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Dual of X_Φ-frames in Banach spaces

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Abstract

Frames in Banach spaces with respect to some sequence space, namely X_d-frames were introduced and studied by Casazza et al [1]. Motivated by [1,15], in this paper, we study dual of X_Φ-frames and independent X_Φ-frame for Banach space. A necessary and sufficient condition for a X_Φ-Bessel sequence to be an independent X_Φ-frame have been given. Further, we proved that an independent X_Φ-frame with respect to model sequence space X_Φ must have a dual frame.

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Key Words: X_Φ-frame, X_Φ^*-frame, X_Φ-Bessel sequence, independent X_Φ-frame.

1. INTRODUCTION

In 1952 the notion of frames in Hilbert spaces were first introduced by Duffin and Schaeffer [7] and reintroduced in 1986 by Daubechies, Grossman, and Mayer [6], and popularized from then on. Frames have many properties of bases but lacks a very important one, namely, uniqueness. This property of frames makes them very useful in the study of function spaces, signal and image processing, filter banks, wireless and communications etc.

Coifman and Weiss [5] introduced a concept, similar to that of frames, called atomic decompositions for function spaces. Later the concept of frames in Hilbert spaces was extended to Banach spaces by Feichtinger and Gröchenig [8], who introduced the concept of atomic decompositions in Banach spaces. This concept was further generalized by Gröchenig [9], who introduced the notion of the Banach frames for Banach spaces. Frames in Banach spaces were further studied in [2,4]. In 2006, Sun [17] introduced the concept of g-fames for a Hilbert space. Recently, various generalization of frames in Banach spaces have been proposed in [3,4,10, 11,12,13,14, and 16].

Model space of sequences have been introduced and studied in [14]. In this paper, we study dual of X_Φ-frames and independent X_Φ-frame for Banach spaces. A necessary and sufficient condition for a X_Φ-Bessel sequence to be an independent X_Φ-frame have been given. Further, we proved that an independent X_Φ-frame with respect to model sequence space X_Φ must have a dual frame.

2. PRELIMINARIES

Throughout this paper, E will denote a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), E^* conjugate space of E, $L(E,F)$ will denote Banach space of all bounded linear operators from E into F and $ran(T)$ is the range of T. The identity operator on E and kernel of T are denoted by I_E and $\ker(T)$, respectively. By a Banach sequence space (often called a BK space), we mean a Banach space of scalar valued sequences, indexed by $\mathbb{N}$, for which coordinate functionals are continuous.

Definition 2.1 ([1]). A sequence space X_d is called a BK-space, if it is a Banach space and the coordinate functional are continuous on X_d, i.e., the relations $x_n = \left\{\alpha_j^{(n)}\right\}$, $x = \left\{\alpha_j\right\} \in X_d$, $\lim_{n\to\infty} x_n = x$ imply $\lim_{n\to\infty} \alpha_j^{(n)} = \alpha_j$ ($j = 1, 2, ...$).

The theory of spaces of sequences of scalars admits a natural generalization to a vector sequence spaces [14]. If $\Phi = \{G_n\}$ is a sequence of Banach spaces, a sequence space X_Φ associated with $\{G_n\}$ is a linear subspace of $\prod_{n=1}^{\infty} G_n$ (the collection of all sequences $\{y_n\}$ with $y_n \in G_n$, $n = 1, 2, ...,$ endowed with product topology).

The coordinate transformations $P_n : X_\Phi \to G_n$ are defined by

$$P_n(\{y_i\}) = y_n,\ n = 1, 2, ...$$

Then X_Φ is called a generalized BK-space induced by $\{G_n\}$ if X_Φ is a Banach space and P_n is a continuous operator, for every $n \in \mathbb{N}$. The scalar BK-spaces containing all unit vectors e_n are generalized by the spaces X_Φ containing all canonical subspaces

$$F_n = \{0\} \times \{0\} \times ... \{0\} \times \underset{\substack{\downarrow \\ n^{th}\ place}}{G_n} \times \{0\} \times ... \left(G_n \neq \{0\}, n = 1, 2...\right).$$

These F_n's closed linear subspaces of X_Φ. We refer to the space X_Φ as a model space.

The following is the example of such type of a model space.

Let $\Phi=\{G_n\}$ be a sequence of closed linear subspaces of a Banach space E. Consider the linear space X_Φ of the system Φ, that is, the space of all elements sequences $y=\{y_n\}_{n=1}^{\infty}$ for which the series $\sum_{n=1}^{\infty} y_n$ is convergent equipped with the norm

$$\|y\|_{X_\Phi}=\sup_{1\le n<\infty}\left\|\sum_{i=1}^{n} y_i\right\|_E, \quad y_n\in G_n\,(n=1,2...). \tag{2.1}$$

Thus the space X_Φ is complete with respect to (2.1). Indeed, clearly (2.1) define a norm on X_Φ. Now, let $\{y_n^{(k)}\}$ $(k=1,2,..........)$ be a Cauchy sequence in X_Φ. Then for every $\varepsilon>0$ there exists a positive integer $\mathbb{N}(\varepsilon)$ such that

$$\left\|\{y_n^{(k)}\}-\{y_n^{(m)}\}\right\|_{X_\Phi}=\sup_{1\le n<\infty}\left\|\sum_{i=1}^{n}\left(y_i^{(k)}-y_i^{(m)}\right)\right\|<\varepsilon \quad \left(k,m>\mathbb{N}(\varepsilon)\right) \tag{2.2}$$

Then

$$\begin{aligned}\left\|y_n^{(k)}-y_n^{(m)}\right\| &\le \left\|\sum_{i=1}^{n}\left(y_i^{(k)}-y_i^{(m)}\right)\right\| + \left\|\sum_{i=1}^{n-1}\left(y_i^{(k)}-y_i^{(m)}\right)\right\| \\ &< 2\varepsilon \qquad \left(k,m>\mathbb{N}(\varepsilon);\, n=1,2,.......\right),\end{aligned}$$

whence, since by our assumption each G_n is complete, $\lim_{k\to\infty} y_n^{(k)}=y_n\in G_n$ $(n=1,2,...)$. Hence, from the inequalities (2.2)

$$\left\|\sum_{i=1}^{n}\left(y_i^{(k)}-y_i^{(m)}\right)\right\|<\varepsilon \qquad \left(k,m>\mathbb{N}(\varepsilon);\, n=1,2,...\right).$$

We obtain, for $m\to\infty$, we obtain

$$\left\|\sum_{i=1}^{n}\left(y_i^{(k)}-y_i\right)\right\|\le\varepsilon \qquad \left(k>\mathbb{N}(\varepsilon), n=1,2,...\right),$$

Then, $$\left\|\sum_{i=n+1}^{n+l} y_i\right\|\le 2\varepsilon+\left\|\sum_{i=n+1}^{n+l} y_i^{(k)}\right\| \qquad \left(k>\mathbb{N}(\varepsilon);\, n,l=1,2,...\right).$$

Consequently, since each series $\sum_{i=1}^{\infty} y_i^{(k)}$ converges and since E is complete, it follows that $\sum_{i=1}^{\infty} y_i$ converges, i.e., $\{y_n\}\in X_\Phi$. Moreover, by the above we have

$$\left\|\{y_n^{(k)}\}-\{y_n\}\right\|_{X_\Phi}=\sup_{1\le n<\infty}\left\|\sum_{i=1}^{n}\left(y_i^{(k)}-y_i\right)\right\|\le\varepsilon \quad \left(k>\mathbb{N}(\varepsilon)\right),$$

which shows that the space X_Φ is complete with respect to this norm.

Lemma 2.2. Let $\{G_n\}$ be a sequence of subspaces of E and $\{v_n\} \subset L(E, G_n)$ be a sequence of operators $\forall n \in \mathbb{N}$. If $\{v_n\}$ is total over E, then $X = \{\{v_n(x)\} : x \in E\}$ is a Banach space with the norm given by $\|\{v_n(x)\}\|_X = \|x\|_E, x \in E.$

Definition 2.3.[15] Let E be a Banach space over $\mathbb{F}$ and X_Φ be a model space induced by $\{G_n\}$. For every $n \in \mathbb{N}$, $\{v_n\}$ be a sequence of bounded linear operator in $L(E, G_n)$. We say that the family $T = \{v_n\}$ of bounded linear operator is a X_Φ-Bessel sequence for E with respect to X_Φ if there exists a positive constant B such that $\|\{v_n(x)\}\|_{X_\Phi} \le B\|x\|_E$, for all $x \in E$.

Define $\quad B_T = \inf\left\{B > 0 : \|\{v_n(x)\}\|_{X_\Phi} \le B\|x\|_E, \forall\, x \in E\right\}.$

We call B_T the Bessel bound of T. For any $T = \{v_n\}$, define $R_T : E \to X_\Phi$ such that $R_T(x) = \{v_n(x)\}$, for all $x \in E$. Then, we call R_T is the analysis operator of T. Clearly, above definition implies that $R_T \in L(E, X_\Phi)$.

Definition 2.4.[15] Let $\Phi = \{G_n\}$ be a sequence of non-trivial subspaces of a Banach space E and $\{v_n : v_n \in L(E, G_n), \forall\, n \in \mathbb{N}\}$ be a sequence of linear operators (not necessarily projections). Let X_Φ be a model space associated with E. Then, we say that $(\{G_n\}, \{v_n\})$ is a X_Φ-frame for E with respect to X_Φ if

(a) $\{v_n(x)\} \in X_\Phi$, for all $x \in E$;

(b) there exist constants A, B with $0 < A \le B < \infty$ such that

$$A\|x\|_E \le \|\{v_n(x)\}\|_{X_\Phi} \le B\|x\|_E,\ x \in E.$$

The positive constants A and B, respectively, are called lower and upper optimum bounds for the X_Φ-frame $(\{G_n\}, \{v_n\})$.

Put
$$A_T = \sup\left\{A > 0 : A\|x\|_E \le \|\{v_n(x)\}\|_{X_\Phi}, \forall\, x \in E\right\}$$
and
$$B_T = \inf\left\{B > 0 : \|\{v_n(x)\}\|_{X_\Phi} \le B\|x\|_E, \forall\, x \in E\right\}.$$

These constants A_T, B_T are called lower and upper optimum bounds of $T = \{v_n\}$.

3. MAIN RESULTS

In this section, we will study and discuss about dual of X_Φ-frames for Banach space E with respect to X_Φ. Let X_Φ be a model space induced by $\{G_n\}$.

Consider the sequence space

$$\mathcal{T}\left(\left\{G_n^*\right\}\right)=\left\{\left\{y_n^*\right\}\in\prod_{n=1}^{\infty}G_n^*:\sum_{n=1}^{\infty}y_n^*\left(y_n\right)\text{ converges, }\forall\left\{y_n\right\}\in X_\Phi\right\}.$$

This is a Banach space equipped with the following norm

$$\left\|\left\{y_n^*\right\}\right\|=\sup_{\substack{\|\{y_i\}\|\le1\\ \{y_i\}\in X_\Phi}}\ \sup_{1\le n<\infty}\left\{\left|\sum_{i=1}^{n}\left\langle y_i^*,y_i\right\rangle\right|\right\},\quad\forall\left\{y_n^*\right\}\in\mathcal{T}\left(\left\{G_n^*\right\}\right).$$

Then, $\mathcal{T}\left(\left\{G_n^*\right\}\right)$ is an isomorphic to the dual of X_Φ under the mapping $\left\{y_n^*\right\}\to h$, where

$$h\left(\left\{y_n\right\}\right)=\sum_{n=1}^{\infty}y_n^*y_n,\quad\left\{y_n\right\}\in X_\Phi,\ h\in X_\Phi^*.$$

Thus, the space X_Φ^* is also a generalized BK-space.

Hence, for any $\left\{y_n\right\}\in X_\Phi$ and $\left\{y_n^*\right\}\in X_\Phi^*$, we have $\left\langle\left\{y_n^*\right\},\left\{y_n\right\}\right\rangle=\sum_{n=1}^{\infty}y_n^*y_n$.

Definition 3.1. Let $T=\left\{v_n\right\}$ be a X_Φ-frame for E with respect to X_Φ and $Q=\left\{u_n\right\}$ be a X_Φ^*-frame for E^* with respect to X_Φ^*. If these two X_Φ-frames satisfy the following conditions:

$$x=\sum_{n=1}^{\infty}u_n^*v_nx,\quad\forall x\in E.\tag{3.1}$$

$$x^*=\sum_{n=1}^{\infty}v_n^*u_nx^*,\quad\forall x^*\in E^*.\tag{3.2}$$

Then, we call (T,Q) is a pair of dual frames for E. Here, one of them is called a dual frame of others. Let $T=\left\{v_n\right\}$ be a X_Φ-Bessel sequence for E with respect to X_Φ and R_T be an analysis operator of $T=\left\{v_n\right\}$. Then, we have $R_T\left(x\right)=\left\{v_n\left(x\right)\right\},\quad\forall x\in E.$

It is easy to see that adjoint operator R_T^* of R_T can be defined as follows;

$$R_T^*:X_\Phi^*\to E^*\text{, such that }R_T^*\left(\left\{y_n^*\right\}\right)=\sum_{n=1}^{\infty}v_n^*y_n^*,\qquad\forall\left\{y_n^*\right\}\in X_\Phi^*.$$

Indeed, suppose that R_T and R_Q are the analysis operators of $T=\left\{v_n\right\}$ and $Q=\left\{u_n\right\}$ respectively, then from (3.1) and (3.2) we find that

$$R_Q^*R_Tx=R_Q^*\left(\left\{v_n\left(x\right)\right\}\right)=\sum_{n=1}^{\infty}u_n^*v_nx=x=I_Ex,\quad\forall x\in E.\ \Rightarrow\ R_Q^*R_T=I_E.$$

Similarly, we find that $R_T^*R_Qx=R_T^*\left(\left\{u_n\left(x\right)\right\}\right)=\sum_{n=1}^{\infty}v_n^*u_nx^*=x^*=I_{E^*}x^*,\quad\forall x^*\in E^*.$

Definition 3.2. A family $T=\{v_n\}$ of operators, where $v_n \in L(E,G_n)\ \forall n\in\mathbb{N}$, is said to be an independent, if the following condition is satisfied:

$$\sum_{n=1}^{\infty} v_n^* x_n^* = 0,\ \{x_n^*\}\in X_\Phi^* \Rightarrow x_n^* = 0,\ \forall n\in\mathbb{N}.$$

Here, we give the necessary and sufficient condition for a X_Φ-Bessel sequence to be an independent X_Φ-frame.

Theorem 3.3. Let $T=\{v_n\}$ be a X_Φ-Bessel sequence for E with respect to X_Φ. Then, $\{v_n\}$ is an independent X_Φ-frame if and only if its analysis operator R_T is invertible.

Proof. Assume that $\{v_n\}$ is an independent X_Φ-frame. Then, we have

$$\sum_{n=1}^{\infty} v_n^* x_n^* = 0,\ \{x_n^*\}\in X_\Phi^* \Rightarrow x_n^* = 0,\ \forall n\in\mathbb{N}.$$

In order to show that R_T is invertible. We first to show that R_T^* is an injective.

So let $\{x_n^*\}\in \ker(R_T^*) \Rightarrow R_T^*(\{x_n^*\})=0.$ This gives $\sum_{n=1}^{\infty} v_n^* x_n^* = 0.$ (By definition of R_T^*)

Hence, $x_n^* = 0,\ \forall n\in\mathbb{N}.$ Thus, $\ker(R_T^*)=\{0\} \Rightarrow R_T^*$ is injective.

Now, we see that $\overline{\operatorname{ran}}(R_T)=(\ker(R_T^*))^{\perp}=\{0\}^{\perp}=X_\Phi$. Therefore, range of R_T is dense in X_Φ.

In addition, from the definition of X_Φ-frames, we know that R_T is bounded below and $\operatorname{ran}(R_T)$ is closed. Hence R_T is invertible.

Conversely, let us assume that R_T is invertible. Then R_T is bounded below. Thus, $\{v_n\}$ is a X_Φ-frame. In order to prove that $\{v_n\}$ is an independent X_Φ-frame let, if possible, $\{v_n\}$ is not an independent X_Φ-frame. Then, there exists a non-zero sequence $\{y_n^*\}\subset X_\Phi^*$ (assume $n_0\in\mathbb{N},\ y_{n_0}\neq 0$) such that $v_{n_0}^* y_{n_0}^* + \sum_{n\neq n_0}^{\infty} v_n^* y_n^* = 0.$

$$v_{n_0}^* y_{n_0}^* = -\sum_{n\neq n_0}^{\infty} v_n^* y_n^*. \tag{3.3}$$

Since $y_{n_0}^* \neq 0$, there exists $y_{n_0}\in G_{n_0}$ such that $y_{n_0}^*(y_{n_0})\neq 0$.

Again, since R_T is invertible, there is $x\in E$ such that

$$R_T(x)=\{\delta_{nn_0}\, y_n\},\ \text{i.e.,}\ v_n(x)=\delta_{nn_0}\, y_n,\quad \forall n\in\mathbb{N}.$$

From L.H.S. of (3.3), we have

$$\left\langle v_{n_0}^* y_{n_0}^*,\, x\right\rangle = y_{n_0}^*\left(v_{n_0}\left(x\right)\right) \neq 0.$$

Thus, we get

$$\left\langle v_{n_0}^* y_{n_0}^*,\, x\right\rangle \neq 0\,. \tag{3.4}$$

Again, from R.H.S. of (3.3), we have

$$\left\langle -\sum_{n\neq n_0}^{\infty} v_n^* y_n^*,\, x\right\rangle = -\sum_{n\neq n_0}^{\infty}\left\langle v_n^* y_n^*,\, x\right\rangle = -\sum_{n\neq n_0}^{\infty}\left\langle y_n^*,\, v_n x\right\rangle$$

$$= -\sum_{n\neq n_0}^{\infty} y_n^*\left(v_n x\right) = -\sum_{n\neq n_0}^{\infty} y_n^*\left(\delta_{n n_0}\, y_n\right) = 0.$$

Thus, we get

$$\left\langle -\sum_{n\neq n_0}^{\infty} v_n^* y_n^*,\, x\right\rangle = 0\,. \tag{3.5}$$

Therefore, from (3.4) and (3.5). We get a contradiction. Hence, $\{v_n\}$ is an independent X_Φ-frame.

In the next result, we have proved that an independent X_Φ-frame with respect to model sequence space X_Φ must have a dual frame.

Theorem 3.4. An independent X_Φ-frame $\{v_n\}$ for E with respect to X_Φ must have a dual frame.

Proof. Let $T = \{v_n\}$ be an independent X_Φ-frame for E with respect to X_Φ Then from Theorem 3.3, its analysis operator R_T is invertible and hence R_T^* is invertible.

For any $n \in \mathbb{N}$, put $u_n = P_n R_T^{*-1}$, where P_n is the coordinate operator on X_Φ^*. Then, $u_n \in L\left(E^*, G_n^*\right)$, $\forall\, n \in \mathbb{N}$ and for any $x^* \in E^*$, there exists a sequence $\left\{y_n^*\right\} \in X_\Phi^*$ such that

$$x^* = \sum_{n=1}^{\infty} v_n^* y_n^*\,.$$

So,

$$\left\{u_n\left(x^*\right)\right\} = \left\{P_n R_T^{*-1}\left(x^*\right)\right\} = \left\{P_n\left(\left\{y_n^*\right\}\right)\right\} = \left\{y_n^*\right\} \in X_\Phi^*$$

and for all $x^* \in E^*$, we have

$$\left\|x^*\right\| = \left\|R_T^{*-1} R_T^*\left(x^*\right)\right\| \leq \left\|R_T^{*-1}\right\|\left\|R_T^*\left(x^*\right)\right\| \leq \left\|R_T^{*-1}\right\|\left\|R_T^*\right\|\left\|x^*\right\|\,.$$

Therefore,

$$\frac{\|x^*\|}{\|R_T^*\|} \le \left\|\left\{u_n\left(x^*\right)\right\}\right\|_{X_\Phi^*} = \left\|\left\{P_n R_T^{*-1}\left(x^*\right)\right\}\right\|_{X_\Phi^*} = \left\|\left\{R_T^{*-1}\left(x^*\right)\right\}\right\|_{X_\Phi^*} \le \left\|R_T^{*-1}\right\|.\left\|x^*\right\|.$$

Hence, $Q=\{u_n\}$ is a X_Φ^*-frame for E^* with respect to X_Φ^*.

For any $x \in E$ and $x^* \in E^*$, we have

$$\left\langle \sum_{n=1}^{\infty} v_n^* u_n x^*,\ x \right\rangle = \sum_{n=1}^{\infty}\left\langle v_n^* u_n x^*,\ x\right\rangle = \sum_{n=1}^{\infty}\left\langle u_n x^*,\ v_n x\right\rangle$$

$$= \sum_{n=1}^{\infty}\left\langle P_n R_T^{*-1} x^*,\ v_n x\right\rangle = \sum_{n=1}^{\infty}\left\langle P_n\left(\left\{y_n^*\right\}\right),\ v_n x\right\rangle$$

$$= \sum_{n=1}^{\infty}\left\langle y_n^*,\ v_n x\right\rangle = \left\langle \sum_{n=1}^{\infty} v_n^* y_n^*,\ x\right\rangle = \left\langle x^*,\ x\right\rangle.$$

And $$\left\langle x^*,\ \sum_{n=1}^{\infty} u_n^* v_n x\right\rangle = \sum_{n=1}^{\infty}\left\langle u_n x^*,\ v_n x\right\rangle$$

$$= \sum_{n=1}^{\infty}\left\langle P_n R_T^{*-1} x^*,\ v_n x\right\rangle = \sum_{n=1}^{\infty}\left\langle P_n\left(\left\{y_n^*\right\}\right),\ v_n x\right\rangle$$

$$= \sum_{n=1}^{\infty}\left\langle y_n^*,\ v_n x\right\rangle = \left\langle \sum_{n=1}^{\infty} v_n^* y_n^*,\ x\right\rangle = \left\langle x^*,\ x\right\rangle.$$

From above, we must have $x = \sum_{n=1}^{\infty} u_n^* v_n x, \quad \forall x \in E$, and $x^* = \sum_{n=1}^{\infty} v_n^* u_n x^*, \quad \forall x^* \in E^*$.

This show that $Q=\{u_n\}$ is a dual frame of $T=\{v_n\}$, i.e., (T,Q) is called a pair of dual frames for E.

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