Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 1 (2017), pp. 1-7

http://www.ripublication.com/gjpam.htm

# Independent Outer-connected Domination in Graphs

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**Abstract**

A dominating set $S$ is an *independent outer-connected dominating set* of $G$ if $S$ is an independent dominating set of $G$ and the subgraph $\langle V(G) \setminus S \rangle$ is connected. The *independent outer-connected domination number* of $G$ denoted by $\widetilde{\gamma}_{ic}(G)$, is the cardinality of the smallest independent outer-connected dominating set of $G$. An independent outer-connected dominating set of $G$ with cardinality equal to $\widetilde{\gamma}_{ic}(G)$ is called $\widetilde{\gamma}_{ic}$*-set* of $G$. This paper presents some properties of the independent outer-connected dominating sets in a connected nontrivial graph. It determines the bounds of the independent outer-connected domination number and characterized those graphs which attained the bounds. It also characterizes the independent outer-connected dominating set in the join and corona of graphs and the corresponding values of the parameter are determined.

**AMS subject classification:** 05C69.
**Keywords:** Domination, independent domination, outer-connected domination, independent outer-connected domination, join and corona.

## 1. Introduction

Let $G = (V(G), E(G))$ be a graph and $v \in V(G)$. The *open neighborhood* of $v$ is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of $v$ is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. If $S \subseteq V(G)$, then the *open neighborhood* of $S$ is the set $N_G(S) = N(S) = \cup_{v \in S} N_G(v)$ and the *closed neighborhood* of $S$ is the set $N_G[S] = N[S] = S \cup N(S)$.

A subset $S$ of $V(G)$ is a *dominating set* of $G$ if for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. The *domination number* of $G$ denoted by $\gamma(G)$ is the smallest cardinality of a dominating set of $G$. A dominating set of $G$ with cardinality equal to $\gamma(G)$ is called a $\gamma$-set of $G$. The binary operations of a dominating set were investigated in [3] and a thorough investigation of this type of set generate some variants of domination.

A subset $S$ of $V(G)$ is an *independent set* of $G$ if for every two elements $x, y \in S$, $xy \notin E(G)$. The *independence number* of $G$, denoted by $\beta(G)$, is the largest cardinality of an independent set in $G$. A nonempty subset $S$ of $V(G)$ is an *independent dominating set* of $G$ if $S$ is both an independent set and a dominating set of $G$. The *independent domination number* of $G$, denoted by $\gamma_i(G)$, is the smallest cardinality of an independent dominating set of $G$. An independent dominating set of $G$ with cardinality equal to $\gamma_i(G)$ is called a $\gamma_i$*-set* of $G$. Independent dominating sets and its variants were investigated in [1] and [5].

Another variant of domination was the introduced and investigated by Cyman, et al. (2007) called the outer-connected domination in graphs. A dominating set $S$ is an *outer-connected dominating set* of $G$ if the subgraph $\langle V(G) \setminus S \rangle$ is connected. The *outer-connected domination number* of $G$ denoted by $\widetilde{\gamma}_c(G)$, is the smallest cardinality of an outer-connected dominating set of $G$. An outer-connected dominating set of $G$ with cardinality equal to $\widetilde{\gamma}_c(G)$ is called $\widetilde{\gamma}_c(G)$-set of $G$. Outer-connected dominating sets in graphs and its variants were investigated and developed in [2] and [4]. In their respective research, they were able to determine the bounds and the exact values of the outer-connected domination numbers in generalized and special types of graphs.

A dominating set $S$ is an *independent outer-connected dominating set* of $G$ if $S$ is an independent dominating set of $G$ and the subgraph $\langle V(G) \setminus S \rangle$ is connected. The *independent outer-connected domination number* of $G$ denoted by $\widetilde{\gamma}_{ic}(G)$, is the smallest cardinality of an independent outer-connected dominating set of $G$. An independent outer-connected dominating set of $G$ with cardinality equal to $\widetilde{\gamma}_{ic}(G)$ is called $\widetilde{\gamma}_{ic}$-set of $G$.

The next section presents some properties of the independent outer-connected dominating sets in a connected nontrivial graph. It determines the bounds of the independent outer-connected domination number and characterized those graphs which attained the bounds. It also characterizes the independent outer-connected dominating set in the join and corona of graphs and the corresponding values of the parameter are determined.

## 2. Results

The independent outer-connected dominating set does not always exist in a connected nontrivial graph $G$. Define $\widetilde{\mathscr{R}}_{ic}$ be the family of all graphs with independent outer-connected dominating set. In this study, it is assumed that all connected nontrivial graphs $G$ belong to the family $\widetilde{\mathscr{R}}_{ic}$.

**Remark 2.1.** For any connected nontrivial graph $G$, $1 \leq \widetilde{\gamma}_{ic}(G) \leq n-1$ and the bounds are sharp.

Consider the graphs $K_3$ and $K_{1,n-1}$ in Figure 1.

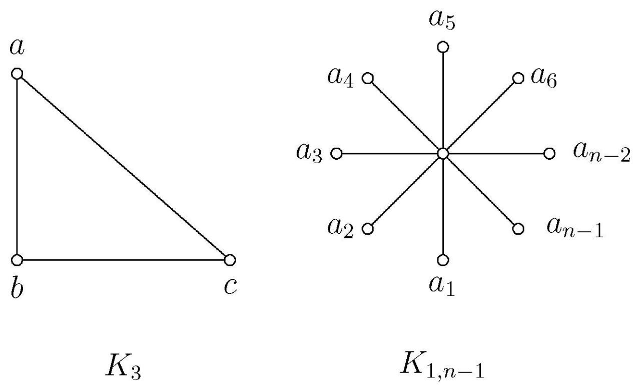


Figure 1: Graphs $K_3$ and $K_{1,n-1}$

The sets $S_1 = \{a\}$ and $S_2 = \{a_1, a_2, a_3, \ldots, a_{n-1}\}$ are independent dominating sets in $K_3$ and $K_{1,n-1}$, respectively, and $\langle V(K_3) \setminus S_1 \rangle$ and $\langle V(K_{1,n-1}) \setminus S_2 \rangle$ are connected subgraphs. Hence, $S_1$ and $S_2$ are independent outer-connected dominating sets in $K_3$ and $K_{1,n-1}$ respectively. Since $|S_1| = 1$, it follows that $\widetilde{\gamma}_{ic}(K_3) = 1$. Furthermore, since there is no independent outer-connected dominating set in $K_{1,n-1}$ with cardinality strictly less than $n-1$, it follows that $\widetilde{\gamma}_{ic}(K_{1,n-1}) = n-1$.

**Theorem 2.2.** Let $G$ be a connected nontrivial graph. Then $\widetilde{\gamma}_{ic}(G) = 1$ if and only if there exists a singleton $\{a\}$ that dominates $G$ and its complement induces a connected subgraph.

*Proof.* Let $G$ be a connected nontrivial graph. Suppose that $\widetilde{\gamma}_{ic}(G) = 1$. Then $G$ has an independent outer-connected dominating set $S$ with $|S| = 1$. Hence, there exists $a \in V(G)$ such that $S = \{a\}$ dominates $G$ and $V(G) \setminus \{a\}$ induces a connected subgraph of $G$.

For the converse, suppose that $S = \{a\}$ is a dominating set in $G$ and its complement induces a connected subgraph of $G$. Then $S$ is an independent outer-connected dominating set of $G$. By Remark 2.1, $\widetilde{\gamma}_{ic}(G) = |S| = 1$. ■

**Corollary 2.3.** For any complete graph $K_n$ of order $n \geq 2$, $\widetilde{\gamma}_{ic}(K_n) = 1$.

*Proof.* Let $K_n$ be a complete graph of order $n \geq 2$. Let $S = \{a\} \subseteq V(K_n)$. Then $S = \{a\}$ dominates $K_n$ and $\langle V(K_n) \setminus S \rangle$ is connected. By Theorem 2.2, $\widetilde{\gamma}_{ic}(K_n) = |S| = 1$. ■

**Theorem 2.4.** For any connected nontrivial graph $G$, $\widetilde{\gamma}_{ic}(G) = n-1$ if and only if there exists a singleton set $\{x\}$ that dominates $G$ and $\langle (V(G) \setminus \{x\}) \rangle$ is an empty subgraph of $G$.

*Proof.* Let $G$ be a connected graph of order $n \geq 2$. Suppose that $\widetilde{\gamma}_{ic}(G) = n - 1$. Then $G$ has an independent outer-connected dominating set $S$ with $|S| = n - 1$. Let $S = \{a_1, a_2, \ldots, a_{n-2}, a_{n-1}\}$ be a $\widetilde{\gamma}_{ic}$-set of $G$. Since the order of $G$ is $n$, there exists $x \in (V(G) \setminus S)$. Since $S$ is an independent dominating set of $G$, $xa_i \in E(G)$ for all $i = 1, 2, \ldots, n - 1$ and $a_i a_j \notin E(G)$ for $i \neq j$. Hence, $\{x\}$ is a singleton set that dominates $G$ and $\langle(V(G) \setminus \{x\})\rangle = \langle S\rangle$ is an empty subgraph of $G$.

For the converse, suppose that there exists a singleton set $\{x\}$ that dominates $G$ and $\langle V(G) \setminus \{x\}\rangle$ is an empty subgraph of $G$. Now suppose that $G$ has an independent outer connected dominating set of $G$, say $S$. If $S = \{x\}$, then this is a contradiction since $\langle V(G) \setminus \{x\}\rangle$ is an empty graph which is not a connected. Suppose $S = \{x, a_i\}$ where $i = 1, 2, \ldots, n-1$. Then still it is a contradiction since $xa_i \in E(G)$ for all $i = 1, 2, \ldots, n-1$, showing that $S$ is not an independent set. Suppose $S = V(G) \setminus \{x\} = \{a_1, a_2, \ldots, a_{n-1}\}$. Since $\{x\}$ dominates $G$ and since $\langle V(G) \setminus \{x\}\rangle$ is an empty subgraph of $G$, it follows that $S$ is an independent dominating set of $G$. Furthermore, since $\langle V(G) \setminus S\rangle = \langle\{x\}\rangle$ is connected, $S$ is an independent outer-connected dominating set of $G$. Since there exists no independent outer-connected dominating set $D$ of $G$ with $|D| \leq n - 2$ and since $S$ is an independent outer-connected dominating set of $G$ with the smallest cardinality, $\widetilde{\gamma}_{ic}(G) = |S| = n - 1$. ■

**Corollary 2.5.** For any star $K_{1,n-1}$, $\widetilde{\gamma}_{ic}(K_{1,n-1}) = n - 1$.

*Proof.* Suppose $V(K_{1,n-1}) = \{x, a_1, a_2, \ldots, a_{n-1}\}$ where $xa_i \in E(K_{1,n-1})$ for all $i = 1, 2, \ldots, n - 1$ and $a_i a_j \notin E(G)$ for $i \neq j$. Then the set $\{x\}$ dominates $K_{1,n-1}$ and $\langle V(K_{1,n-1}) \setminus S\rangle = \langle\{x\}\rangle$ is connected. By Theorem 2.4, $\widetilde{\gamma}_{ic}(K_{1,n-1}) = |S| = n - 1$. ■

The next theorem involves the join of two connected graphs. The *join* of two graphs $G$ and $H$ denoted by $G + H$ is the graph with vertex-set $V(G + H) = V(G) \mathbin{\dot\cup} V(H)$ and edge-set $E(G + H) = E(G) \mathbin{\dot\cup} E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

**Theorem 2.6.** Let $G$ and $H$ be connected nontrivial graphs. Then $S \subseteq V(G + H)$ is an independent outer-connected dominating set of $G + H$ if and only if $S$ is an independent dominating set in $G$ or $S$ is an independent dominating set in $H$.

*Proof.* Suppose that $S$ is an independent outer-connected dominating set of $G + H$. Since $S$ is an independent set of $G + H$, either $S \subseteq V(G)$ or $S \subseteq V(H)$. Suppose that $S \subseteq V(G)$. Then $S$ is an independent set in $G$. Since $S$ is a dominating set of $G + H$, it follows that $S$ is a dominating set in $G$. Hence, $S$ is an independent dominating set in $G$. Similarly, if $S \subseteq V(H)$, then $S$ is an independent dominating set in $H$.

For the converse, suppose first that $S$ is an independent dominating set in $G$. By definition of the join of graphs, $S$ is an independent dominating set of $G + H$. Moreover, $\langle V(G + H) \setminus S\rangle$ is connected. Hence $S$ is an independent outer-connected dominating set in $G + H$. Similarly, if $S$ is an independent dominating set of $H$, it follows that $S$ is an independent outer-connected dominating set of $G + H$. ■

**Corollary 2.7.** Let $G$ and $H$ be connected nontrivial graphs. Then $\widetilde{\gamma}_{ic}(G + H) = min\left\{\gamma_i(G), \gamma_i(H)\right\}$.

*Proof.* Suppose that $S$ is a $\widetilde{\gamma}_{ic}$-set in $G+H$. By Theorem 2.6, either $S$ is an independent dominating set in $G$ or $S$ is an independent dominating set in $H$. Suppose that $S$ is an independent dominating set in $G$, then $\widetilde{\gamma}_{ic}(G + H) = |S| \geq \gamma_i(G)$. Also, if $S$ is an independent dominating set of $H$, then $\widetilde{\gamma}_{ic}(G + H) = |S| \geq \gamma_i(H)$. Hence, $\widetilde{\gamma}_{ic}(G + H) \geq min\{\gamma_i(G), \gamma_i(H)\}$.

On the other hand, suppose that $S$ is a $\gamma_i$-set of $G$. By Theorem 2.6, $S$ is an independent outer-connected dominating set in $G+H$. Hence, $\gamma_i(G) = |S| \geq \widetilde{\gamma}_{ic}(G+H)$. Also, if $S$ is a $\gamma_i$-set of $H$, then $S$ is an independent outer-connected dominating set in $G+H$. Hence, $\gamma_i(H) = |S| \geq \widetilde{\gamma}_{ic}(G+H)$. Thus, $min\left\{\gamma_i(G), \gamma_i(H)\right\} \geq \widetilde{\gamma}_{ic}(G+H)$. Therefore, $\widetilde{\gamma}_{ic}(G + H) = min\left\{\gamma_i(G), \gamma_i(H)\right\}$. ■

**Theorem 2.8.** Let $H$ be a connected nontrivial graph. Then $S \subseteq V(K_1 + H)$ is an independent outer-connected dominating set of $K_1 + H$ if $S = V(K_1)$ or $S$ is an independent dominating set in $H$.

*Proof.* Let $H$ be a connected nontrivial graph. Suppose that $S$ is an independent outer-connected dominating set of $K_1 + H$. Since $S$ is an independent set, either $S = V(K_1)$ or $S \subseteq V(H)$. Suppose $S \neq V(K_1)$. Then $S \subseteq V(H)$. Since $S$ is an independent dominating set of $K_1 + H$, it follows that $S$ is an independent dominating set in $H$.

For the converse, suppose that $S = V(K_1)$. Then by definition of the join of graphs, $S$ is an independent dominating set in $K_1+H$ and by assumption, $\langle V(K_1 + H) \setminus V(K_1)\rangle = H$ is connected. Hence, $S$ is an independent outer-connected dominating set of $K_1 + H$. Suppose that $S$ is an independent dominating set of $H$. Then by definition of the join of graphs, $S$ is an independent dominating set of $K_1 + H$ and $\langle V(K_1 + H) \setminus S\rangle = K_1 + \langle V(H) \setminus S\rangle$ is connected. Hence, $S$ is an independent outer-connected dominating set of $K_1 + H$. ■

**Corollary 2.9.** For any connected nontrivial graph $H$, $\widetilde{\gamma}_{ic}(K_1 + H) = 1$.

*Proof.* Let $H$ be a connected nontrivial graph $H$. Choose $S = V(K_1)$. By Theorem 2.8, $S$ is an independent outer-connected dominating set of $K_1 + H$. By Remark 2.1, it follows that $\widetilde{\gamma}_{ic}(K_1 + H) = |S| = 1$. ■

The next theorem involves the corona of two connected graphs. The *corona* of two graphs $G$ and $H$, denoted by $G \circ H$, is the graph obtained by taking one copy of $G$ of order $n$ and $n$ copies of $H$, and then joining the $i$th vertex of $G$ to every vertex in the $i$th copy of $H$. For every $v \in V(G)$, we denote by $H^v$ the copy of $H$ whose vertices are joined or attached to the vertex $v$.

**Theorem 2.10.** Let $G$ and $H$ be connected nontrivial graphs of orders $m \geq 2$ and $n \geq 2$, respectively. A subset $C$ of $V(G \circ H)$ is an independent outer-connected dominating set of $G \circ H$ if and only if $C = \bigcup_{w \in V(G)} D_w$ where $D_w$ is an independent dominating set of $H^w$ for all $w \in V(G)$.

*Proof.* Suppose that $C$ is an independent outer-connected dominating set of $G \circ H$. Suppose $C \bigcap V(G) \neq \varnothing$. Let $w \in C \bigcap V(G)$. Since $C$ is an independent set, $wz \notin E(H^w)$ for all $z \in V(H^w)$. This implies that $\langle V(H^w) \rangle$ is a component of $\langle V(G \circ H) \setminus C \rangle$. This contradicts the assumption that $C$ is an outer-connected dominating set of $G \circ H$. Hence, $C \bigcap V(G) = \varnothing$. This implies that $C \bigcap V(H^w) \neq \phi$ for all $w \in V(G)$. Let $D_w = C \bigcap V(H^w)$. Since $C$ is an independent dominating set in $G \circ H$, $D_w$ is also an independent dominating set of $H^w$ for each $w \in V(G)$.

For the converse, suppose that $C = \bigcup_{w \in V(G)} D_w$ where $D_w$ is an independent dominating set of $H^w$ for each $w \in V(G)$. Then $C$ is an independent dominating set of $G \circ H$ and $\langle V(G \circ H) \setminus C \rangle = G \circ \langle V(H^w) \setminus D_w \rangle$ is connected. Therefore, $C$ is an independent outer-connected dominating set of $G \circ H$. ■

**Corollary 2.11.** Let $G$ and $H$ be connected nontrivial graphs of order $m \geq 2$ and $n \geq 2$, respectively. Then $\widetilde{\gamma}_{ic}(G \circ H) = m\gamma_i(H)$.

*Proof.* Suppose that $C$ is a $\widetilde{\gamma}_{ic}$-set of $G \circ H$. By Theorem 2.10, $C = \bigcup_{w \in V(G)} D_w$ where $D_w$ is an independent dominating set of $H^w$ for each $w \in V(G)$. Hence, $\widetilde{\gamma}_{ic}(G \circ H) = |C| = |\bigcup_{w \in V(G)} D_w| = |V(G)||D_w| \geq m\gamma_i(H)$. On the other hand, let $D$ be a $\gamma_i$-set of $H$. Then for each $w \in V(G)$, assume that $D_w \cong D$. Then by Theorem 2.10, $C = \bigcup_{w \in V(G)} D_w$ is an independent outer-connected dominating set of $G \circ H$. Hence, $\widetilde{\gamma}_{ic}(G \circ H) \leq |C| = |\bigcup_{w \in V(G)} D_w| = |V(G)||D_w| = m\gamma_i(H)$. Therefore, $\widetilde{\gamma}_{ic}(G \circ H) = m\gamma_i(H)$. ■

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