1-movable Independent Outer-connected Domination in Graphs

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Abstract

A nonempty subset $S$ of $V(G)$ is a 1-movable independent outer-connected dominating set of $G$ if $S$ is an independent outer-connected dominating set of $G$ and for every $v \in S$, there exists $u \in (V(G) \setminus S) \cap N_G(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is an independent outer-connected dominating set of $G$. The cardinality of the smallest 1-movable independent outer-connected dominating set of $G$ is called 1-movable independent outer-connected domination number of $G$ denoted by $\tilde{\gamma}_{mic}^1(G)$. A 1-movable independent outer-connected dominating set with cardinality equal to $\tilde{\gamma}_{mic}^1(G)$ is called $\tilde{\gamma}_{mic}^1$-set of $G$. This paper investigates the properties of the 1-movable independent outer-connected dominating set in graphs through characterization of those sets in the join and corona of graphs and the corresponding values or bounds of the parameter are determined.

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1. Introduction

Let $G = (V(G), E(G))$ be a graph and $v \in V(G)$. The open neighborhood of $v$ is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood of $v$ is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. If $S \subseteq V(G)$, then the open neighborhood of $S$ is the set $N_G(S) = N(S) = \cup_{u \in S} N_G(u)$ and the closed neighborhood of $S$ is the set $N_G[S] = N[S] = S \cup N(S)$.

A subset $S$ of $V(G)$ is a dominating set of $G$ if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. The domination number of $G$ denoted by $\gamma(G)$ is the cardinality of the smallest dominating set of $G$. A dominating set of $G$ with cardinality equal to $\gamma(G)$ is called a $\gamma$-set of $G$. The join and corona of a dominating set were investigated in [4].

An independent dominating set of $G$ is an independent set of $G$ if for every two elements $x, y \in S$, $xy \notin E(G)$. The independence number of $G$, denoted by $\beta(G)$, is the largest cardinality of an independent set in $G$. A nonempty subset $S$ of $V(G)$ is an independent dominating set of $G$ if $S$ is both an independent set and a dominating set of $G$. The independent domination number of $G$, denoted by $\gamma_i(G)$, is the cardinality of the smallest independent dominating set of $G$. An independent dominating set of $G$ with cardinality equal to $\gamma_i(G)$ is called a $\gamma_i$-set of $G$. Independent dominating sets and its variant were investigated in [2] and [13].

In 2007, Cyman defined outer-connected domination in graphs. A dominating set $S$ is an outer-connected dominating set of $G$ if the subgraph $\langle V(G) \setminus S \rangle$ is connected. The outer-connected domination number of $G$ denoted by $\gamma_c(G)$, is the cardinality of the smallest outer-connected dominating set of $G$. An outer-connected dominating set of $G$ with cardinality equal to $\gamma_c(G)$ is called $\gamma_c$-set of $G$. Outer-connected dominating sets in graphs were investigated in [3] and [5].

A dominating set $S$ is an independent outer-connected dominating set of $G$ if $S$ is an independent dominating set of $G$ and the subgraph $\langle V(G) \setminus S \rangle$ is connected. The independent outer-connected domination number of $G$ denoted by $\gamma_{ic}(G)$, is the cardinality of the smallest independent outer-connected dominating set of $G$. An independent outer-connected dominating set of $G$ with cardinality equal to $\gamma_{ic}(G)$ is called $\gamma_{ic}$-set of $G$.

In 2011, Blair, Gera and Horton introduced 1-movable domination in graphs. A nonempty set $S \subseteq V(G)$ is a 1-movable dominating set of $G$ if $S$ is a dominating set of $G$ and for every $v \in S$, $S \setminus \{v\}$ is a dominating set of $G$ or there exists a vertex $u \in (V(G) \setminus S) \cap N_G(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G$. The 1-movable domination number of a graph $G$, denoted by $\gamma_m^1(G)$, is the cardinality of the smallest 1-movable dominating set of $G$. A 1-movable dominating set of $G$ with cardinality equal to $\gamma_m^1(G)$ is called $\gamma_m^1$-set of $G$. This concept was investigated in [1] and [6]. Moreover, the movability of some variants of domination were investigated in [7], [8], [9], [10], [11] and [12].

A nonempty subset $S$ of $V(G)$ is a 1-movable independent outer-connected dominating set of $G$ if $S$ is an independent outer-connected dominating set of $G$ and for every $v \in S$, there exists $u \in (V(G) \setminus S) \cap N_G(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is an indepen-
dent outer-connected dominating set of $G$. The cardinality of the smallest 1-movable independent outer-connected dominating set of $G$ is called 1-movable independent outer-connected domination number of $G$ and is denoted by $\tilde{\gamma}_{mic}^1(G)$. A 1-movable independent outer-connected dominating set with cardinality equal to $\tilde{\gamma}_{mic}^1(G)$ is called $\tilde{\gamma}_{mic}^1$-set of $G$.

The graph in the figure shows that the set $S_1 = \{a\}$ of a graph $G$ is a $\tilde{\gamma}_{ic}$-set of $G$. Thus, $\tilde{\gamma}_{ic}(G) = |S_1| = 1$. Moreover, the set $S_2 = \{e, f\}$ is a $\tilde{\gamma}_{mic}^1$-set of $G$. Hence, $\tilde{\gamma}_{mic}^1(G) = |S_2| = 2$.

2. Results

A 1-movable independent outer-connected dominating set does not always exist in a connected nontrivial graph $G$. Here, we denote $\tilde{\mathcal{R}}_{mic}^1$ be the family of all graphs with a 1-movable independent outer-connected dominating set. In this paper, it is considers all connected nontrivial graphs which belong to the family $\tilde{\mathcal{R}}_{mic}^1$.

**Remark 2.1.** Let $G$ be a connected nontrivial graph of order $n \geq 2$. Then $1 \leq \tilde{\gamma}_{mic}^1(G) \leq \beta(G)$.

**Theorem 2.2.** Let $G$ be a connected nontrivial graph of order $n \geq 2$. Then, $\tilde{\gamma}_{mic}^1(G) = 1$ if and only if one of the following holds:

(i) $G$ is a complete graph of order 2; or

(ii) there exists two adjacent vertices $x, y$ and each dominates $G$.

**Proof.** Let $G$ be a connected nontrivial graph of order $n \geq 2$. Assume that $\tilde{\gamma}_{mic}^1(G) = 1$. Then the graph $G$ has a 1-movable independent outer-connected dominating set of $G$, say $S$. If $|V(G)| = 2$, then $G$ is a complete graph of order 2. Suppose $|V(G)| \geq 3$. Let $S = \{a\}$ be a 1-movable independent outer-connected dominating set of $G$. Since $S$ is a 1-movable independent outer-connected dominating set of $G$, there exists $b \in (V(G) \setminus S) \cap N_G(a)$ such that $(S \setminus \{a\}) \cup \{b\} = \{b\}$ is an independent outer-connected dominating set of $G$. So, $ab \in E(G)$ and each vertex $a$ and $b$ dominates $G$.

For the converse, suppose that (i) holds. Then $G = K_2$. Let $V(K_2) = \{x, y\}$ and suppose $S = \{x\}$. Then $S$ is an independent dominating set of $G$ and $(V(G) \setminus S) = \{y\}$ is connected. Hence, $S$ is an independent outer-connected dominating set of $G$. Moreover,
Theorem 2.4. \( S \setminus \{x\} \cup \{y\} \) is an independent dominating set and \( (V(G) \setminus (S \setminus \{x\} \cup \{y\})) = \{\{x\}\} \) is connected. Hence, \( S \setminus \{x\} \cup \{y\} \) is an independent outer-connected dominating set of \( G \). Thus, \( S \) is a 1-movable independent outer-connected dominating set of \( G \). Since \( |S| = 1 \), \( S \) is a \( \gamma_{mic} \)-set by Remark 2.1. Therefore, \( \gamma_{mic}^{-1}(G) = |S| = 1 \). Suppose (ii) holds. Let \( x, y \) be the vertices with \( xy \in E(G) \) and each dominates \( G \) and suppose \( S = \{x\} \). Then, \( S \) is an independent dominating set of \( G \) and \( (V(G) \setminus S) = \{\{y\}\} + (V(G) \setminus \{x, y\}) \) is connected. Hence, \( S \) is an independent outer-connected dominating set of \( G \). Also, since \( y \) dominates \( G \), \( S \setminus \{x\} \cup \{y\} \) is an independent dominating set of \( G \) and \( (V(G) \setminus (S \setminus \{x\} \cup \{y\})) = \{\{x\}\} + (V(G) \setminus \{x, y\}) \) is connected. Hence, \( S \) is an independent outer-connected dominating set of \( G \). Thus, \( S \) is a 1-movable independent outer-connected dominating set of \( G \). Since \( |S| = 1 \), by Remark 2.1, \( \gamma_{mic}^{-1}(G) = |S| = 1 \).

\[ \square \]

Corollary 2.3. For any complete graph \( K_n \) of order \( n \geq 2 \), \( \gamma_{mic}^{-1}(K_n) = 1 \).

The join of two graphs \( G \) and \( H \) denoted by \( G + H \) is the graph with vertex-set \( V(G + H) = V(G) \cup V(H) \) and edge-set \( E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\} \). The next results characterizes the 1-movable independent outer-connected dominating sets in the join of two connected graphs and the exact values or bounds of the parameter are determined.

Theorem 2.4. Let \( G \) and \( H \) be connected nontrivial graphs. A subset \( S \) of \( V(G + H) \) is a 1-movable independent outer-connected dominating set of \( G + H \) if and only if one of the following holds:

(i) \( S \) is an independent dominating set of \( G \) such that

(a) if \( |S| = 1 \), then either \( S \) is a 1-movable independent dominating set of \( G \) or there exists \( z \in V(H) \) that dominates \( H \); and

(b) if \( |S| \geq 2 \), then \( S \) is a 1-movable independent dominating set of \( G \)

(ii) \( S \) is an independent dominating set of \( H \) such that

(a) if \( |S| = 1 \), then either \( S \) is a 1-movable independent dominating set of \( H \) or there exists \( w \in V(G) \) that dominates \( G \); and

(b) if \( |S| \geq 2 \), then \( S \) is a 1-movable independent dominating set of \( H \).

Proof. Assume that \( S \) is a 1-movable independent outer-connected dominating set of \( G + H \). Then either \( S \subseteq V(G) \) or \( S \subseteq V(H) \). Suppose \( S \subseteq V(G) \). Then \( S \) is an independent dominating set of \( G \). Suppose \( |S| = 1 \), say \( S = \{a\} \) for some \( a \in V(G) \). Since \( S \) is a 1-movable outer-connected dominating set of \( G + H \), there exists \( u \in (V(G + H) \setminus S) \cap N(a) \) such that \( (S \setminus \{a\}) \cup \{u\} \) is an independent outer-connected dominating set of \( G + H \). If \( u \in V(G) \), then \( S \) is a 1-movable independent dominating set of \( G \). If \( u \in V(H) \), then \( u \) dominates \( H \). Suppose that \( |S| \geq 2 \) and let \( v \in S \). Since \( S \) is an independent set, and \( S \subseteq V(G) \). Since \( S \) is a 1-movable independent
outer-connected dominating set of $G + H$, there exists $u \in (V(G + H) \setminus S) \cap N(v)$ such that $S \setminus \{v\} \cup \{u\}$ is an independent outer-connected dominating set of $G + H$. Moreover, $S$ being an independent set, it follows that $u \in V(G) \setminus S$. Hence, $S$ is a 1-movable independent dominating set of $G$. Thus, (i) holds. Similarly, (ii) holds if $S \subseteq V(H)$.

For the converse suppose (i) holds. Then $S$ is an independent dominating set of $G + H$. Suppose $|S| = 1$, say $S = \{x\}$ for some $x \in V(G)$. Then $(V(G + H) \setminus S) = (V(G) \setminus S) + H$ is connected. Hence $S$ is an independent outer-connected dominating set of $G + H$. Let $v \in S$. By assumption, suppose first that $S$ is a 1-movable independent dominating set of $G$. Then there exists $u \in (V(G) \setminus S) \cap N_G(x)$ such that $S \setminus \{x\} \cup \{u\} = \{u\}$ is an independent dominating set of $G$ and hence of $G + H$ and the subgraph $(V(G + H) \setminus \{u\}) = (V(G) \setminus \{u\}) + H$ is connected. Hence, $(S \setminus \{x\}) \cup \{u\} = \{u\}$ is an independent outer-connected dominating set of $G + H$. Suppose $|S| \geq 2$. Then $(V(G + H) \setminus S) = (V(G) \setminus S) + H$ is connected. Hence $S$ is an independent outer-connected dominating set of $G + H$. Let $v \in S$. By assumption, there exists $u \in (V(G) \setminus S) \cap N_G(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is an independent dominating set of $G + H$. Moreover, $(V(G + H) \setminus (S \setminus \{v\})) \cup \{u\} = (V(G) \setminus (S \setminus \{v\})) \cup \{u\} + H$ is connected. Thus, $(S \setminus \{v\}) \cup \{u\}$ is an independent outer-connected dominating set of $G + H$. Hence, in either case, $S$ is a 1-movable independent outer-connected dominating set of $G + H$. Similarly, if (ii) holds, then $S \subseteq V(H)$ is a 1-movable independent outer-connected dominating set of $G + H$.

**Corollary 2.5.** Let $G$ and $H$ be connected nontrivial graphs. Then

\[
\tilde{\gamma}_{mic}^1(G + H) = \begin{cases} 
1 & \text{if } \gamma(G) = 1 = \gamma(H) \text{ or } \gamma_{mi}^1(G) = 1 \text{ or } \gamma_{mi}^1(H) = 1 \\
\min \{\gamma_{mi}^1(G), \gamma_{mi}^1(H)\} & \text{otherwise.}
\end{cases}
\]

**Proof.** Let $G$ and $H$ be connected nontrivial graphs. Then consider the following cases:

**Case 1:** $\gamma(G) = 1 = \gamma(H)$.
Let $S = \{a\}$ be a $\gamma$-set of $G$ and $\{z\}$ be a $\gamma$-set of $H$. By Theorem 2.4 (ia), $S$ is a 1-movable independent outer-connected dominating set of $G + H$. So, $1 \leq \tilde{\gamma}_{mic}^1(G + H) \leq |S| = 1$. Thus $\tilde{\gamma}_{mic}^1(G + H) = 1$.

**Case 2:** $\gamma_{mi}^1(G) = 1$ or $\gamma_{mi}^1(H) = 1$.
Let $S = \{a\}$ be a $\gamma_{mi}$-set of $G$. By Theorem 2.4 (ia), $S$ is a 1-movable independent outer-connected dominating set of $G + H$. So, $1 \leq \tilde{\gamma}_{mic}^1(G + H) \leq |S| = 1$. So, $\tilde{\gamma}_{mic}^1(G + H) = 1$. Similarly, if $D = \{w\}$ is a $\tilde{\gamma}_{mic}^1(H)$, then by Theorem 2.4 (iiia), $S$ is a 1-movable independent outer-connected dominating set of $G + H$. So, $1 \leq \tilde{\gamma}_{mic}^1(G + H) \leq |S| = 1$. Thus, $\tilde{\gamma}_{mic}^1(G + H) = 1$.

**Case 3:** $\gamma_{mi}^1(G) \geq 2$ and $\gamma_{mi}^1(H) \geq 2$.
Let $S$ be a $\gamma_{mi}$-set of $G$ with $|S| \geq 2$. By Theorem 2.4 (ib), $S$ is a 1-movable independent
outer-connected dominating set of $G + H$. Also, if $S$ is a $\gamma_{mi}^1$-set of $H$, then by Theorem 2.4(ii), $S$ is a 1-movable independent outer-connected dominating set of $G + H$. Hence, 
\[ \widetilde{\gamma}_{mic}^1(G + H) \leq \min \{ \gamma_{mi}^1(G), \gamma_{mi}^1(H) \}. \]

Suppose that $S$ is a $\gamma_{mi}^1$-set of $G + H$. By Theorem 2.4(iii), $S$ is a 1-movable independent dominating set of $G$ or by Theorem 2.4(ii), $S$ is a 1-movable independent dominating set of $H$. Hence, \( \widetilde{\gamma}_{mic}^1(G + H) = |S| \geq \min \{ \gamma_{mi}^1(G), \gamma_{mi}^1(H) \} \). Therefore, \( \widetilde{\gamma}_{mic}^1(G + H) = \min \{ \gamma_{mi}^1(G), \gamma_{mi}^1(H) \} \).

**Theorem 2.6.** Let $H$ be a connected nontrivial graph. Then $S \subseteq V(K_1 + H)$ is a 1-movable independent outer-connected dominating set of $K_1 + H$ if and only if one of the following holds:

(i) $S = V(K_1)$ and there exists $a \in V(H)$ that dominates $H$.

(ii) $S$ is an independent dominating set of $H$ such that if $|S| \geq 2$, then $S$ is a 1-movable independent dominating set of $H$.

**Proof.** Suppose $S$ is a 1-movable independent outer-connected dominating set in $K_1 + H$ and $S = V(K_1) = \{x\}$. Then there exists $a \in V(H)$ such that $(S \setminus V(K_1)) \cup \{a\} = \{a\}$ is an independent outer-connected dominating set of $K_1 + H$. Thus, $a \in V(H)$ dominates $H$. Suppose $S \neq V(K_1)$. Then $S \subseteq V(H)$ and $S$ is an independent dominating set in $H$. Suppose $|S| \geq 2$. Since $S$ is a 1-movable independent outer-connected dominating set of $K_1 + H$, for all $v \in S$, there exists $u \in (V(K_1 + H) \setminus S) \cap N_G(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is an independent outer-connected dominating set of $K_1 + H$. Since $S$ is an independent set, $u \neq x$. Hence $u \in (V(H) \setminus S) \cap N_G(v)$ for all $v \in S$. Therefore, $S$ is a 1-movable independent dominating set of $H$.

For the converse, suppose (i) holds. Then $S = V(K_1) = \{x\}$ is an independent dominating set of $K_1 + H$ and $(V(K_1 + H) \setminus V(K_1)) = H$ is connected. Hence, $S$ is an independent outer-connected dominating set of $K_1 + H$. By assumption, there exists $a \in V(H)$ that dominates $H$. Thus, $(S \setminus \{x\}) \cup \{a\} = \{a\}$ is an independent dominating set of $K_1 + H$ and $(V(K_1 + H) \setminus \{a\}) = K_1 + (V(H) \setminus \{a\})$ is connected. Hence, $(S \setminus \{x\}) \cup \{a\} = \{a\}$ is an independent outer-connected dominating set of $K_1 + H$. This concludes that $S$ is a 1-movable independent outer-connected dominating set of $K_1 + H$. Suppose (ii) holds. Then $S$ is an independent dominating set of $K_1 + H$ and $(V(K_1 + H) \setminus S) = K_1 + (V(H) \setminus S)$ is connected. Thus $S$ is an independent outer-connected dominating set of $K_1 + H$. Suppose $|S| = 1$, say $S = \{w\}$ for some $w \in V(H)$. Then $(S \setminus \{w\}) \cup \{x\} = \{x\}$ is an independent dominating set of $K_1 + H$ and $(V(K_1 + H) \setminus \{x\}) = H$ is connected. Thus, $(S \setminus \{w\}) \cup \{x\} = \{x\}$ is an independent outer-connected dominating set of $K_1 + H$. Suppose $|S| \geq 2$. Let $v \in S$. By assumption, $S$ is a 1-movable independent dominating set of $H$. Hence there exists $u \in (V(H) \setminus S) \cap N_H(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is an independent dominating set of $H$ and hence of $K_1 + H$ and $(V(K_1 + H) \setminus S) = K_1 + (V(H) \setminus (S \setminus \{v\}) \cup \{u\})$ is connected. Therefore, $S$ is a 1-movable independent outer-connected dominating set of $K_1 + H$. \[\square\]
Corollary 2.7. Let $H$ be a connected nontrivial graph. Then

$$\tilde{\gamma}_{\text{mic}}^1(K_1 + H) = \begin{cases} 
1 & \text{if } \gamma(H) = 1 \\
\gamma_{\text{mi}}^1(H) & \text{if } \gamma(H) \neq 1.
\end{cases}$$

Proof. Consider the following cases:

Case 1: $\gamma(H) = 1$

Let $S = V(K_1)$. Since $\gamma(H) = 1$, there exists $a \in V(H)$ that dominates $H$. Hence $S$ is a 1-movable independent outer-connected dominating set of $K_1 + H$ by Theorem 2.6(i). Hence, $1 \leq \tilde{\gamma}_{\text{mic}}^1(K_1 + H) \leq |S| = 1$. So, $\tilde{\gamma}_{\text{mic}}^1(K_1 + H) = 1$. Similarly, if $S = \{z\}$ is a $\gamma$-set of $H$, then $S$ is a 1-movable independent outer-connected dominating set of $K_1 + H$ by Theorem 2.6(ii). Thus, $1 \leq \tilde{\gamma}_{\text{mic}}^1(K_1 + H) \leq |S| = 1$. Therefore, $\gamma_{\text{mic}}^1(K_1 + H) = 1$.

Case 2: $\gamma(H) \neq 1$

Let $S$ be a $\tilde{\gamma}_{\text{mic}}^1$-set of $K_1 + H$. Since $\gamma(H) \neq 1$, $|S| \geq 2$. By Theorem 2.6(ii), $S$ is a 1-movable independent dominating set of $H$. Hence, $\tilde{\gamma}_{\text{mic}}^1(K_1 + H) = |S| \geq \gamma_{\text{mi}}^1(H)$. On the other hand, suppose that $S$ is a $\gamma_{\text{mi}}^1$-set of $H$. By Theorem 2.6(ii), $S$ is a 1-movable independent outer-connected dominating set of $K_1 + H$. Hence, $\tilde{\gamma}_{\text{mic}}^1(K_1 + H) \leq |S| = \gamma_{\text{mi}}^1(H)$. Therefore, $\tilde{\gamma}_{\text{mic}}^1(K_1 + H) = \gamma_{\text{mi}}^1(H)$. \hfill \blacksquare

The corona of two graphs $G$ and $H$, denoted by $G \circ H$, is the graph obtained by taking one copy of $G$ of order $n$ and $n$ copies of $H$, and then joining the $i$th vertex of $G$ to every vertex in the $i$th copy of $H$. For every $v \in V(G)$, we denote by $H^v$ the copy of $H$ whose vertices are joined or attached to the vertex $v$. The next results present the characterization of the 1-movable independent outer-connected dominating sets in the corona of two connected nontrivial graphs and the exact values or bounds of the corresponding parameter are determined.

Theorem 2.8. Let $G$ and $H$ be connected nontrivial graphs of orders $n \geq 2$ and $m \geq 2$, respectively. A subset $C$ of $V(G \circ H)$ is a 1-movable independent outer-connected dominating set of $G \circ H$ if and only if $C = \bigcup_{v \in V(G)} D_v$ where $D_v$ is a 1-movable independent dominating set of $H$ for each $v \in V(G)$.

Proof. Assume that $C$ is a 1-movable independent outer-connected dominating set of $G \circ H$. Suppose $C \bigcap V(G) \neq \phi$. Let $v \in C \bigcap V(G)$. Since $C$ is an independent set, $w \notin C$ for all $w \in V(H^v)$. Hence, $\langle V(H^v) \rangle$ is a component subgraph of $\langle V(G \circ H) \rangle \backslash C$ which contradicts the assumption that $\langle V(G \circ H) \rangle \backslash C$ is connected. Hence, $C \bigcap V(G) = \phi$. This leads to a conclusion that $C \bigcap V(H^v) \neq \phi$ for all $v \in V(G)$. Let $D_v = C \bigcap V(H^v)$ for each $v \in V(G)$. Since $C$ is a 1-movable independent outer-connected dominating set of $G \circ H$, by Theorem 2.6(ii), $D_v$ is a 1-movable independent dominating set of $H^v$ for each $v \in V(G)$.\hfill \blacksquare
For the converse, suppose \( C = \bigcup_{v \in V(G)} D_v \) where \( D_v \) is a 1-movable independent dominating set of \( H^v \) for each \( v \in V(G) \). Then, \( C = \bigcup_{v \in V(G)} D_v \) is an independent dominating set of \( G \circ H \) and \( \langle V(G \circ H) \setminus C \rangle = G \circ \langle V(H^v) \setminus D_v \rangle \) is connected. Hence, \( C = \bigcup_{v \in V(G)} D_v \) is an independent outer-connected dominating set of \( G \circ H \). Let \( w \in C \). Then \( w \in D_v \) for all \( v \in V(G) \). Since \( D_v \) is a 1-movable independent dominating set of \( H^v \), there exists \( u \in V(H^v) \setminus D_v \cap N(w) \) such that \( C \setminus \{w\} \cup \{u\} \) is an independent dominating set of \( G \circ H \) and the subgraph \( \langle V(G \circ H) \setminus (C \setminus \{w\} \cup \{u\}) \rangle = G \circ \langle V(H^v) \setminus (D_v \setminus \{w\}) \cup \{u\} \rangle \) is connected. Thus, \( C \setminus \{w\} \cup \{u\} \) is an independent outer-connected dominating set of \( G \circ H \). Therefore, \( C \) is a 1-movable independent outer-connected dominating set of \( G \circ H \). □

**Corollary 2.9.** Let \( G \) and \( H \) be connected nontrivial graphs of orders \( m \geq 2 \) and \( n \geq 2 \), respectively. Then, \( \tilde{\gamma}^1_{mic}(G \circ H) = m \gamma^1_{mi}(H) \).

**Proof.** Suppose that \( C \) is a \( \tilde{\gamma}^1_{mic} \)-set of \( G \circ H \). By Theorem 2.8, \( C = \bigcup_{v \in V(G)} D_v \) where \( D_v \) is a 1-movable independent dominating set of \( H^v \) for each \( v \in V(G) \). Hence, \( \tilde{\gamma}^1_{mic}(G \circ H) = |C| = |\bigcup_{v \in V(G)} D_v| = |V(G)| |D_v| \geq m \gamma^1_{mi}(H) \). Also, suppose \( D \) is a \( \gamma^1_{mi} \)-set of \( H \). For each \( v \in V(G) \), let \( D_v \cong D \). Then, \( C = \bigcup_{v \in V(G)} D_v \) is a 1-movable independent outer-connected dominating set of \( G \circ H \). Hence, \( \tilde{\gamma}^1_{mic}(G \circ H) \leq |C| = \bigg| \bigcup_{v \in V(G)} D_v \bigg| = |V(G)| |D_v| = m \gamma^1_{mi}(H) \). Therefore, \( \tilde{\gamma}^1_{mic}(G \circ H) = m \gamma^1_{mi}(H) \). □

**References**


