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The Numerical Computation of Three Step Hybrid Block Method for Directly Solving Third Order Ordinary Differential Equations

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Abstract

This paper consider a three step block method with two hybrid points chosen within grid interval $[x_n, x_{n+1}]$. The new hybrid block method is adopted for directly solving third order ordinary differential equations (ODEs). The numerical properties of the method were investigated with the method exhibiting convergence following from satisfying the properties of consistency and zero-stability. Tow certain numerical examples are tested to demonstrate the accuracy of the method. The new results show improved accuracy in terms of error comparison when compared with existing method in this field.

AMS subject classification: 65L05, 65L06, 65L20.
Keywords: Block method, Hybrid method.

1. Introduction

In this work we study the solution to general third order initial value problem of the form

$$y^{'''} = f(x, y, y^{'}, y^{'''}), \;\; y(a) = \delta_0, \;\; y^{'}(a) = \delta_1, \, y^{''}(a) = \delta_2. \; x \in [a, b]. \tag{1}$$

These equations have been found to adequately model real life application ranging from, fluid flows, mechanics,electric circuits model, vibrations. In literature, scholars proposed several numerical methods for solving equation (1). Famous authors as [1], [2], [11] have discussed the direct solution of general third ordinary differential equation to eliminate the burden inherent in the method of reduction. sequentially, researchers have developed direct block methods generally to numerically approximate these equations include [3], [10], [8], [9] amongst others. This research developed a hybrid block method for the numerical solution of (1) above with the aim of outperforming existing methods in terms of error.

2. Methodology

In this part, derivation of a three-step hybrid block method with two off-step points; $x_{n+\frac{1}{3}}$ and $x_{n+\frac{3}{5}}$ for solving (1) is shown.

Let the power series polynomial of the form:

$$y(x) = \sum_{i=0}^{q+d-1} a_i \left(\frac{x - x_n}{h}\right)^i . \tag{2}$$

as an approximate solution of (1), where

i $x \in [x_n, x_{n+1}]$ for $n = 0, 1, 2, \ldots, N - 1$,

ii q is the number of interpolation points which is same as the order of the differential equation; which is 3 in this case,

iii d is the number of collocation points,

iv $h = x_n - x_{n-1}$ is the step size of partitioning the interval $[a, b]$ given by $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$.

The next step in the derivation of the new hybrid block method includes differentiating (1) thrice to give

$$y^{'''}(x) = f(x, y, y^{'}, y^{''}) = \sum_{i=2}^{q+d-1} \frac{i(i-1)(i-2)}{h^3} a_i \left(\frac{x - x_n}{h}\right)^{i-3} . \tag{3}$$

The approximate solution (2) is interpolated at x_{n+k}, $k = 0, 1, 2$ while differential system (3) is collocated at all points in the selected interval to produce nine equations which can be written as a system in matrix form $AX = B$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 & 64 & 128 & 256 \\ 0 & 0 & 0 & \frac{6}{h^3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{8}{h^3} & \frac{20}{(3h^3)} & \frac{40}{(9h^3)} & \frac{70}{(27h^3)} & \frac{112}{(81h^3)} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24}{h^3} & \frac{60}{h^3} & \frac{120}{h^3} & \frac{210}{h^3} & \frac{336}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{72}{(5h^3)} & \frac{108}{(5h^3)} & \frac{648}{(25h^3)} & \frac{3402}{(125h^3)} & \frac{81648}{(3125h^3)} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{48}{h^3} & \frac{240}{h^3} & \frac{960}{h^3} & \frac{3360}{h^3} & \frac{10752}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{72}{h^3} & \frac{540}{h^3} & \frac{3240}{h^3} & \frac{17010}{h^3} & \frac{81648}{h^3} \end{pmatrix},$$

$$X = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} \text{ and } B = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ f_n \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{3}{5}} \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} \tag{4}$$

The unknown values of $a_i^{\prime}s$, $i = 0(1)8$ in (4) can be obtained by matrix inverse approach, where $X = A^{-1}B$. The values obtained are then substituted back into equation (2) to give a continuous implicit scheme of the form

$$y(x) = \sum_{i=0}^{2} \alpha_i(x) y_{n+i} + \sum_{i=0}^{3} \beta_i(x) f_{n+i} + \beta_{\frac{1}{3}}(x) f_{n+\frac{1}{3}} + \beta_{\frac{3}{5}}(x) f_{n+\frac{3}{5}} \tag{5}$$

It is necessary to also obtain the corresponding first and second derivative schemes of equation (5), and these are obtained as given below

$$y^{'}(x) = \sum_{i=0}^{2} \alpha_i^{'}(x) y_{n+i} + \sum_{i=0}^{3} \beta_i^{'}(x) f_{n+i} + \beta_s^{'}(x) f_{n+s} + \beta_r^{'}(x) f_{n+r} \tag{6}$$

$$y^{''}(x) = \sum_{i=0}^{2} \alpha_i^{''}(x) y_{n+i} + \sum_{i=0}^{3} \beta_i^{''}(x) f_{n+i} + \beta_s^{''}(x) f_{n+s} + \beta_r^{''}(x) f_{n+r} \tag{7}$$

where

$$\alpha_1 = \frac{(2x-2x_n)}{h} - \frac{(x-x_n)^2}{h^2}$$

$$\alpha_2 = \frac{(x-x_n)^2}{(2h^2)} - \frac{(x-x_n)}{(2h)}$$

$$\alpha_0 = \frac{(x-x_n)^2}{(2h^2)} - \frac{(3x-3x_n)}{(2h} + 1$$

$$\beta_0 = -\frac{(13(x-x_n)^4)}{(48h)} + \frac{(x-x_n)^3}{6} + \frac{(131(x-x_n)^5)}{(540h^2)} - \frac{(7(x-x_n)^6)}{(60h^3)} + \frac{(26(x-x_n)^7)}{(945h^4)} - \frac{(5(x-x_n)^8)}{(2016h^5)} - \frac{(451h(x-x_n)^2)}{10080} - \frac{(31h^2(x-x_n))}{15120}$$

$$\beta_1 = \frac{(3(x-x_n)^4)}{(32h)} - \frac{(33(x-x_n)^5)}{(160h^2)} + \frac{(163(x-x_n)^6)}{(960h^3)} - \frac{(89(x-x_n)^7)}{(1680h^4)} + \frac{(5(x-x_n)^8)}{(896h^5)} - \frac{(2651h(x-x_n)^2)}{13440} + \frac{(1259h^2(x-x_n))}{6720}$$

$$\beta_{\frac{1}{3}} = \frac{(729(x-x_n)^4)}{(1280h)} - \frac{(5103(x-x_n)^5)}{(6400h^2)} + \frac{(5913(x-x_n)^6)}{(12800h^3)} - \frac{(2673(x-x_n)^7)}{(22400h^4)} + \frac{(81(x-x_n)^8)}{(7168h^5)} - \frac{(9477h(x-x_n)^2)}{35840} + \frac{(12393h^2(x-x_n))}{89600)}$$

$$\beta_3 = \frac{(x-x_n)^4}{(2304h} - \frac{(37(x-x_n)^5)}{(34560h^2)} + \frac{(5(x-x_n)^6)}{(4608h^3)} - \frac{(59(x-x_n)^7)}{(120960h^4)} + \frac{(5(x-x_n)^8)}{(64512h^5)} + \frac{(29h(x-x_n)^2)}{107520} - \frac{(149h^2(x-x_n))}{483840}$$

$$\beta_{\frac{3}{5}} = \frac{(18125(x-x_n)^5)}{(24192h^2)} - \frac{(3125(x-x_n)^4)}{(8064h)} - \frac{(8125(x-x_n)^6)}{(16128h^3)} + \frac{(11875(x-x_n)^7)}{(84672h^4)} - \frac{(3125(x-x_n)^8)}{(225792h^5)} + \frac{(3125h(x-x_n)^2)}{225792} + \frac{(625h^2(x-x_n))}{338688}$$

$$\beta_2 = -\frac{(3(x-x_n)^4)}{(560h)} + \frac{(9(x-x_n)^5)}{(700h^2)} - \frac{(13(x-x_n)^6)}{(1050h^3)} + \frac{(37(x-x_n)^7)}{(7350h^4)} - \frac{(x-x_n)^8}{(1568h^5} - \frac{(181h(x-x_n)^2)}{23520} + \frac{(481h^2(x-x_n))}{58800}$$

Evaluating equation (5) at the non-interpolating points $x_{n+\frac{1}{3}}$, $x_{n+\frac{3}{5}}$ and x_{n+3} and evaluating equation (7) at all points give the discrete schemes and its derivative. The discrete scheme and its derivatives are combined in a matrix form as below

$$AY_M = BR_1 + h^3\,[CR_2 + DR_3] \tag{8}$$

$$A=\begin{bmatrix}
1 & \frac{-5}{9} & 0 & \frac{1}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{-21}{25} & 1 & \frac{3}{25} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 3 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{-2}{h} & 0 & \frac{1}{(2h)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{-4}{(3h)} & 0 & \frac{1}{(6h)} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{-1}{(2h)} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{-4}{(5h)} & 0 & \frac{-1}{(10h)} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{2}{h} & 0 & \frac{-3}{(2h)} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{4}{h} & 0 & \frac{-5}{(2h)} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{2}{h^2} & 0 & \frac{-1}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{2}{h^2} & 0 & \frac{-1}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & \frac{2}{h^2} & 0 & \frac{-1}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & \frac{2}{h^2} & 0 & \frac{-1}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & \frac{2}{h^2} & 0 & \frac{-1}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & \frac{2}{h^2} & 0 & \frac{-1}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

$$B = \begin{bmatrix} \frac{5}{9} & 0 & 0 \\ \frac{7}{25} & 0 & 0 \\ 1 & 0 & 0 \\ \frac{-3}{(2h)} & -1 & 0 \\ \frac{-7}{(6h)} & 0 & 0 \\ \frac{-1}{(2h)} & 0 & 0 \\ \frac{-9}{(10h)} & 0 & 0 \\ \frac{1}{(2h)} & 0 & 0 \\ \frac{3}{(2h)} & 0 & 0 \\ \frac{1}{h^2} & 0 & -1 \\ \frac{1}{h^2} & 0 & 0 \\ \frac{1}{h^2} & 0 & 0 \\ \frac{1}{h^2} & 0 & 0 \\ \frac{1}{h^2} & 0 & 0 \\ \frac{1}{h^2} & 0 & 0 \end{bmatrix}, \ Y_M = \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{3}{5}} \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y'_{n+\frac{1}{3}} \\ y'_{n+\frac{3}{5}} \\ y'_{n+1} \\ y'_{n+2} \\ y'_{n+3} \\ y''_{n+\frac{1}{3}} \\ y''_{n+\frac{3}{5}} \\ y''_{n+1} \\ y''_{n+2} \\ y''_{n+3} \end{bmatrix},$$

$$R_1 = \begin{bmatrix} y_n \\ y_n' \\ y_n'' \end{bmatrix}, \quad C = \begin{bmatrix} -\frac{311)}{157464} \\ -\frac{1932014216144431)}{844424930131968000} \\ -\frac{17)}{360} \\ -\frac{31)}{15120h} \\ -\frac{22579)}{5511240h} \\ \frac{41)}{3780h} \\ \frac{206756055532931)}{14186338826217062 4h} \\ -\frac{817)}{15120h} \\ \frac{1033)}{7560h} \\ -\frac{451)}{5040h^2} \\ \frac{87413)}{3674160h^2} \\ \frac{439)}{15120h^2} \\ \frac{30892227186823)}{1688849860263936h^2} \\ -\frac{2809)}{15120h^2} \\ \frac{3581)}{5040h^2} \end{bmatrix}, \quad R_2 = \begin{bmatrix} f_n \end{bmatrix},$$

$$D = \begin{bmatrix} \frac{2183}{103680} & \frac{2873}{69984} & \frac{-30406247484175}{171597954441921588} & \frac{1129}{612360} & \frac{-355}{5038848} \\ \frac{9003879}{500000000} & \frac{6198968448095049}{140737488355328000} & \frac{-5731}{1008000} & \frac{983095266258333}{492581209243648000} & \frac{-4128960186368303}{54043195528445952000} \\ \frac{2187}{6400} & \frac{127}{160} & \frac{-625}{1152} & \frac{89}{200} & \frac{107}{11520} \\ \frac{12393}{89600h} & \frac{1259}{6720h} & \frac{35184372088832}{19066439382435729h} & \frac{481}{58800h} & \frac{-149}{483840h} \\ \frac{9059}{1209600h} & \frac{59813834750055}{985162418487296h} & \frac{-156148243330236416}{13899434309795646441h} & \frac{384830477418243}{137922738588221440h} & \frac{-3793}{35271936h} \\ \frac{-81}{1120h} & \frac{-41}{240h} & \frac{3125}{42336h} & \frac{-5}{588h} & \frac{1}{3024h} \\ \frac{-17158473}{700000000h} & \frac{-38252993519933}{985162418487296h} & \frac{-147557439391042}{6355479794145243h} & \frac{-897478534702293}{5516909543528857 60h} & \frac{1112477978993159}{18158513697557839872h} \\ \frac{29079}{89600h} & \frac{3221}{6720h} & \frac{-9042383626829824}{19066439382435729h} & \frac{3503}{58800h} & \frac{-25}{13824h} \\ \frac{-729}{1280h} & \frac{1513}{3360h} & \frac{115625}{169344h} & \frac{6289}{5880h} & \frac{15383}{241920h} \\ \frac{-9477}{17920h^2} & \frac{-2651}{6720h^2} & \frac{3125}{112896h^2} & \frac{-181}{11760h^2} & \frac{29}{53760h^2} \\ \frac{-32906574933547761}{157625986957967360h^2} & \frac{-1450607258448441}{3940649673949184h^2} & \frac{-451521047015981056}{4633144769931882147h^2} & \frac{-330515220668921}{19703248369745920h^2} & \frac{76261}{117573120h^2} \\ \frac{-14499}{89600h^2} & \frac{-1433}{6720h^2} & \frac{6931321301499904h^2}{19066439382435729} & \frac{-1073}{58800h^2} & \frac{359}{483840h^2} \\ \frac{-3039249564193869}{39406496739491840h^2} & \frac{-186579933893557}{492581209243648h^2} & \frac{211915473091035136}{3972174871340776875h^2} & \frac{-1127787290386461}{68961369294110720h^2} & \frac{1402029110795591}{22698142121947299 84h^2} \\ \frac{97767}{89600h^2} & \frac{1889}{1344h^2} & \frac{-30997431810260992}{19066439382435729h^2} & \frac{18919}{58800h^2} & \frac{-3323}{483840h^2} \\ \frac{-322947}{89600h^2} & \frac{-12857}{6720h^2} & \frac{554375}{112896h^2} & \frac{94351}{58800h^2} & \frac{3139}{10752h^2} \end{bmatrix}$$

$$R_3 = \begin{bmatrix} f_{n+s} \\ f_{n+1} \\ f_{n+r} \\ f_{n+2} \\ f_{n+3} \end{bmatrix}$$

Multiplying Equation (8) by the inverse of A gives

$$IY_m \;=\; \bar{B}R_1 + h^2\left[\bar{C}R_2 + \bar{D}R_3\right] \qquad (9)$$

where I is 8×8 identity matrix and

$$\bar{B} = \begin{bmatrix} 1 & \frac{h}{3} & \frac{h^2}{18} \\ 1 & h & \frac{h^2}{2} \\ 1 & \frac{3h)}{5} & \frac{9h^2}{50} \\ 1 & 2h & 2h^2 \\ 1 & 3h & \frac{9h^2)}{2} \\ 0 & 1 & \frac{h}{3} \\ 0 & 1 & h \\ 0 & 1 & \frac{3h}{5} \\ 0 & 1 & 2h \\ 0 & 1 & 3h \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \bar{C} = \begin{bmatrix} \frac{81119}{22044960} \\ \frac{283}{6048} \\ \frac{53373748954636193}{354658470655426560} \\ \frac{173}{945} \\ \frac{81}{224} \\ \frac{15311}{551124h} \\ \frac{43}{420h} \\ \frac{608574480094322017}{10639754119662796800h} \\ \frac{8}{63h} \\ \frac{57}{140h} \\ \frac{3716}{32805h^2} \\ \frac{16}{135h^2} \\ \frac{191118356666679647}{1773292353277132 80h^2} \\ -\frac{13h}{135h^2} \\ \frac{4}{5h^2} \end{bmatrix},$$

$$\bar{D} = \begin{bmatrix} \frac{62863}{14515200} & \frac{5077}{9797760} & \frac{-2276905104986785373009615724674029}{9771662496263705862840071967284720 64} & \frac{-49}{1749600} & \frac{1573}{705438720} \\ \frac{22599}{179200} & \frac{19}{1920} & \frac{-7502996979199245341}{478338831226547569152} & \frac{-19}{39200} & \frac{37}{967680} \\ \frac{422967987}{14000000000} & \frac{2605295792686847}{985162418487296000} & \frac{-704066407075204174733}{5979235390331844614400 0} & \frac{-489459088801973}{3448068464705536000} & \frac{1827605764783219}{16212958658533785600 0} \\ \frac{2187}{2800} & \frac{29}{70} & \frac{-784627135079659637}{13287189756292988032} & \frac{53}{3675} & \frac{-1}{2160} \\ \frac{59049}{25600} & \frac{8991}{4480} & \frac{-3217461645786029272 73}{478338831226547569152} & \frac{19197}{39200} & \frac{279}{35840} \\ \frac{27493}{604800h} & \frac{30788911142989}{6333186975989760h} & \frac{-388920055180610321191 1}{1743545039820765889559 04h} & \frac{-1611820699970681}{6206523236469964800h} & \frac{1817}{88179840h} \\ \frac{891}{2800h} & \frac{61}{1680h} & \frac{31777398970726214801}{717508246839821353728h} & \frac{-19}{14700h} & \frac{1}{10080h} \\ \frac{54069201}{350000000h} & \frac{777006043710241}{73887181386547200h} & \frac{-9966409193986778027}{239169415613273784576h} & \frac{-4736453742504059}{8275364315293286400h} & \frac{62043750194477069}{1361888527316837990400h} \\ \frac{3483}{2800h} & \frac{227}{210h} & \frac{-63554797941452438557}{1195847078066368922 88h} & \frac{302}{3675h} & \frac{-13}{5040} \\ \frac{19683}{22400h} & \frac{81}{56h} & \frac{42899488610480390230 1}{717508246839821353728h} & \frac{5427}{4900h} & \frac{279}{4480h} \\ \frac{1441542818226213}{4503599627370496h^2} & \frac{1559333725146793}{59109745109237760h^2} & \frac{-7272610836661481000839}{5811816799402552965196 8h^2} & \frac{-572448285968749}{413768215764664320h^2} & \frac{917}{8398080h^2} \\ \frac{2349}{6400h^2} & \frac{29}{160h^2} & \frac{8925133661531129739}{26574379512585976064h^2} & \frac{-1}{350h^2} & \frac{7}{34560h^2} \\ \frac{3560178765734487}{7881299347898368h^2} & \frac{116106316830821}{7388718138654720h^2} & \frac{383712092571519165982 7}{149480884758296115360000h^2} & \frac{-199176197120087}{206884107882332160h^2} & \frac{888064810521187}{113490710609736499 20h^2} \\ \frac{81}{50h^2} & \frac{9}{5h^2} & \frac{-39545207608014851177 3}{239169415613273784576h^2} & \frac{59}{175h^2} & \frac{-1}{135h^2} \\ \frac{-19683}{6400h^2} & \frac{-243}{160h^2} & \frac{625h}{128h^2} & \frac{81}{50h^2} & \frac{373}{1280h^2} \end{bmatrix}$$

3. Analysis of the Method

3.1. Order of the Methhod

The linear difference operator L associated with (9) is defined as

$$L[y(x); h] = IY_M - \bar{B}R_1 - h^3\left[\bar{C}R_2 + \bar{D}R_3\right] \tag{10}$$

where $y(x)$ is an arbitrary test function continuously differentiable on [a,b]. Y_M and R_3 component's are expanded in Taylor's series respectively and its terms are collected in powers of h to give

$$L[y(x), h] = \bar{C_0}y(x) + \bar{C_1}hy^{'}(x) + \bar{C_2}hy^{''}(x) + \cdots \tag{11}$$

Definition 3.1. Hybrid block method (9) and associated linear operator (10) are said to be of order p, if $\bar{C_0} = \bar{C_1} = \bar{_2} = \cdots = \bar{C}_{p+2} = 0$ and $\bar{C}_{p+3} \neq 0$ with error vector constants $\bar{C}_{p+2}$.

Expanding (9) in Taylor series about x_n gives

$$
\begin{bmatrix}
\sum_{j=0}^{\infty} \frac{(\frac{1}{3})^j h^j}{j!} - y_n^j - y_n - \left(\frac{h}{3}\right) y_n' - \frac{h^2}{18} y_n'' - \frac{81119h^3}{22044960} y_n''' \\
- \frac{62863}{14515200} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+3}}{j!} y_n^{j+3} \\
- \frac{5077}{9797760} \sum_{j=0}^{\infty} \frac{(1)^j h^{j+3}}{j!} y_n^{j+3} \\
+ \frac{2276905104986785373009615724674029}{977166249626370586284007196728472064} \sum_{j=0}^{\infty} \frac{(\frac{3}{5})^j h^{j+3}}{j!} y_n^{j+2} \\
+ \frac{49}{1749600} \sum_{j=0}^{\infty} \frac{(2)^j h^{j+3}}{j!} y_n^{j+3} \\
- \frac{1573}{705438720} \sum_{j=0}^{\infty} \frac{(3)^j h^{j+3}}{j!} y_n^{j+3} \\
\\
\sum_{j=0}^{\infty} \frac{(1)^j h^j}{j!} - y_n^j - y_n - (h) y_n' - \frac{h^2}{2} y_n'' - \frac{283h^3}{6048} y_n''' \\
- \frac{22599}{179200} \sum_{j=0}^{\infty} \frac{(\frac{1}{3})^j h^{j+3}}{j!} y_n^{j+3} - \frac{19}{1920} \sum_{j=0}^{\infty} \frac{(1)^j h^{j+3}}{j!} y_n^{j+3} \\
+ \frac{7502996979199245341}{478338831226547569152} \sum_{j=0}^{\infty} \frac{(\frac{3}{5})^j h^{j+3}}{j!} y_n^{j+2} + \frac{19}{39200} \sum_{j=0}^{\infty} \frac{(2)^j h^{j+3}}{j!} y_n^{j+3} \\
- \frac{37}{967680} \sum_{j=0}^{\infty} \frac{(3)^j h^{j+3}}{j!} y_n^{j+3} \\
\\
\sum_{j=0}^{\infty} \frac{(\frac{3}{5})^j h^j}{j!} - y_n^j - y_n - (\frac{3h}{5}) y_n' - \frac{h^2}{2} y_n'' - \frac{53373748954636193h^3}{3546584706554265600} y_n''' \\
- \frac{422967987}{14000000000} \sum_{j=0}^{\infty} \frac{(\frac{1}{3})^j h^{j+3}}{j!} y_n^{j+3} - \frac{2605295792686847}{985162418487296000} \sum_{j=0}^{\infty} \frac{(1)^j h^{j+3}}{j!} y_n^{j+3} \\
+ \frac{704066407075204174733}{59792353903318446144000} \sum_{j=0}^{\infty} \frac{(\frac{3}{5})^j h^{j+3}}{j!} y_n^{j+2} \\
+ \frac{489459088801973}{3448068464705536000} \sum_{j=0}^{\infty} \frac{(2)^j h^{j+3}}{j!} y_n^{j+3} \\
- \frac{1827605764783219}{162129586585337856000} \sum_{j=0}^{\infty} \frac{(3)^j h^{j+3}}{j!} y_n^{j+3} \\
\\
\sum_{j=0}^{\infty} \frac{(2)^j h^j}{j!} - y_n^j - y_n - (2h) y_n' - 2h^2 y_n'' - \frac{173h^3}{945} y_n''' \\
- \frac{2187}{2800} \sum_{j=0}^{\infty} \frac{(\frac{1}{3})^j h^{j+3}}{j!} y_n^{j+3} - \frac{29}{70} \sum_{j=0}^{\infty} \frac{(1)^j h^{j+3}}{j!} y_n^{j+3} \\
+ \frac{784627135079659637}{13287189756292988032} \sum_{j=0}^{\infty} \frac{(\frac{3}{5})^j h^{j+3}}{j!} y_n^{j+2} - \frac{53}{3675} \sum_{j=0}^{\infty} \frac{(2)^j h^{j+3}}{j!} y_n^{j+3} \\
+ \frac{1}{2160} \sum_{j=0}^{\infty} \frac{(3)^j h^{j+3}}{j!} y_n^{j+3} \\
\\
\sum_{j=0}^{\infty} \frac{(3)^j h^j}{j!} - y_n^j - y_n - (3h) y_n' - \frac{9h^2}{2} y_n'' - \frac{81h^3}{224} y_n''' \\
- \frac{59049}{25600} \sum_{j=0}^{\infty} \frac{(\frac{1}{3})^j h^{j+3}}{j!} y_n^{j+3} - \frac{8991}{4480} \sum_{j=0}^{\infty} \frac{(1)^j h^{j+3}}{j!} y_n^{j+3} \\
+ \frac{321746164578602927273}{478338831226547569152} \sum_{j=0}^{\infty} \frac{(\frac{3}{5})^j h^{j+3}}{j!} y_n^{j+2} - \frac{19197}{39200} \sum_{j=0}^{\infty} \frac{(2)^j h^{j+3}}{j!} y_n^{j+3} \\
- \frac{279}{35840} \sum_{j=0}^{\infty} \frac{(3)^j h^{j+3}}{j!} y_n^{j+3}
\end{bmatrix}
$$

Comparing the coefficients to power of h yields,

$$\bar{C_1} = \bar{C_2} = \cdots = \bar{C_{10}} = 0, \quad \text{and } \bar{C_{11}} \neq 0.$$

This yields, the main method

$$\hat{B} = [y_{n+\frac{1}{3}}, y_{n+\frac{3}{5}}, y_{n+1}, y_{n+2}, y_{n+3}]^T$$

having order $[8, 8, 8, 8, 8]^T$

3.2. Zero Stability

The hybrid block method $\hat{B}$ is said to be zero stable if the first characteristic polynomial $\pi(x)$ having roots such that $|x_z| = 1$, and if $|x_z| = 1$, then, the multiplicity of x_z must not exceed three. This is clear below,

$$\begin{aligned}
\Pi(x) &= |x I - \hat{B}| \\
&= \left| x \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| \\
&= x^4(x-1)
\end{aligned}$$

which implies $x = 0, 0, 0, 0, 1$. Hence, our method is zero stable for all $s, r \in (0, 3)$

3.3. Consistency

The hybrid block method $\hat{B}$ is said to be consistent if its order greater than or equal one i.e. $P \geq 1$. This proves that our method is consistent for all $s, r \in (0, 3)$.

3.4. Convergence

Theorem 3.2. thm1 Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent.

Since the method is consistent and zero stable, it implies the method is convergent for all $s, r \in (0, 3)$.

3.5. Numerical Results

In finding the accuracy of our methods, the following third order ODEs are examined. The new block methods solved the same problems the existing methods solved in order to compare results in terms of error.

Problem 1: $y^{'''} = 2y^{''} + 3y^{'} - 10y + 34xe^{-2x} - 16e^{-2x} - 10x^2 + 6x + 34,\ y(0) = 3,$

$y^{'}(0) = 0,\ y^{''}(0) = 0$

Exact solution: $y(x) = 1 + \frac{1}{2}ln\left(\frac{2+x}{2-x}\right)$

Problem 2: $y^{'''} + y^{'} = 0, \quad y(0) = 0, \quad y^{'}(0) = 1, y^{''}(0) = 2.$

Exact solution: $y(x) = 1 - e^{x}$

Problem 3: $y^{'''} - 3\sin x = 0, \quad y(0) = 1, \quad y^{'}(0) = 0, y^{''}(0) = -2.$

Exact solution: $y(x) = 1 - e^{x}$

Table 1: Comparison of the new method with some existing methods for solving problem 1

h	Method	Error at x=1.
	New method	$7.56e^{-12}$
0.0125	Adams(2014)	$3.56e^{-8}$
	Awoyemi(2005)	$7.60e^{-6}$
	New method	$1.19e^{-13}$
0.00625	Adams(2014)	$3.23e^{-9}$
	Awoyemi(2005)	$9.54e^{-7}$
h		Error at x=4.
	New method	$5.80e^{-10}$
0.01	Adams(2014)	$2.91e^{-6}$
	Awoyemi(2003)	$1.16e^{-3}$

Table 2: Comparison of the new method with some existing methods for solving problem 2

h	Method	Error at x=1.
	New method	$1.62e^{-10}$
0.01	Awoyemi(2006)	$1.07e^{-6}$
	Adesanya(2012)	$5.14e^{-5}$
h		Error at x=5.
	New method	$1.41e^{-12}$
0.025	Awoyemi(2003)	$3.53e^{-6}$
	Adams(2014)	$9.72e^{-8}$

Table 3: Comparison of the new method with some existing methods for solving problem 2

h	Method	Error at $x = 1.$
	New method	$1.11e^{-15}$
0.01	Adams(2014)	$6.40e^{-10}$
	Adesanya(2012)	$1.75e^{-14}$

4. Conclusion

We have developed a new hybrid block method for numerically approximating the solution of third order initial value problems. This method has satisfied possessing properties that will prove its convergence when employed to solve third order ODEs as seen in the numerical results above. It is clear to see that, the new hybrid block method outperformed the existing methods in terms of error.

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