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Generalized Near Rough Connected Topologized Approximation Spaces

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Abstract

Rough set theory has been introduced by Pawlak [18]. It is considered as a base for all researches in the rough set area introduced after this date. Most of these researches concentrated on developing results and techniques based on Pawlak's results [18]. In this paper, we apply topological concepts to introduce definitions for generalized b-approximations, generalized b-boundary regions and generalized near rough connected topologized approximation spaces from topological view. The basic concepts of some generalized near open, generalized near rough, and generalized near exact sets are sufficiently illustrated. Moreover, proved results, examples and counter examples are provided.

Keywords: Generalized b- approximations, Generalized b-rough set, Generalized b-connected spaces.

1. INTRODUCTION

One of the most powerful notions in system analysis is the concept of topological structures [10] and their generalizations. Rough set theory, introduced by Pawlak in 1982 [18], is a mathematical tool that supports also the uncertainty reasoning but qualitatively. In this paper, we shall integrate some ideas in terms of concepts in topology. Topology is a branch of mathematics, whose concepts exist not only in almost all branches of mathematics, but also in many real life applications. We believe that topological structure will be an important base for modification of knowledge extraction and processing.

2. PRELIMINARIES

A topological space [10] is a pair (X,τ) consisting of a set X and family τ of subsets of X satisfying the following conditions:

(T1) $\phi \in \tau$ and $X \in \tau$.

(T2) τ is closed under arbitrary union.

(T3) τ is closed under finite intersection.

Throughout this paper (X,τ) denotes a topological space, the elements of X are called points of the space, the subsets of X belonging to τ are called open sets in the space, the complement of the subsets of X belonging to τ are called closed sets in the space, and the family of all open sets of (X,τ) is denoted by τ and the family of all closed sets of (X,τ) is denoted by $C(X)$.

For a subset A of a space (X,τ), $Cl(A)$ denote the closure of A and is given by $Cl(A)=\cap\{F \subseteq X : A \subseteq F \text{ and } F \in C(X)\}$. Evidently, $Cl(A)$ is the smallest closed subset of X which contains A. Note that A is closed iff $A = Cl(A)$. $Int(A)$ denote the interior of A and is given by $Int(A)=\cup\{G \subseteq X : G \subseteq A \text{ and } G \in \tau\}$ Evidently, $Int(A)$ is the largest open subset of X which contained in A. Note that A is open iff $A = Int(A)$. The boundary of a subset $A \subseteq X$ is denoted by $BN(A)$ and is given by $BN(A) = Cl(A) - Int(A)$.

We shall recall some concepts about some near open sets which are essential for our present study.

Definition 2.1. A subset A of a space (X,τ) is called:

i) Semi-open [12] (briefly $s-$open) if $A \subseteq Cl(Int(A))$.

ii) Pre-open [14] (briefly $p-$open) if $A \subseteq Int(Cl(A))$.

iii) α-open [15] if $A \subseteq Int(Cl(Int(A)))$.

iv) β-open [1] (= semi-pre-open [2]) if $A \subseteq Cl(Int(Cl(A)))$.

v) b - open [2] $A \subseteq Int(Cl(A)) \cup Cl(Int(A))$.

The complement of a $s-$open (resp. $p-$open, α-open, β-open and b-open) set is called a $s-$closed (resp. $p-$closed , α-closed, β-closed and b-closed) set .

The family of all $s-$open (resp. $p-$open, α-open, β-open and b-open) sets of (X,τ) is denoted by $SO(X)$ (resp. $PO(X)$, $\alpha O(X)$, $\beta O(X)$ and $BO(X)$).

The family of all $s-$closed (resp. $p-$closed, α- closed, β-closed and b-closed) sets of (X,τ) is denoted by $SC(X)$ (resp. $PC(X)$, $\alpha C(X)$, $\beta C(X)$ and $BC(X)$).

The semi-closure (resp. α-closure, pre-closure, β-closure and b-closure) of a subset A of (X,τ), denoted by ${}_sCl(A)$ (resp. ${}_\alpha Cl(A)$, ${}_pCl(A)$, ${}_\beta Cl(A)$ and ${}_bCl(A)$) and defined to be the intersection of all semi-closed (resp. α-closed, p-closed, β-closed and b-closed) sets containing A. The semi-interior (resp. α-interior, pre-interior, β-interior and b-interior) of a subset A of (X,τ), denoted by ${}_sInt(A)$ (resp. ${}_\alpha Int(A)$, ${}_pInt(A)$, ${}_\beta Int(A)$ and ${}_bInt(A)$) and defined to be the union of all semi-open (resp. α-open, p-open, β-open and b-open) sets contained in A .

Definition 2.2. A subset A of a space (X,τ) is said to be:

(i) generalized closed[11] (briefly, g-closed) if $Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in (X,τ),

(ii) generalized semi-closed[6] (briefly, gs-closed) if ${}_sCl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in (X,τ),

(iii) generalized β-closed[7] (briefly, $g\beta$-closed) if ${}_\beta Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in (X,τ),

(iv) α-generalized closed[13] (briefly, αg-closed) if ${}_\alpha Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in (X,τ),

(v) generalized pre-closed[16] (briefly, gp-closed) if ${}_pCl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in (X,τ),

(vi) generalized B-closed[2] (briefly, gB-closed) if ${}_BCl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in (X,τ).

The complement of a g-closed (resp. gs-closed, $g\beta$-closed, gp-closed, αg-closed and gb-closed) set is called g-open (resp. gs-open, $g\beta$-open, gp-open, αg-open and gb-open). The family of all g-open (resp. gs-open, gp-

open, αg -open, $g\beta$-open and gb -open) sets of (X,τ) is denoted by $gO(X)$ (resp. $gSO(X)$, $gPO(X)$, $\alpha gO(X)$, $g\beta O(X)$ and $gBO(X)$). The family of all g -closed (resp. gs -closed, gp -closed, αg -closed, $g\beta$-closed and gb -closed) sets of (X,τ) is denoted by $gC(X)$ (resp. $gSC(X)$, $gPC(X)$, $\alpha gC(X)$, $g\beta C(X)$ and $gBC(X)$).

The generalized interior (briefly g -interior) of A is denoted by ${}_gInt(A)$ and is defined by ${}_gInt(A)=\cup\{G\subseteq X : G\subseteq A, G$ is a g -open set$\}$, the generalized near interior (briefly gj -interior) of A is denoted by ${}_{gj}Int(A)$ for all $j\in\{s,p,\beta,b\}$ and is defined by ${}_{gj}Int(A)=\cup\{G\subseteq X : G\subseteq A, G$ is a gj -open set$\}$ and the generalized $\alpha-$interior (briefly $g\alpha$-interior) of A is denoted by ${}_{g\alpha}Int(A)$ and is defined by ${}_{g\alpha}Int(A)=\cup\{G\subseteq X : G\subseteq A, G$ is a αg -open set$\}$

The generalized closure (briefly g -closure) of A is denoted by ${}_gCl(A)$ and is defined by ${}_gCl(A)=\cap\{F\subseteq X : A\subseteq F, F$ is a g -closed set$\}$, the generalized near closure (briefly gj -closure) of A is denoted by ${}_{gj}Cl(A)$ for all $j\in\{s,p,\beta,b\}$ and is defined by $gjCl(A)=\cap\{F\subseteq X : A\subseteq F, F$ is a gj closed set$\}$ and the generalized $\alpha-$closure (briefly $g\alpha$-closure) of A is denoted by ${}_{g\alpha}Cl(A)$ and is defined by ${}_{g\alpha}Cl(A)=\cap\{F\subseteq X : A\subseteq F, F$ is a αg -closed set$\}$

The generalized boundary (briefly g -boundary) region of A is denoted by ${}_gBN(A)$ and is defined by ${}_gBN(A)={}_gCl(A)-{}_gInt(A)$ and the generalized near boundary (briefly gj -boundary) region of A is denoted by ${}_{gj}BN(A)$ for all $j\in\{s,p,\alpha,\beta,B\}$ and is defined by

$${}_{gj}BN(A)={}_{gj}Cl(A)-{}_{gj}Int(A).$$

The generalized exterior (briefly g -exterior) of A is denoted by ${}_gExt(A)$ and is defined by ${}_gExt(A)=X-{}_gCl(A)$ and the generalized near exterior (briefly gj -exterior) of A is denoted by ${}_{gj}Ext(A)$ for all $j\in\{s,p,\alpha,\beta,b\}$ and is defined by

$${}_{gj}Ext(A)=X-{}_{gj}Cl(A)$$

3. GENERALIZED NEAR APPROXIMATIONS

The present section is devoted to introduce the concept of generalized near approximations.

3.1. Generalized near lower and generalized near upper approximations

Definition 3.1.1 [9]. Let $K=(X,R)$ be an approximation space with general relation R and τ_k is the topology associated to K. Then the triple (X,R,τ_k) is called a topologized approximation space.

Definition 3.1.2 [9]. Let $K=(X,R,\tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the lower approximation (resp. upper approximation) of A is defined by

$$\underline{R}A = Int(A) \text{ (resp. } \overline{R}A = Cl(A)\text{).}$$

Definition 3.1.3 [2]. Let $K=(X,R,\tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized lower approximation (briefly g-lower approximation) of A is denoted by $\underline{R}_g A$ and is defined by $\underline{R}_g A = {}_g Int(A)$.

Definition 3.1.4 [2]. Let $K=(X,R,\tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized near lower approximation (briefly gj-lower approximation) of A is denoted by $\underline{R}_{gj} A$ and is defined by

$$\underline{R}_{gj} A = {}_{gj} Int(A), \text{ where } j \in \{s, p, \alpha, \beta\}.$$

Definition 3.1.5. Let $K=(X,R,\tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized $b-$lower approximation (briefly gb-lower approximation) of A is denoted by $\underline{R}_{gb} A$ and is defined by $\underline{R}_{gb} A = {}_{gb} Int(A)$.

Definition 3.1.6 [2]. Let $K=(X,R,\tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized upper approximation (briefly g-upper approximation) of A is denoted by $\overline{R}_g A$ and is defined by $\overline{R}_g A = {}_g Cl(A)$.

Definition 3.1.7 [2]. Let $K=(X,R,\tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized near upper approximation (briefly gj-upper approximation) of A is denoted by $\overline{R}_{gj} A$ and is defined by $\overline{R}_{gj} A = {}_{gj} Cl(A)$, where $j \in \{s, p, \alpha, \beta\}$.

Definition 3.1.8. Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized $b-$upper approximation (briefly gb-upper approximation) of A is denoted by $\overline{R}_{gb}A$ and is defined by $\overline{R}_{gb}A = {}_{gb}Cl(A)$.

Theorem 3.1.1. Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the implications between lower approximation and gj-lower approximations of A are given by the following diagram for all $j \in \{s, p, \alpha, \beta, B\}$.

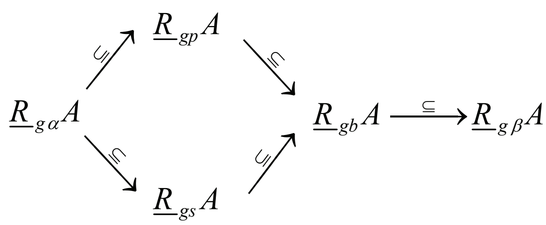

Figure 3.1.1.
Relation between the gj-lower approximations

Proof.

$$\underline{R}_{gp}A = {}_{gp}Int(A) = \cup\{G \in gpO(X) : G \subseteq A\} \subseteq \cup\{G \in g\beta O(X) : G \subseteq A\}$$
$$\underline{R}_{gp}A = {}_{gp}Int(A) \subseteq {}_{g\beta}Int(A) = \underline{R}_{g\beta}A$$

Theorem 3.1.2. Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the implications between gj-upper approximations of A are given by the following diagram for all $j \in \{s, p, \alpha, \beta, B\}$.

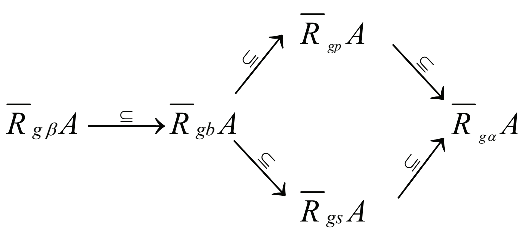

Figure 3.1.2.
Relation between the gj-upper approximations

Proof.

(i) $\overline{R}_{gs}A = {}_{gs}Cl(A) = \cap\{F \in gsC(X) : A \subseteq F\} \supseteq \cap\{F \in gbC(X) : A \subseteq F\}$

$\overline{R}_{gs}A \supseteq {}_{gb}Cl(A) = \overline{R}_{gb}A$.

(ii) $\overline{R}_{gp}A = {}_{gp}Cl(A) = \cap\{F \in gpC(X) : A \subseteq F\} \supseteq \cap\{F \in gbC(X) : A \subseteq F\}$

$\overline{R}_{gp}A \supseteq {}_{gb}Cl(A) = \overline{R}_{gb}A$.

(iii) $\overline{R}_{gb}A = {}_{gb}Cl(A) = \cap\{F \in gbC(X) : A \subseteq F\} \supseteq \cap\{F \in g\beta C(X) : A \subseteq F\}$

$\overline{R}_{gb}A \supseteq {}_{g\beta}Cl(A) = \overline{R}_{g\beta}A$

3.2. Generalized b-boundary regions

In this section we obtain some rules to find generalized b-boundary regions.

Definition 3.2.1 [2]. Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized near boundary (briefly gj -boundary) region of A is denoted by $BN_{Rgj}(A)$ and is defined by

$$BN_{Rgj}(A) = \overline{R}_{gj}(A) - \underline{R}_{gj}(A), \text{ where } j \in \{s, p, \alpha, \beta\}.$$

Definition 3.2.2. Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized $b-$boundary (briefly gb -boundary) region of A is denoted by $BN_{Rgb}(A)$ and is defined by

$$BN_{Rgb}(A) = \overline{R}_{gb}(A) - \underline{R}_{gb}(A).$$

Definition 3.2.3 [**2**]**.** Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized near positive (briefly gj -positive) region of A is denoted by $POS_{Rgj}(A)$ and is defined by $POS_{Rgj}(A) = \underline{R}_{gj}A$, where $j \in \{s, p, \alpha, \beta\}$.

Definition 3.2.4. Let $K = (X, R, \tau_k)$ be a topologized approximation space. If $A \subseteq X$, then the generalized $b-$positive (briefly gb -positive) region of A is denoted by $POS_{Rgb}(A)$ and is defined by $POS_{Rgb}(A) = \underline{R}_{gb}A$.

Definition 3.2.5 [2] Let $K=(X,R,\tau_k)$ be a topologized approximation space. If $A\subseteq X$, then the generalized near negative (briefly gj-negative) region of A is denoted by $NEG_{Rgj}(A)$ and is defined by

$$NEG_{Rgj}(A)=X-\overline{R}_{gj}A\ ,\ \text{where } j\in\{s,p,\alpha,\beta\}.$$

Definition 3.2.6. Let $K=(X,R,\tau_k)$ be a topologized approximation space. If $A\subseteq X$, then the generalized $b-$negative (briefly gb-negative) region of A is denoted by $NEG_{Rgb}(A)$ and is defined by

$$NEG_{Rgb}(A)=X-\overline{R}_{gb}A\ .$$

Theorem 3.2.1. $k=(X,R,\tau_k)$ be a topologized approximation space. If $A\subseteq X$, then $BN_{Rgj}(A)\subseteq BN_R(A)$, for all $j\in\{s,p,\alpha,\beta,B\}$.

Proof. obvious.

Theorem 3.2.2. Let $k=(X,R,\tau_k)$ be a topologized approximation space. If $A\subseteq X$, then the implications between boundary and gj-boundary of A are given by the following diagram for all $j\in\{s,p,\alpha,\beta,B\}$.

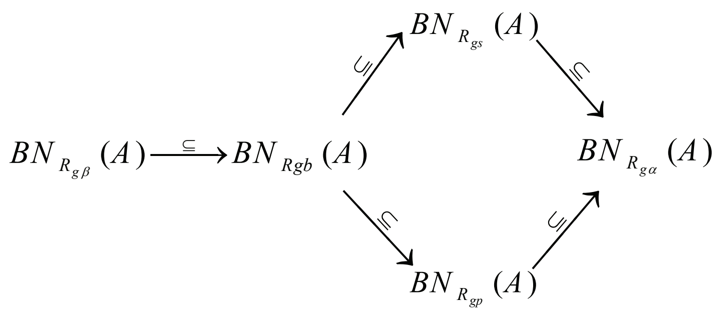

Figure 3.2.1.
Relations between the gj-boundaries

Proof. Obvious.

3.3. Generalized near rough and generalized near exact sets

In this section, we used topological concepts to introduce definitions to generalized b-rough and generalized b-exact sets.

Definition 3.3.1 [2]. Let (X,τ) be a topological space and $A\subseteq X$. Then

i) A is totally gj -definable (gj -exact) set if ${}_{gj}Int(A)=A={}_{gj}Cl(A)$,

ii) A is internally gj -definable set if $A={}_{gj}Int(A)$,

iii) A is externally gj -definable set if $A={}_{gj}Cl(A)$,

iv) A is gj - indefinable set if $A\neq{}_{gj}Int(A), A\neq{}_{gj}Cl(A)$,

where $j\in\{s,p,\alpha,\beta\}$.

Definition 3.3.2. Let (X,τ) be a topological space and $A\subseteq X$. Then

v) A is totally gb -definable (gb -exact) set if ${}_{gb}Int(A)=A={}_{gb}Cl(A)$,

vi) A is internally gb -definable set if $A={}_{gb}Int(A)$,

vii) A is externally gb -definable set if $A={}_{gb}Cl(A)$,

viii) A is gb - indefinable set if $A\neq{}_{gb}Int(A), A\neq{}_{gb}Cl(A)$.

Theorem 3.3.1. Let (X,τ) be a topological space and $A\subseteq X$. If A is an exact set then it is gb - exact.

Proof. Let A be exact set, then $Cl(A)=A=Int(A)$. Now,

$Cl(A)=\cap\{F\subseteq X:A\subseteq F, F\in C(X)\}\supseteq\cap\{F\subseteq X:A\subseteq F, F\in{}_{gb}C(X)\}$

Also,

$$Int(A)=\cup\{G\subseteq X, G\in\tau\}\subseteq\cup\{G\subseteq X:G\subseteq A, G\in{}_{gb}O(X)\}={}_{gb}Int(A).$$

Therefore, $Cl(A)\supseteq{}_{gb}Cl(A)\supseteq A\supseteq{}_{gb}Int(A)\supseteq Int(A)$. Since A is exact we get ${}_{gb}Cl(A)=A={}_{gb}Int(A)$; Hence A is gb -exact.

4. GENERALIZED NEAR ROUGH CONNECTED TOPOLOGIZED APPROXIMATION SPACES

The present section is devoted to introduce various level of connectedness in approximation spaces with general binary relations using some classes of generalized near closed sets.

Proposition 4.1. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. If A and B are two subsets of X, then:

i) $\underline{R}_{gb} A \subseteq A \subseteq \overline{R}_{gb} A$

ii) $\underline{R}_{gb}\, \varphi = \overline{R}_{gb}\, \varphi = \varphi$ and $\underline{R}_{gb}\, X = \overline{R}_{gb}\, X = X$.

iii) If $A \subseteq B$, then $\underline{R}_{gb}\, A \subseteq \underline{R}_{gb}\, B$.

iv) If $A \subseteq B$, then $\overline{R}_{gb}\, A \subseteq \overline{R}_{gb}\, B$.

v) $\underline{R}_{gb}\,(X - A) = X - \overline{R}_{gb}\, A$.

vi) $\overline{R}_{gb}\,(X - A) = X - \underline{R}_{gb}\, A$.

Definition 4.1 Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and $A \subseteq X$. Then

i) A is called totally gb -definable (gb -exact) set if $\underline{R}_{gb}\, A = A = \overline{R}_{gb}\, A$,

ii) A is called internally gb -definable set if $A = \underline{R}_{gb}\, A$,

iii) A is called externally gb -definable set if $A = \overline{R}_{gb}\, A$,

iv) A is called gb -indefinable (gb -rough) set if $A \neq \underline{R}_{gb}\, A$ and $A \neq \overline{R}_{gb}\, A$.

Definition 4.2. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. Then κ is said to be generalized j-rough (briefly gj -rough) disconnected if there are two nonempty subsets A and B of X such that

$$A \cup B = X \quad \text{and} \quad A \cap \overline{R}_{gb}\, B = \overline{R}_{gb}\, A \cap B = \varphi, \text{ Where } j \in \{s, p, \alpha, \beta, b\}.$$

The space $\kappa = (X, R, \tau_K)$ is said to be gj -rough connected if it is not gj -rough disconnected.

Proposition 4.2. A topologized approximation space $\kappa = (X, R, \tau_K)$ is gj -rough disconnected If and only if there exists a nonempty gj -exact proper subset X , where $j \in \{s, p, \alpha, \beta, b\}$ **.**

Proof. We shall prove this theorem in the case of $j = b$.

Suppose $\kappa = (X, R, \tau_K)$ is gb -rough disconnected topologized approximation space, then there exist two nonempty subsets A and B of X such that $A \cup B = X$ and $A \cap \overline{R}_{gb} B = \overline{R}_{gb} A \cap B = \varphi$. But $A \subseteq \overline{R}_{gb} A$, hence $A \cap B = \varphi$. Thus $A = X - B$. Also $A = X - \overline{R}_{gb} B$, since $A \cap \overline{R}_{gb} B = \phi$ and $A \cup \overline{R}_{gb} B \supseteq A \cup B = X$. Hence $A = \underline{R}_{gb} A$ and $B = \overline{R}_{gb} B$. Similarly $B = \underline{R}_{gb} B$ and $A = \overline{R}_{gb} A$. Therefore there exists a nonempty gb -exact proper subset A of X .

Conversely, suppose that A is a nonempty gb -exact proper subset of X , then we get $B = X - A$ is also a nonempty gb -exact proper subset of X . Hence $A \cup B = X$ and $A \cap \overline{R}_{gb} B = A \cap B = \overline{R}_{gb} A \cap B = \varphi$.

Thus $\kappa = (X, R, \tau_K)$ is gb -rough disconnected.

Example 4.1. Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space such that $X = \{1, 2, 3, 4\}$ and R be a binary relation defined on X such that

$R = \{(1,1), (1,3), (2,3), (3,3), (3,4), (4,4)\}$, Thus $X / R = \{\{1,3\}, \{3\}, \{3,4\}, \{4\}\}$ and

$\tau_K = \{X, \phi, \{3\}, \{4\}, \{3,4\}, \{1,3\}, \{1,3,4\}\}$, So

$$BO(X) = \{X, \phi, \{3\}, \{4\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}\},$$

$$BC(X) = \{\phi, X, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,4\}, \{1,2,3\}, \{1,2,4\}\} ,$$

$${}_gBO(X) = \{X, \phi, \{1\}, \{3\}, \{4\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}\},$$

$${}_gBC(X) = \{X, \phi, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{1,2,3\}, \{1,2,4\}, \{2,3,4\}\}$$

Since $A = \{1\}$ is a nonempty gb -exact proper subset of X , then the space $\kappa = (X, R, \tau_K)$ is gb -rough disconnected.

Proposition 4.3. The implications between generalized near rough disconnected topologized approximation spaces are given by the following figure.

$$
\begin{array}{ccc}
g\alpha-\textit{rough disconnected} & \Rightarrow & gs-\textit{rough disconnected} \\
\Downarrow & & \Downarrow \\
gp-\textit{rough disconnected} & \Rightarrow & gb-\textit{rough disconnected} \\
 & & \Downarrow \\
 & & g\beta-\textit{rough disconnected}
\end{array}
$$

Figure 4.2.1

Definition 4.3 Let $\kappa=(X,R_1,\tau_K)$, $\mathcal{Q}=(Y,R_2,\tau_Q)$ be two topologized approximation spaces. Then for every $j\in\{s,p,\alpha,\beta,b\}$ a mapping $f:\kappa\to\mathcal{Q}$ is called gj -rough continuous if $f^{-1}(\underline{R_2}V)\subseteq\underline{R_1}_{gj}f^{-1}(V)$ for every subset V of Y in $\mathcal{Q}$. Where f^{-1} means the inverse image of V.

Theorem 4.1. Let $f:\kappa\to\mathcal{Q}$ be a mapping from a topologized approximation space $\kappa=(X,R_1,\tau_K)$ to a topologized approximation space $\mathcal{Q}=(Y,R_2,\tau_Q)$. Then the following statements are equivalent. Where $j\in\{s,p,\alpha,\beta,b\}$

i) f is gj -rough continuous.

ii) The inverse image of each internally R_2-definable set in $\mathcal{Q}$ is internally gj -definable set in κ.

iii) The inverse image of each externally R_2-definable set in $\mathcal{Q}$ is externally gj -definable set in κ.

iv) $f\left(\overline{R_1}_{gj}A\right)\subseteq\overline{R_2}f(A)$ for every subset A of X in κ.

v) $\overline{R_1}_{gj}f^{-1}(B)\subseteq f^{-1}\left(\overline{R_2}B\right)$ for every subset B of Y in $\mathcal{Q}$.

Proof. We shall prove this theorem in the case of $j=b$.

(i)$\Rightarrow$(ii) Let f be gb-rough continuous and let V be an internally R_2 – definable set in Q. Then $\underline{R_2}V = V$ and $f^{-1}(V)$ is a subset of X in κ. By (i), we get

$f^{-1}(V) = f^{-1}(\underline{R_2}V) \subseteq \underline{R_1}_{gb} f^{-1}(V)$. Then

$f^{-1}(V) \subseteq \underline{R_1}_{gb} f^{-1}(V)$. But $\underline{R_1}_{gb} f^{-1}(V) \subseteq f^{-1}(V)$. Hence

$f^{-1}(V) = \underline{R_1}_{gb} f^{-1}(V)$. Therefore $f^{-1}(V)$ is internally gb-definable set in κ.

(ii)$\Rightarrow$(i) Let A be a subset of Y in Q. Since $\underline{R_2}A \subseteq A$, then $f^{-1}(\underline{R_2}A) \subseteq f^{-1}(A)$. But $\underline{R_2}A$ is internally R_2 – definable set in Q, then by (ii), we get $f^{-1}(\underline{R_2}A)$ is internally gb-definable set in κ contained in $f^{-1}(A)$. Hence $f^{-1}(\underline{R_2}A) \subseteq \underline{R_1}_{gb} f^{-1}(A) \subseteq f^{-1}(A)$, since $\underline{R_1}_{gb} f^{-1}(A)$ is the largest internally gb-definable set contained in $f^{-1}(A)$. Thus

$f^{-1}(\underline{R_2}A) \subseteq \underline{R_1}_{gb} f^{-1}(A)$ for every subset A of Y in Q. Therefore f is gb-rough continuous.

(ii)$\Rightarrow$(iii) Let F be an externally R_2 – definable set in Q, then $Y - F$ is internally R_2 – definable. Thus by (ii), we have $f^{-1}(Y - F)$ is internally gb-definable set in κ.

Since $f^{-1}(Y - F) = X - f^{-1}(F)$, then $X - f^{-1}(F)$ is internally gb-definable set in κ. Hence $f^{-1}(F)$ is externally gb-definable set in κ.

Similarly we can prove (iii)$\Rightarrow$(ii).

(ii)$\Rightarrow$(iv) Let A be a subset of X in κ, then $\overline{R_2} f(A)$ is an externally R_2 – definable set in Q. Hence $Y - \overline{R_2} f(A)$ is internally R_2 – definable set in Q. Thus by (ii), we get $f^{-1}(Y - \overline{R_2} f(A)) = X - f^{-1}(\overline{R_2} f(A))$ is internally gb-definable set in κ, and so $f^{-1}(\overline{R_2} f(A))$ is externally gb-definable set containing A in κ. Thus $A \subseteq \overline{R_1}_{gb} A \subseteq f^{-1}\left(\overline{R_2} f(A)\right)$, since $\overline{R_1}_{gb} A$ is the smallest externally gb-definable set containing A in κ. Hence

$$f\left(\overline{R_1}_{gb} A\right) \subseteq f\left[f^{-1}\left(\overline{R_2} f(A)\right)\right] \subseteq \overline{R_2} f(A).$$

Therefore $f\left(\overline{R_1}_{gb} A\right) \subseteq \overline{R_2} f(A)$ for every subset A in κ.

(iv)$\Rightarrow$(v) Let B be a subset of Y in $\mathcal{Q}$. Let $A = f^{-1}(B)$, then A is a subset of X in κ. By (iv), we get

$$f\left(\overline{R_1}_{gb} A\right) \subseteq \overline{R_2} f(A) = \overline{R_2} f\left(f^{-1}(B)\right) \subseteq \overline{R_2} B .$$

Hence $\overline{R_1}_{gb} A \subseteq f^{-1}\left(\overline{R_2} B\right)$. Thus $\overline{R_1}_{gb} A = \overline{R_1}_{gb} f^{-1}(B) \subseteq f^{-1}\left(\overline{R_2} B\right)$.

Therefore $\overline{R_1}_{gb} f^{-1}(B) \subseteq f^{-1}\left(\overline{R_2} B\right)$ for every subset B of Y in $\mathcal{Q}$.

(v)$\Rightarrow$(ii) Let G be an internally R_2 – definable set in $\mathcal{Q}$, then $B = Y - G$ is externally R_2 – definable set in $\mathcal{Q}$. Thus by (v), we get

$$\overline{R_1}_{gb} f^{-1}(B) \subseteq f^{-1}\left(\overline{R_2} B\right).$$

Since B is externally R_2 – definable set, then $f^{-1}\left(\overline{R_2} B\right) = f^{-1}(B)$. Thus $\overline{R_1}_{gb} f^{-1}(B) \subseteq f^{-1}(B)$. But $f^{-1}(B) \subseteq \overline{R_1}_{gb} f^{-1}(B)$, then $\overline{R_1}_{gb} f^{-1}(B) = f^{-1}(B)$. Hence $f^{-1}(B)$ is externally gb-definable set in κ.

Since $f^{-1}(B) = f^{-1}(Y - G) = X - f^{-1}(G)$, then $X - f^{-1}(G)$ is externally gb-definable set in κ. Therefore $f^{-1}(G)$ is internally gb-definable set in κ.

Example 4.2. Let $\kappa = (X, R_1, \tau_K)$, $\mathcal{Q} = (Y, R_2, \tau_Q)$ be two topologized approximation spaces such that $X = \{1,2,3,4\}$, $Y = \{a,b,c,d\}$,

$R_1 = \{(1,1),(2,2),(4,4)(1,2),(2,1)\}$ and $R_2 = \{(a,a),(d,d),(a,b)\}$. Then

$\tau_K = \{X, \varphi, \{4\}, \{1,2\}, \{1,2,4\}\}$ and $\tau_Q = \{Y, \varphi, \{d\}, \{a,b\}, \{a,b,d\}\}$. Hence

Define a mapping $f : K \to Q$ such that

$$f(1) = a, f(2) = d, f(3) = b \text{ and } f(4) = c .$$

Then f is not a gb-rough continuous mapping, since $V=\{a,b\}$ is an internally R_2 – definable set in $\mathcal{Q}$, but $f^{-1}(V)=\{1,4\}$ is not an internally gb-definable set in κ.

Proposition 2.4. Let $\kappa=(X,R_1,\tau_K)$ and $\mathcal{Q}=(Y,R_2,\tau_Q)$ be two topologized approximation spaces. If $f:\kappa\to\mathcal{Q}$ is a gb-rough continuous mapping, then the inverse image of each exact set in $\mathcal{Q}$ is gb-exact set in κ.

Proof.

Let A be an exact set in $\mathcal{Q}$, then A is both internally and externally R_2 – definable set in $\mathcal{Q}$. Hence $f^{-1}(A)$ is both internally and externally gb-definable set in κ. Therefore $f^{-1}(A)$ is a gb-exact set in κ.

3. CONCLUSIONS

In this paper, we used gb-open sets to introduce the definition of gb-rough connected topologized approximation space.

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