^2\|x_n - u_n\|^2 + 2\gamma_n\langle R^n x_n - u_n, J(z_n - u_n)\rangle \\
&\le \|x_n - u_n\|^2 + 8\gamma_n M^2. \qquad (8)
\end{aligned}$$

Substituting (8) into (7), we have

$$\begin{aligned}
&\|y_n - u_n\|^2 \\
&\le (1-\beta_n+\beta_n^2)\|x_n - u_n\|^2 + 2\beta_n\|S^n z_n - u_n\|^2 \\
&\quad + \beta_n\|y_n - z_n\|^2 + 8\beta_n\gamma_n M^2. \qquad (9)
\end{aligned}$$

From (1) and (3), we have

$$\begin{aligned}
&x_{n+1} - u_{n+1} - (y_n - u_n) \\
&= (1-\alpha_n)x_n + \alpha_n T^n y_n - ((1-\alpha_n)u_n + \alpha_n T^n u_n) - (y_n - u_n) \\
&= x_n - y_n - \alpha_n(x_n - u_n) + \alpha_n(T^n y_n - T^n u_n)
\end{aligned}$$

and

$$\begin{aligned}
x_n - y_n &= \beta_n(x_n - S^n z_n) \\
&= \beta_n(x_n - u_n) + \beta_n(u_n - S^n z_n).
\end{aligned}$$

Since $\alpha_n \to 0$, $\beta_n \to 0$ and $\{T^n y_n - T^n u_n\}$, $\{x_n - u_n\}$ and $\{S^n z_n - u_n\}$ are all bounded, we have

$$x_{n+1} - u_{n+1} - (y_n - u_n) \to 0 \quad (n \to \infty).$$

By the uniformly continuity of J, we know that

$$e_n := \langle T^n y_n - T^n u_n, J(x_{n+1} - u_{n+1}) - J(y_n - u_n)\rangle \to 0 \tag{10}$$

as $n \to \infty$. Substituting (9) and (10) into (6), we have

$$\begin{aligned}
&\|x_{n+1} - u_{n+1}\|^2 \\
&\le (1-\alpha_n)^2\|x_n - u_n\|^2 \\
&\quad + 2\alpha_n[k_n\{(1-\beta_n+\beta_n^2)\|x_n - u_n\|^2 + 2\beta_n\|S^n z_n - u_n\|^2 \\
&\quad + \beta_n\|y_n - z_n\|^2 + 8\beta_n\gamma_n M^2\} - \phi(\|y_n - u_n\|)] \\
&\quad + 2\alpha_n e_n \\
&= \|x_n - u_n\|^2 \\
&\quad - \alpha_n(2-\alpha_n)\|x_n - u_n\| + 2\alpha_n k_n(1-\beta_n+\beta_n^2)\|x_n - u_n\|^2 \\
&\quad + 2\alpha_n\beta_n k_n(2\|S^n z_n - u_n\|^2 + \|y_n - z_n\|^2 + 8\gamma_n M^2) \\
&\quad - 2\alpha_n\phi(\|y_n - u_n\|) + 2\alpha_n e_n \\
&\le \|x_n - u_n\|^2 - \alpha_n\phi(\|y_n - u_n\|) \\
&\quad - \alpha_n\phi(\|y_n - u_n\|) - \alpha_n\{2-\alpha_n - 2k_n(1-\beta_n+\beta_n^2)\}4M^2 \\
&\quad + 2\alpha_n\beta_n k_n(2\|S^n z_n - u_n\|^2 + \|y_n - z_n\|^2 + 8\gamma_n M^2) \\
&\quad + 2\alpha_n e_n \\
&\le \|x_n - u_n\|^2 - \alpha_n\phi(\|y_n - u_n\|) \\
&\quad - \alpha_n[\phi(\|y_n - u_n\|) + \{2-\alpha_n - 2k_n(1-\beta_n+\beta_n^2)\}4M^2 \\
&\quad - 2\beta_n k_n(2\|S^n z_n - u_n\|^2 + \|y_n - z_n\|^2 + 8\gamma_n M^2) - 2e_n].
\end{aligned} \tag{11}$$

Let $\sigma = \inf_{n\ge 0}\{\|y_n - u_n\|\}$. Now, we prove that $\sigma = 0$. Suppose the contrary; if $\sigma > 0$, then $\|y_n - u_n\| \ge \sigma > 0$ for all $n \ge 0$. Hence $\phi(\|y_n - u_n\|) \ge \phi(\sigma) > 0$. From (11), we have

$$\begin{aligned}
&\|x_{n+1} - u_{n+1}\|^2 \\
&\le \|x_n - u_n\|^2 - \alpha_n\phi(\sigma) \\
&\quad - \alpha_n\Big[\phi(\sigma) + 4(2-\alpha_n - 2k_n(1-\beta_n+\beta_n^2))M^2 \\
&\quad - 2\beta_n k_n(2\|S^n z_n - u_n\|^2 + \|y_n - z_n\|^2 + 8\gamma_n M^2) - 2e_n\Big].
\end{aligned} \tag{12}$$

Since $\alpha_n \to 0$, $\beta_n \to 0$, $\gamma_n \to 0$ and $k_n \to 1$ $(n \to \infty)$, there exists n_1 such that for all $n \ge n_1$,

$$\begin{aligned}
&\phi(\sigma) + 4(2-\alpha_n - 2k_n(1-\beta_n+\beta_n^2))M^2 \\
&- 2\{\beta_n k_n(2\|S^n z_n - u_n\|^2 + \|y_n - u_n\|^2 + 8\gamma_n M^2) + e_n\} > 0.
\end{aligned}$$

Hence, from (12), we have

$$\|x_{n+1} - u_{n+1}\|^2 \le \|x_n - u_n\|^2 - \alpha_n \phi(\sigma), \quad \forall\, n \ge n_1,$$

that is,

$$\alpha_n \phi(\sigma) \le \|x_n - u_n\|^2 - \|x_{n+1} - u_{n+1}\|^2, \quad \forall\, n \ge n_1.$$

Therefore, for any $m \ge n_1$, we have

$$\begin{aligned}\sum_{n=n_1}^{m} \alpha_n \phi(\sigma) &\le \|x_{n_1} - u_{n_1}\|^2 - \|x_{m+1} - u_{m+1}\|^2 \\ &\le \|x_{n_1} - u_{n_1}\|^2.\end{aligned}$$

Let $m \to \infty$. By condition (ii), we have

$$\infty = \sum_{n=n_1}^{\infty} \alpha_n \phi(\sigma) \le \|x_{n_1} - u_{n_1}\|^2 < \infty.$$

This is a contradiction. Hence we have $\sigma = 0$. Therefore there exists a subsequence $\{y_{n_j} - u_{n_j}\} \subset \{y_n - u_n\}$ such that

$$\|y_{n_j} - u_{n_j}\| \to 0 \quad (j \to \infty), \tag{13}$$

that is,

$$\|(1 - \beta_{n_j}) x_{n_j} + \beta_{n_j} S^{n_j} z_{n_j} - u_{n_j}\| \to 0 \quad (j \to \infty).$$

Since $\beta_n \to 0$ and $\{x_{n_j} - S^{n_j} z_{n_j}\}$ is bounded, from (13), we get

$$\begin{aligned}\|x_{n_j} - u_{n_j}\| &\le \|x_{n_j} - y_{n_j}\| + \|y_{n_j} - u_{n_j}\| \\ &= \beta_{n_j} \|x_{n_j} - S^{n_j} z_{n_j}\| + \|y_{n_j} - u_{n_j}\| \\ &\to 0 \quad (j \to \infty).\end{aligned} \tag{14}$$

Since $\gamma_n \to 0$ and $\{R^{n_j} x_{n_j} - u_{n_j}\}$ is bounded, from (14), we obtain

$$\begin{aligned}\|z_{n_j} - u_{n_j}\| &= \|(1 - \gamma_{n_j})(x_{n_j} - u_{n_j}) + \gamma_{n_j}(R^{n_j} x_{n_j} - u_{n_j})\| \\ &\le (1 - \gamma_{n_j}) \|x_{n_j} - u_{n_j}\| + \gamma_{n_j} \|R^{n_j} x_{n_j} - u_{n_j}\| \\ &\to 0 \quad (j \to \infty).\end{aligned}$$

From (14), condition (i) and $\{T^{n_j} y_{n_j} - T^{n_j} u_{n_j}\}$, $\{S^{n_j+1} z_{n_j+1} - u_{n_j+1}\}$, $\{R^{n_j+1} x_{n_j+1} - u_{n_j+1}\}$ are all bounded, we have

$$\begin{aligned}&\|x_{n_j+1} - u_{n_j+1}\| \\ &= \|(1 - \alpha_n)(x_{n_j} - u_{n_j}) + \alpha_{n_j}(T^{n_j} y_{n_j} - T^{n_j} u_{n_j})\| \\ &\le (1 - \alpha_{n_j}) \|x_{n_j} - u_{n_j}\| + \alpha_{n_j} \|T^{n_j} y_{n_j} - T^{n_j} u_{n_j}\| \\ &\to 0 \quad (j \to \infty).\end{aligned} \tag{15}$$

Also, from (15), we obtain

$$\begin{aligned}
&\|y_{n_j+1} - u_{n_j+1}\| \\
&= \|(1-\beta_{n_j+1})(x_{n_j+1} - u_{n_j+1}) + \beta_{n_j+1}(S^{n_j+1}z_{n_j+1} - u_{n_j+1})\| \\
&\le (1-\beta_{n_j+1})\|x_{n_j+1} - u_{n_j+1}\| + \beta_{n_j+1}\|S^{n_j+1}z_{n_j+1} - u_{n_j+1}\| \\
&\to 0 \quad (j \to \infty),
\end{aligned}$$

and

$$\begin{aligned}
&\|z_{n_j+1} - u_{n_j+1}\| \\
&= \|(1-\gamma_{n_j+1})(x_{n_j+1} - u_{n_j+1}) + \gamma_{n_j+1}(T^{n_j+1}x_{n_j+1} - u_{n_j+1})\| \\
&\le (1-\gamma_{n_j+1})\|x_{n_j+1} - u_{n_j+1}\| + \gamma_{n_j+1}\|T^{n_j+1}x_{n_j+1} - u_{n_j+1}\| \\
&\to 0 \quad (j \to \infty).
\end{aligned}$$

By induction, we can prove that

$$\|x_{n_j+i} - u_{n_j+i}\| \to 0\,, \quad \|y_{n_j+i} - u_{n_j+i}\| \to 0\,, \quad \|z_{n_j+i} - u_{n_j+i}\| \to 0$$

as $j \to \infty$, for all $i \ge 0$. This implies that

$$\|x_n - u_n\| \to 0, \quad \text{as} \quad n \to \infty.$$

If $u_n \to p \in F(R) \cap F(S) \cap F(T) \subset F(T)$, then we have

$$\|x_n - p\| \le \|x_n - u_n\| + \|u_n - p\| \to 0 \quad (n \to \infty).$$

Conversely, if $x_n \to p \in F(R) \cap F(S) \cap F(T)$, then we have

$$\|u_n - p\| \le \|u_n - x_n\| + \|x_n - p\| \to 0 \quad (n \to \infty).$$

This completes the proof. ■

As a direct consequence of Theorem 2.3, we obtain the following.

Corollary 2.4. Let E be a real uniformly smooth Banach space, K be a nonempty closed convex subset of E. Let $S, T : K \to K$ are asymptotically pseudo-contractive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{k\to\infty} k_n = 1$ and $F(S) \cap F(T) \neq \emptyset$. Let $\{r_n\}$, $\{u_n\}$ be the modified double Ishikawa iterative sequence, modified single Mann iterative sequence defined by (2), (3), respectively, and $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in [0, 1] satisfying the following conditions;

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

If $r_0 = u_0 \in K$, range of S, T are bounded and there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ satisfying

$$\langle T^n s_n - T^n u_n, J(s_n - u_n)\rangle \leq k_n \|s_n - u_n\|^2 - \phi(\|s_n - u_n\|),$$

for all $n \geq 0$, then the following statements are equivalent:

(1) The modified single Mann iterative sequence $\{u_n\}$ converge to $p \in F(S) \cap F(T) (\subset F(T))$.

(2) The modified double Ishikawa iterative sequence $\{r_n\}$ converge to $p \in F(S) \cap F(T)$.

Corollary 2.5. Let E be a real uniformly smooth Banach space, K be a nonempty closed convex subset of E. Let $R, S, T : K \to K$ are asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{k\to\infty} k_n = 1$ and $F(R) \cap F(S) \cap F(T) = \{x \in K : Rx = Sx = Tx = x\} \neq \emptyset$. Let $\{x_n\}, \{u_n\}$ be the modified triple three-step iterative sequence, modified single Mann iterative sequence defined by (1), (3), respectively, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be three real sequences in [0, 1] satisfying the following conditions;

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

If $x_0 = u_0 \in K$, range of R, S, T are bounded and there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ satisfying

$$\|T^n y_n - T^n u_n\| \leq k_n \|y_n - u_n\| - \phi(\|y_n - u_n\|),$$

for all $n \geq 0$, then the following statements are equivalent:

(1) The modified single Mann iterative sequence $\{u_n\}$ converges to $p \in F(R) \cap F(S) \cap F(T) (\subset F(T))$.

(2) The modified triple three-step iterative sequence $\{x_n\}$ converges to $p \in F(R) \cap F(S) \cap F(T)$.

Proof. If T is an asymptotically nonexpansive mapping, then T is an asymptotically pseudo-contractive mapping. Hence the conclusion follws from Theorem 2.3, immediately. ■

Corollary 2.6. Let E be a real uniformly smooth Banach space, K be a nonempty closed convex subset of E. Let $S, T : K \to K$ are asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{k\to\infty} k_n = 1$ and $F(S) \cap F(T) \neq \emptyset$. Let $\{r_n\}, \{u_n\}$ be the modified double Ishikawa iterative sequence, modified single Mann iterative sequence defined by (2), (3), respectively, and $\{\alpha_n\}, \{\beta_n\}$ be real sequences in [0, 1] satisfying the following conditions;

(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

If $r_0 = u_0 \in K$, range of S, T are bounded and there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ satisfying

$$\|T^n s_n - T^n u_n\| \leq k_n \|s_n - u_n\| - \phi(\|s_n - u_n\|),$$

for all $n \geq 0$, then the following statements are equivalent:

(1) The modified single Mann iterative sequence $\{u_n\}$ converge to $p \in F(S) \cap F(T) (\subset F(T))$.

(2) The modified double Ishikawa iterative sequence $\{r_n\}$ converge to $p \in F(S) \cap F(T)$.

Competing interests
The author declares to have no competing interests.

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