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From The Zeros of the Riemann Zeta Function to Its Analytical Continuation Formula

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Abstract

The zeros of the Riemann Zeta Function are considered in this work and from the multiplication of these zeros; the analytic continuation formula of the Riemann Zeta Function is derived. The Riemann Zeta Function is eventually obtained through this. A general formula for the zeros of the analytic continuation formula of the Riemann Zeta Function is obtained and it is shown that these zeros will always be real.

Keywords: Riemann zeta function; Non-trivial zeros, analytic continuation formula. MSC:11M06,11M20,11M26

1. INTRODUCTION

It is a common norm to obtain a desired polynomial from its known roots just as it is possible to move from the polynomial to its unknown roots. It is an established fact that the Riemann Zeta Function has the following roots, namely;$z = \frac{1}{2} \pm it \; and \; z = -2n$.

The author seeks to proof the Riemann Hypothesis by showing that if the Riemann Zeta function can be derived from its zeros then, one can proof that all the non-trivial zeros will always have their real parts as $\frac{1}{2}$ since the zeros are unique and that the zeros of the analytic continuation formula will always be real. It is good to note that the solutions to algebraic functions are algebraic numbers and that the solutions to L functions are algebraic functions. Thus, one is justified for treating the solutions of the Riemann Zeta function as we would have treated the solutions of algebraic functions, since the Riemann Zeta function is a type of L Function.

2. Theorem: If

1. **ALL** the ***nontrivial*** zeros are on the Critical Line [which is the ***Riemann Hypothesis***, ***RH***] and

2. **ALL** the *trivial* zeros are on the negative real axe (at even negative integer numbers) .***THEN*** the Analytical Continuation Formula of the Riemann Zeta Function $\zeta(z)$ can be derived from them.

Proof: If $z = \frac{1}{2} \pm it \; and \; z = -2n$

are the zeros of the Riemann Zeta Function, then it follows that:

$$\sigma_n(z) = \left(z - \frac{1}{2} + it_n\right)\left(z - \frac{1}{2} - it_n\right)(z + 2n) \tag{1}$$

We can show (1) that

$$\sigma_n(z) = \left(z^2 - z + \frac{1}{4} + {t_n}^2\right)(z + 2n) \tag{2}$$

If we choose $P(t) = \frac{1}{4} + {t_n}^2$

$$\sigma_n(z) = \left(z^2 - z + P(t_n)\right)(z + 2n) \tag{3}$$

One obtains:

$$\sigma_n(z) = 2n\left(1 + \frac{z}{2n}\right)\left(z^2 - z + P(t_n)\right) \tag{4}$$

By the process of discretization.one obtains;

$$\mu_n(z) = \sum_{n=1}^{\infty} 2n\left(1 + \frac{z}{2n}\right) z(z-1)\left(1 + \frac{P(t_n)}{z(z-1)}\right) \tag{5}$$

We write, by the method of partial summation [3],

$$\sum_{dq\le n} 2n\left(1+\frac{z}{2n}\right)=\sum_{d\le n} 2n \sum_{q\le n/d}\left(1+\frac{z}{2n}\right) \tag{5a}$$

$$\mu_n(z)=z(z-1)\left(1+\frac{P(t_n)}{z(z-1)}\right)\sum_{dq\le n} 2n\left(1+\frac{z}{2n}\right) \; ; d=2n \; and \; q=\left(1+\frac{z}{2n}\right) \tag{5b}$$

From Riemann 1859 [1];

$$-\log\prod\left({}^{z}/_{2}\right)=\lim\left(\sum_{n=1}^{n=m}\log\left(1+\frac{z}{2n}\right)-\frac{z}{2}\log m\right) \tag{6}$$

$for\ m=\infty$

$$-\frac{d\frac{1}{z}\log\Pi\left({}^{z}/_{2}\right)}{dz}=\sum_{1}^{\infty}\frac{d\frac{1}{z}\log\left(1+\frac{z}{2n}\right)}{dz} \tag{7}$$

From the above

$$-\frac{1}{z}\log\prod\left({}^{z}/_{2}\right)=\sum_{1}^{\infty}\frac{1}{z}\log\left(1+\frac{z}{2n}\right) \tag{8}$$

$$-\frac{z}{2}\Gamma\left({}^{z}/_{2}\right)=-\prod\left({}^{z}/_{2}\right)=\sum_{q\le n/d}\left(1+\frac{z}{2n}\right) \tag{9}$$

$$\mu_n(z)=-\frac{z}{2}\Gamma\left({}^{z}/_{2}\right)z(z-1)\left(1+\frac{P(t_n)}{z(z-1)}\right)\sum_{d\le n} 2n \tag{10}$$

3. Using Discretization Method

Let us choose $\pi^{z/2}\pi^{-z/2}=1$ then:

$$\mu_n(z)=-\frac{z}{2}\Gamma\left({}^{z}/_{2}\right)\pi^{z/2}\pi^{-z/2}z(z-1)\left(1+\frac{P(t_n)}{z(z-1)}\right)\sum_{d\le n} 2n \quad ; n=1,2,3,\ldots \tag{11}$$

It follows from (11) that one can write

$$\sum_{d\le n} 2n=\sum_{d\le n} 2\,n^{-z+z+1}=2\sum_{d\le n} n^{-z+(z+1)}$$

$$\mu_n(z) = -\frac{z}{2}\Gamma({}^{z}/_{2})\pi^{{}^{z}/_{2}}\pi^{-{}^{z}/_{2}}2\sum_{d\leq n}\left(n^{-z}n^{(z+1)}\right)z(z-1)\left(1+\frac{P(t_n)}{z(z-1)}\right) \quad (12)$$

$$\mu_1(z) = -\frac{z}{2}\Gamma({}^{z}/_{2})\pi^{{}^{z}/_{2}}\pi^{-{}^{z}/_{2}}2\sum_{d\leq n}\left(1^{-z}1^{(z+1)}\right)z(z-1)\left(1+\frac{P(t_n)}{z(z-1)}\right) \quad (13)$$

$$\mu_2(z) = -\frac{z}{2}\Gamma({}^{z}/_{2})\pi^{{}^{z}/_{2}}\pi^{-{}^{z}/_{2}}2\sum_{d\leq n}\left(2^{-z}2^{(z+1)}\right)z(z-1)\left(1+\frac{P(t_n)}{z(z-1)}\right) \quad (14)$$

$$\mu_3(z) = -\frac{z}{2}\Gamma({}^{z}/_{2})\pi^{{}^{z}/_{2}}\pi^{-{}^{z}/_{2}}2\sum_{d\leq n}\left(3^{-z}3^{(z+1)}\right)z(z-1)\left(1+\frac{P(t_n)}{z(z-1)}\right) \quad (15)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\mu_\infty(z) = -\frac{z}{2}\Gamma({}^{z}/_{2})\pi^{{}^{z}/_{2}}\pi^{-{}^{z}/_{2}}2\sum_{d\leq n}(\infty^{-z}\infty^{z+1})\,z(z-1)\left(1+\frac{P(t_n)}{z(z-1)}\right) = 0 \quad (16)$$

Elegantly, if one discretizes (11), then it implies:

$$\vartheta_n(z) = \sum_{n=1}^{\infty}\mu_n(z) = \mu_1(z) + \mu_2(z) + \cdots + \mu_\infty(z) \quad (17)$$

$$\vartheta_n(z) = \frac{z(z-1)}{2}\pi^{-{}^{z}/_{2}}\Gamma\left(\frac{z}{2}\right)\sum_{n\geq 1}^{\infty}\sum_{d\leq n} n\left[-2z\pi^{{}^{z}/_{2}}\left(1+\frac{P(t_n)}{z(z-1)}\right)\right] \quad (17a)$$

Such that;

$$\vartheta_n(z) = \frac{z(z-1)}{2}\pi^{-{}^{z}/_{2}}\Gamma\left(\frac{z}{2}\right)\sum_{n\geq 1}^{\infty}\sum_{d\leq n} n^{-z}n^{z+1}\left[-2z\pi^{{}^{z}/_{2}}\left(1+\frac{P(t_n)}{z(z-1)}\right)\right] \quad (18)$$

4. An Expression in Term of Prime Number

If we write and make use of the expression;

$$2 = \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} \quad and \quad n = \prod_{i=1}^{k} p_i^{\propto_i} \quad (18a)$$

One can write (18) as;

$$\vartheta_n(z) = \frac{z(z-1)}{2}\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\left[-z\pi^{z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\right]\left[\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}\sum_{n\geq 1}^{\infty}\sum_{d\leq n}\prod_{i=1}^{k}p_i^{\propto_i}\right] \quad (19)$$

Again, from (19) one obtains;

$$\vartheta_n(z) = \frac{z(z-1)}{2}\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\left[-2z\pi^{z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\right]\left[\sum_{n\geq 1}^{\infty}\sum_{d\leq n}\prod_{i=1}^{k}p_i^{-\propto_i z}p_i^{\propto_i(1+z)}\right] \quad (20)$$

If Equation (11) is written as a product instead of discretizing (11) then one arrivals at

$$\prod_{n=1}^{\infty}\mu_n(z) = \prod_{n=1}^{\infty}\frac{z(z-1)}{2}\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\left[-\sum_{d\leq n}2n\,z\pi^{z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\right] \quad (21)$$

And

$$\prod_{n=1}^{\infty}\mu_n(z) = \prod_{n=1}^{\infty}\frac{z(z-1)}{2}\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\left[-2z\pi^{z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\sum_{d\leq n}\prod_{i=1}^{k}p_i^{-\propto_i z}p_i^{\propto_i(1+z)}\right] \quad (22)$$

5. Product over Prime Numbers

$$Let \qquad \left(\frac{1}{p^z}-1\right)\sum_{n=0}^{\infty}\frac{1}{p^{nz}} = -1, [8,9] \quad (23)$$

Then (11) will become;

$$\mu_n(z) = \frac{z}{2}(z-1)\Gamma(z/2)\pi^{z/2}\left(\sum_{d\leq n}2n\,z\pi^{-z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\left[\left(\frac{1}{p^z}-1\right)\sum_{n=0}^{\infty}\frac{1}{p^{nz}}\right]\right) \quad (24)$$

$$\mu_n(z)\left(\frac{1}{p^z}-1\right)^{-1} = \frac{z}{2}(z-1)\Gamma(z/2)\pi^{z/2}\left(\sum_{d\leq n}2n\,z\pi^{-z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\left[\sum_{n=0}^{\infty}\frac{1}{p^{nz}}\right]\right) \quad (25)$$

If the equation in (25) is multiplied over prime numbers, we will arrival at;

$$\mu_n(z)\prod_{p}^{\infty}\left(\frac{1}{p^z}-1\right)^{-1} = \frac{z}{2}(z-1)\Gamma(z/2)\pi^{-z/2}\prod_{p}^{\infty}\left[\sum_{n=0}^{\infty}\frac{1}{p^{nz}}\right]\left[\sum_{d\leq n}2n\,z\pi^{z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\right] \quad (26)$$

And eventually, we have a function that gives the product the Riemann zeta analytic continuation formula with another function whose zeros are the non-trivial zeros of the zeta function. With this, one obtains an equation whose RHS is the analytic continuation Formula of Riemann zeta function;

$$\left[\sum_{d\le n} 2n\, z\pi^{z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\right]^{-1} \mu_n(z)\prod_p^{\infty}\left(\frac{1}{p^z}-1\right)^{-1} = \frac{z}{2}(z-1)\Gamma(z/2)\pi^{-z/2}\prod_p^{\infty}\sum_{n=0}^{\infty}\frac{1}{p^{nz}} \quad (27)$$

Riemann gave

$$\varepsilon(z) = \frac{z}{2}(z-1)\pi^{-z/2}\Gamma(z/2)\,\zeta(z) \quad (28)$$

$$\Rightarrow \zeta(z) = \frac{2\varepsilon(z)\pi^{z/2}}{z(z-1)\Gamma(z/2)} \quad (29)$$

For equation (26) and (27) to be equal to (28), the quantity;

$$\mu_n(z)\left[\sum_{d\le n} 2n\, z\pi^{z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\right]^{-1}\prod_p^{\infty}\left(\frac{1}{p^z}-1\right)^{-1} = \omega(z) \quad (30)$$

Equation (30) will hold and (27) will imply (28) if and only if;

$$P(t_n) = z(z-1)\left[\left(\frac{\mu_n(z)}{\omega(z)\sum_{d\le n} 2n\, z\pi^{z/2}}\prod_p^{\infty}\left(\frac{1}{p^z}-1\right)^{-1} - 1\right)\right] \quad (31)$$

The only conditions that make (26) and (27) to vanish are if;

$z = \frac{1}{2} \pm it_n\,, z = 1, or -2n$ as $P(t_n) = \frac{1}{4} + t_n{}^2$

This establishes the claim of Riemann that the non-trivial zeros will always have their real parts to be $\frac{1}{2}$.

6. Using the Mellin's Transformation

If we make use of [17];

$$\int_0^{\infty} frac\left(\frac{1}{t}\right)t^{z-1}dt = -\frac{\zeta(z)}{z}; \qquad for\ \ 0 < \mathbb{R}(z) < 1 \quad (32)$$

frac. means the fractional part.

Such that

$$\frac{z}{\zeta(z)}\int_0^\infty frac\left(\frac{1}{t}\right)t^{z-1}dt = -1;\ for\ \ 0 < \mathbb{R}(z) < 1 \tag{33}$$

From (33),

$$\zeta(z)\left(\int_0^\infty frac\left(\frac{1}{t}\right)t^{z-1}dt\right)^{-1} = -z;\quad for\ \ 0 < \mathbb{R}(z) < 1 \tag{34}$$

Then by using (33) in (24), one has

$$\vartheta_n(z) = \frac{z(z-1)}{2}\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\left[\sum_{d\leq n} 2n\pi^{z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\right]\frac{z^2}{\zeta(z)}\int_0^\infty frac\left(\frac{1}{t}\right)t^{z-1}dt \tag{35}$$

And it follows that

$$\vartheta_n(z) = \frac{z(z-1)}{2}\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\frac{1}{\zeta(z)}\left(z^2\sum_{d\leq n} 2n\pi^{z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\int_0^\infty frac\left(\frac{1}{t}\right)t^{z-1}dt\right) \tag{36}$$

If we equate (28) to (36) and the compare them, we see that:

$$\frac{1}{\zeta(z)}\left[\sum_{d\leq n} 2n\pi^{z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\right]\int_0^\infty frac\left(\frac{1}{t}\right)t^{z-1}dt \cong \zeta(z),\quad for\ \ 0 < \mathbb{R}(z) < 1 \tag{37}$$

And the zeta function is obtained as

$$\left(z^2\left[\sum_{d\leq n} 2n\pi^{z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\right]\int_0^\infty frac\left(\frac{1}{t}\right)t^{z-1}dt\right)^{\frac{1}{2}} = \zeta(z),\quad for\ \ 0 < \mathbb{R}(z) < 1 \tag{38}$$

By working with equation (34), one obtains;

$$\vartheta_n(z) = \frac{z(z-1)}{2}\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\zeta(z)\sum_{d\leq n} 2n\pi^{z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\left(\int_0^\infty frac\left(\frac{1}{t}\right)t^{z-1}dt\right)^{-1} \tag{39}$$

And again if (39) is compared with (28)

$$\left[\sum_{d\leq n} 2n\pi^{z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\right]\left(\int_0^\infty frac\left(\frac{1}{t}\right)t^{z-1}dt\right)^{-1} = 1 \tag{40}$$

From (40), we can see that,

$$P(t_n) = z(z-1)\left(\frac{1}{\sum_{d\le n} 2n\pi^{z/2}}\int_0^{\infty} frac\left(\frac{1}{t}\right) t^{z-1}dt - 1\right) \tag{41}$$

7. Lemma:

Let $z = \frac{1}{2} \pm it_n$ then (31) implies $P(t_n) = \frac{1}{4} + {t_n}^2 \; as \; n \to \infty$.

Proof

We want to establish that:

$$P(t_n) = z(z-1)\left[\left(\frac{\mu_n(z)}{\omega(z)\sum_{d\le n} 2n\pi^{z/2}}\prod_p^{\infty}\left(\frac{1}{p^z}-1\right)^{-1} - 1\right)\right] = \frac{1}{4} + {t_n}^2 \tag{42}$$

From

$$z(z-1)\left[\frac{\mu_n(z)}{\omega(z)\sum_{d\le n} 2n\pi^{z/2}}\prod_p^{\infty}\left(\frac{1}{p^z}-1\right)^{-1} - 1\right] \tag{43}$$

Substitute $z = \frac{1}{2} \pm it$ into (42) to obtain

$$\left(\frac{1}{2} + it_n\right)\left(-\frac{1}{2} + it_n\right)\left[\frac{\mu_n(z)}{\omega(z)\sum_{d\le n} 2n\pi^{z/2}}\prod_p^{\infty}\left(\frac{1}{p^z}-1\right)^{-1} - 1\right] \tag{44}$$

The above expression will give

$$\left(-\frac{1}{4} - {t_n}^2\right)\left[\frac{\mu_n(z)}{\omega(z)\sum_{d\le n} 2n\pi^{z/2}}\prod_p^{\infty}\left(\frac{1}{p^z}-1\right)^{-1} - 1\right] \tag{45}$$

After which (45) becomes

$$\left(-\frac{1}{4} - {t_n}^2\right)\left[\frac{\mu_n(z)}{\omega(z)\sum_{d\le n} 2\left(\frac{1}{2} + it_n\right)n\pi^{\left(\frac{1}{4}+\frac{it_n}{2}\right)}}\prod_p^{\infty}\left(\frac{1}{p^z}-1\right)^{-1} - 1\right] \tag{46}$$

Taking the limit of $P(t_n)$as $n \longrightarrow \infty$

$$\lim_{n\to\infty} P(t_n) = \left(-\frac{1}{4} - {t_\infty}^2\right)\left(\frac{\mu_n(z)}{\omega(z)\sum_{d\le n} 2z\infty\pi^{\left(\frac{z}{2}\right)}}\prod_p^{\infty}\left(\frac{1}{p^z}-1\right)^{-1} - 1\right) \tag{47}$$

$$P(t_\infty) = -\left(\frac{1}{4} + {t_\infty}^2\right)(-1) = \frac{1}{4} + {t_\infty}^2 \tag{48}$$

Thus (42) Implies that;

$$\frac{1}{4} + t_n{}^2 = z(z-1)\left[\frac{\mu_n(z)}{\omega(z)\sum_{d\leq n} 2zn\pi^{\frac{z}{2}}}\prod_p^{\infty}\left(\frac{1}{p^z}-1\right)^{-1} - 1\right] \tag{49}$$

From (33), we can obtain a general equation for generating the zeros of the analytic continuation formula of the Riemann Zeta function as follows:

$$t_n{}^2 = \frac{(z-1)\mu_n(z)}{\omega(z)\sum_{d\leq n} 2n\pi^{\frac{z}{2}}}\prod_p^{\infty}\left(\frac{1}{p^z}-1\right)^{-1} - z(z-1) - \frac{1}{4} \tag{50}$$

Such that

$$t_n = \pm\left[\frac{(z-1)\mu_n(z)}{\omega(z)\sum_{d\leq n} 2n\pi^{\frac{z}{2}}}\prod_p^{\infty}\left(\frac{1}{p^z}-1\right)^{-1} - z(z-1) - \frac{1}{4}\right]^{1/2} = \pm\left[P(t_n) - \frac{1}{4}\right]^{1/2} \tag{51}$$

8. Zeros of the Analytic Continuation Formula of the Riemann Zeta Function

Enoch has shown previously that (38) is always real. If one takes the limit of (51) as $n \to \infty$ one obtains;

$$\lim_{n\to\infty} t_n = \pm \lim_{n\to\infty}\left[\frac{(z-1)\mu_n(z)}{\omega(z)\sum_{d\leq n} 2n\pi^{\frac{z}{2}}}\prod_p^{\infty}\left(\frac{1}{p^z}-1\right)^{-1} - z(z-1) - \frac{1}{4}\right]^{1/2} \tag{52}$$

$$\lim_{n\to\infty} t_n = \pm\left[\frac{(z-1)\mu_n(z)}{\omega(z)\sum_{d\leq n} 2n\pi^{\frac{z}{2}}}\prod_p^{\infty}\left(\frac{1}{p^z}-1\right)^{-1} - z(z-1) - \frac{1}{4}\right]^{1/2} \tag{53}$$

$$\lim_{n\to\infty} t_n = \pm\left[0 - \lim_{n\to\infty} z(z-1) - \lim_{n\to\infty}\frac{1}{4}\right]^{1/2} \tag{54}$$

$$\lim_{n\to\infty} t_n = \pm\left[0 - \lim_{n\to\infty} z(z-1) - \lim_{n\to\infty}\frac{1}{4}\right]^{1/2} \tag{55}$$

If we choose $z = \frac{1}{2} \pm it_n \;\; in$ (55) , we arrival at:

$$\lim_{n\to\infty} t_n = \pm\left[-\lim_{n\to\infty}\left(\frac{1}{2}\pm it_n\right)\left(\frac{1}{2}\pm it_n - 1\right) - \lim_{n\to\infty}\frac{1}{4}\right]^{1/2} \tag{56}$$

The RHS is shown that

$$RHS = \pm\left[-\lim_{n\to\infty}\left(\frac{1}{4} + it_n - \frac{1}{2} - it_n - t_n{}^2\right) - \lim_{n\to\infty}\frac{1}{4}\right]^{1/2} \tag{57}$$

$$RHS = \pm\left[\lim_{n\to\infty}\frac{1}{4} + \lim_{n\to\infty} t_n{}^2 - \lim_{n\to\infty}\frac{1}{4}\right]^{1/2} \tag{58}$$

$$RHS = \pm\left[\lim_{n\to\infty} t_n{}^2\right]^{1/2} = \pm t_\infty, \qquad t \in \mathbb{R} \qquad \textit{as postulated by Riemann} \tag{59}$$

9. An Equivalent Formula for the Riemann Zeta Function

If one compares Equations (19) and (21), one can deduces that

$$\pi^{-z/2}\Gamma(z/2)\left[-\prod_p^{\infty}\sum_{d\leq n} 2nz\pi^{z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\right] \cong \pi^{-z/2}\,\Gamma(z/2)\zeta(z) \tag{60}$$

And by comparing the RHS and the LHS of (47), one obtains;

$$\left[-\prod_p^{\infty}\sum_{d\leq n} 2nz\pi^{z/2}\left(1+\frac{P(t_n)}{z(z-1)}\right)\right] \cong \zeta(z) \tag{61}$$

The LHS of (60) will be equal to the RHS of (60) if we substitute (23) for -1 in (61) such that;

$$\left(1+\frac{P(t_n)}{z(z-1)}\right)\prod_p^{\infty}\left[\left(\frac{1}{p^z}-1\right)\sum_{n=0}^{\infty}\frac{1}{p^{nz}}\right]\sum_{d\leq n} 2nz\pi^{z/2} \cong \zeta(z) \tag{62}$$

Another way to show that the LHS and RHS of (61) are equal is that the non-trivial zeros of the Riemann Zeta function are also the zeros of the term on the LHS of (61). But it is not obvious that the LHS of (61) has the same trivial zeros as that of its RHS until the LHS is written as;

$$\zeta(z) \cong \prod_{p}^{\infty}\left[\left(\frac{1}{p^{z}}-1\right)\sum_{n=0}^{\infty}\frac{1}{p^{nz}}\right]\left[\sum_{d\leq}2n\frac{(z+2n)}{(z+2n)}\left(z+\frac{P(t_k)}{(z-1)}\right)\right]\pi^{z/2} \quad (63)$$

Terence Tao in[14] showed how the other properties of the Zeta Function can be shown through analytical continuation method and in the same vain, it will be possible to show that (61) has all other properties of the Riemann Zeta Function.

10. Conclusion

Since there are no other classes of zeros that can lead to the derivation of the Riemann Zeta Function and its analytic continuation formula as the combination presented in this work, concludes the proof of the Riemann Hypothesis. With this, my next presentation will show us how to derive the Riemann operator and its connections to the Schrödinger operator.

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