Approximation of the Common Solution of Equilibrium Problems and Fixed Point Problems of Multi-valued Pseudocontractive-type Mappings in Hilbert Spaces

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Abstract
A combined Ishikawa and Reich-Sabach iteration scheme is introduced in a real Hilbert space $H$ for the approximation of the common solutions of equilibrium problems of bifunctions and fixed point problems of multi-valued pseudocontractive-type mappings. It is also proved that the iteration scheme converges strongly to a common element of the sets of fixed points of a finite family of multi-valued pseudocontractive-type mappings and the sets of solutions of a finite family of equilibrium problems. A numerical example for the computation of this algorithm is presented with concrete examples. The obtained results improve, complement and extend many results on equilibrium problems, multi-valued and single-valued mappings in the contemporary literature.

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1. Introduction

Let $X$ be a normed space, $K$ a subset of $X$ and $T : D(T) \subseteq X \to 2^X$ a multi-valued map.

**Definition 1.1.** $K$ is called proximinal if for each $x \in X$ there exists $k \in K$ such that

$$||x - k|| = \inf\{||x - y|| : y \in K\} = d(x, K). \quad (1.1)$$

It is known that every closed convex subset of a uniformly convex Banach space is proximinal. We shall denote the family of all nonempty closed and bounded subsets of $X$ by $CB(X)$, the family of all nonempty closed and convex subsets of $X$ by $CC(X)$, the family of all nonempty subsets of $X$ by $2^X$ and the family of all proximinal subsets of $X$ by $P(X)$, while $H$ denotes the Hausdorff metric induced by the metric $d$ on a normed space $X$, that is, for every $A, B \in CB(X)$,

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$  

**Definition 1.2.** A point $x \in D(T)$ is called a fixed point of $T$ if $x \in Tx$. If $Tx = \{x\}$, $x$ is called a strict fixed point of $T$. The set $F(T) = \{x \in D(T) : x \in Tx\}$ is called the fixed point set of the multi-valued map $T$ while the set $Fs(T) = \{x \in D(T) : Tx = \{x\}\}$ is called the strict fixed point set of $T$.

**Definition 1.3.** $T$ is called $L$-Lipschitzian if there exists $L \geq 0$ such that for all $x, y \in D(T)$

$$H(Tx, Ty) \leq L||x - y||. \quad (1.2)$$

In (1.2), if $L \in [0, 1)$ $T$ is said to be a contraction while $T$ is nonexpansive if $L = 1$. $T$ is called quasi-nonexpansive if $F(T) = \{x \in D(T) : x \in Tx\} \neq \emptyset$ and for all $p \in F(T)$,

$$H(Tx,Tp) \leq ||x - p||. \quad (1.3)$$

Clearly, every nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive.

**Definition 1.4.** ([1]) $T$ is said to be $k$-strictly pseudocontractive-type of Isiogugu [1] if there exists $k \in (0, 1)$ such that given any pair $x, y \in D(T)$ and $u \in Tx$, there exists $v \in Ty$ satisfying $||u - v|| \leq H(Tx, Ty)$ and

$$H^2(Tx, Ty) \leq ||x - y||^2 + k||x - u - (y - v)||^2. \quad (1.4)$$

If $k = 1$ in (1.4), $T$ is said to be pseudocontractive-type while $T$ is nonexpansive-type if $k = 0$.

**Definition 1.5.** ([15]) A multi-valued mapping $T : K \to P(K)$ is said to satisfy condition (1) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x, Tx) \geq f(d(x, F(T))), \quad \forall x \in K.$$
Let $H$ be a real Hilbert space with an inner product $\langle.,.\rangle$ and a norm $\|\|$, respectively and let $K$ be a nonempty closed convex subset of $H$. Let $A : H \to H$ be an operator on $H$ and $F : K \times K \to \mathbb{R}$ be a bifunction on $K$, where $\mathbb{R}$ is the set of real numbers. The variational inequality problem of $A$ in $K$ denoted by $VIP(A,K)$ is to find an $x^* \in K$ such that
\[
\langle x - x^*, A(x^*) \rangle \geq 0, \quad \forall x \in K,
\] (1.5)
while the equilibrium problem for $F$ is to find $x^* \in K$ such that
\[
F(x^*, x) \geq 0, \quad \forall x \in K. \tag{1.6}
\]

The set of solutions of (1.6) is denoted by $EP(F)$. Suppose $F(x, y) = \langle y - x, Ax \rangle$ for all $x, y \in K$, then $w \in EP(F)$ if and only if $w$ is a solution of (1.5). Many problems in optimization, economics and physics reduce to finding a solution of (1.5), (see for examples, [3], [4] [6] and the references therein). The following conditions are assumed for solving the equilibrium problems for a bifunction $F : K \times K \to \mathbb{R}$:

(A1) $F(x, x) = 0$ for all $x \in K$.

(A2) $F$ is monotone, that is, $F(x, y) + F(y, x) \leq 0$, for all $x, y \in K$.

(A3) For each $x, y, z \in K$, $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$.

(A4) For each $x \in K$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Several algorithms have been introduced by authors for approximating solutions of equilibrium problems for a bifunction (or finite family of bifunctions) as well as a common element of the fixed point sets of finite family of multi-valued (or single-valued) mappings and the set of solutions of finite family of equilibrium problems (see for examples [7], [8], [9], [10], [11] and references therein).

In [7], Reich and Sabach proposed three algorithms for solving (common) equilibrium problems of bifunction(s) $g$ in a general reflexive Banach spaces using the well chosen convex function $f$, the Bregman distance and the projection associated with it. They proposed one of the algorithms as follows.

Let $X$ be a reflexive Banach space, $\{K_i\}_{i=1}^N$ a finite family of nonempty, closed and convex subsets of $X$. Let $\{\lambda^i\}_{i=1}^N$ be a finite family of positive real numbers and $\{g_i\}_{i=1}^N$ a finite family of bifunctions, with $g_i : K_i \times K_i \to \mathbb{R}$ for each $i = 1, 2, \ldots, N$. Suppose $f : X \to \mathbb{R}$ is a coercive Legendre function which is bounded, uniformly Fr\'{e}chet differentiable and totally convex on bounded subsets of $X$, $Res_{\lambda^i, g_i}^f$ is the resolvent of $g_i$ with respect to $\lambda^i$ and $f$ for each $i = 1, 2, \ldots, N$ and $D_f$ is the Bregman distance on $X$.

If $E = \bigcap_{i=1}^N EP(g_i) \neq \emptyset$, then the sequences $\{x_n\}_{n=1}^\infty$ were generated from an arbitrary $x_0 \in X$ as follows:
Algorithm 1 (Algorithm II [7])

\[
\begin{align*}
x_0 &\in X, \\
K_i^0 &= X, \quad i = 1, 2, \ldots, N \\
y_n^i &= \text{Res}_{x_n^i}^f (x_n^i + e_n^i), \\
K_{n+1}^i &= \{ z \in K_n^i : D_f (z, y_n^i) \leq D_f (z, x_n^i + e_n^i) \}, \\
K_{n+1} &= \bigcap_{i=1}^N K_{n+1}^i, \\
x_{n+1} &= \text{proj}_{K_{n+1}}^f (x_0), \quad n = 0, 1, 2, \ldots
\end{align*}
\]

Furthermore, they proved that if \( \lim\inf_{n \to \infty} \lambda_n^i > 0 \) and \( \lim_{n \to \infty} e_n = 0 \), the sequences converge strongly to \( \text{proj}_{E}^f (x_0) \).

Recently, Isiogugu [8] obtained a strong convergence of a hybrid algorithm to a common element of the fixed point sets of two multi-valued strictly pseudocontractive-type mappings and the set of solutions of an equilibrium problem in Hilbert spaces using a strict fixed point set condition. She proved the following theorem:

**Theorem 1.6.** ([8]) Let \( K \) be a nonempty closed convex subset of a real Hilbert space \( H \), let \( f : K \times K \to \mathbb{R} \) be a bifunction satisfying (A1)-(A4) and let \( S, T : K \to P(K) \) be two strictly pseudocontractive-type mappings with contractive coefficients \( \lambda_1 \) and \( \lambda_2 \) respectively such that \( \mathcal{F} = F_\delta (T) \bigcap F_\delta (S) \bigcap EP(f) \neq \emptyset \). Let \( \{x_n\} \) be a sequence generated from an arbitrary \( x_0 \in K \) as follows:

\[
\begin{align*}
x_0 &\in H, \\
K_1 &= K, \\
x_1 &= P_K x_0, \\
y_n &= \alpha_n x_n + (1 - \alpha_n) [\beta_n v_n + (1 - \beta_n) z_n], \\
u_n &\in K \text{ such that } f (u_n, y) + \frac{1}{r_n} (y - u_n, u_n - y_n) \geq 0, \quad \forall y \in K, \\
K_{n+1} &= \{ z \in K_n : \| z - u_n \|^2 \leq \| z - x_n \|^2 \}, \\
x_{n+1} &= \text{proj}_{K_{n+1}}^f (x_0),
\end{align*}
\]

where \( v_n \in Tx_n, z_n \in Sx_n \). \((\alpha_n)_{n=1}^\infty, (\beta_n)_{n=1}^\infty\) are sequences in \([0,1]\) satisfying

(i) \( \alpha_n \geq \max [\lambda_1, \lambda_2] \).

(ii) \( \lim\inf_{n \to \infty} (1 - \alpha_n)(1 - \beta_n)(\alpha_n - \lambda_1) > 0 \), \( \lim\inf_{n \to \infty} (1 - \alpha_n)(\alpha_n - \lambda_2)\beta_n > 0 \)

(iii) \( \{r_n\} \subset [a, \infty) \) for some \( a > 0 \).

Then \( \{x_n\} \) converges strongly to \( p \in P_\mathcal{F} x_0 \).

Isiogugu et al. [13] observed that in Algorithms 1, if \( T^i : K^i \to K^i \) is the identity map on \( K^i, i = 1, 2, \ldots, N \), respectively and \( u_n^i = \alpha_n x_n + (1 - \alpha_n) T^i x_n + e_n^i = x_n + e_n^i \), then we can rewrite Algorithm 1 in the following form:
Algorithm 2.

\[
\begin{aligned}
& x_0 \in X, \\
& K_0^i = X, \quad i = 1, 2, \ldots, N \\
& \upsilon_n^i = \alpha_n x_n + (1 - \alpha_n) T^i x_n + e_n^i, \\
& y_n^i = \text{Res}^f_{\upsilon_n^i, y_n^i}(\upsilon_n^i), \\
& K_{n+1}^i = \{z \in K_n^i : D_f(z, y_n^i) \leq D_f(z, \upsilon_n^i)\}, \\
& K_{n+1} = \bigcap_{i=1}^N K_{n+1}^i, \\
& x_{n+1} = \text{proj}^f_{K_{n+1}}(x_0),
\end{aligned}
\]

Motivated by the above observations in Algorithms 2 and the iteration scheme in Theorem 1.1 considered by Isiogugu in [8], Isiogugu et al. [13] constructed a hybrid algorithm for approximating a common element of the fixed point sets of a finite family of multi-valued nonexpansive mappings and the set of solutions of a finite family of equilibrium problems in Hilbert spaces without error terms. They studied the following iteration scheme:

Let \( H \) be a real Hilbert space and \( \{K^i\}_{i=1}^N \) a finite family of nonempty closed convex subsets of \( H \). Let \( \{f^i\}_{i=1}^N \) be a finite family of bifunctions and \( \{T^i\}_{i=1}^N \) a finite family of nonexpansive mappings such that \( f^i : K^i \times K^i \to \mathbb{R} \) and \( T^i : K^i \to P(K^i) \) for all \( i = 1, 2, \ldots, N \), respectively. Let \( \{\alpha_n^i\}_{n=1}^\infty \) be sequences in \([0,1]\) and \( \{r_n^i\}_{n=1}^\infty \subset [a, \infty) \) for some \( a > 0 \) for all \( i=1, 2, \ldots, N \), then from an arbitrary \( x_0 \in H \) the sequence \( \{x_n\}_{n=1}^\infty \) is generated as follows:

Algorithm 3 (Algorithm 7 [13]).

\[
\begin{aligned}
& x_0 \in H, \\
& K_1^i = K^i, \quad \forall \ i = 1, 2, \ldots, N, \\
& K_1 = \bigcap_{i=1}^N K_1^i, \\
& x_1 = P_{K_1}x_0, \\
& y_n^i = \alpha_n^i x_n + (1 - \alpha_n^i) y_n^i, \\
& u_n^i \in K^i \text{ such that } f^i(u_n^i, y) + \frac{1}{r_n^i} (y - u_n^i, u_n^i - y_n^i) \geq 0, \quad \forall y \in K^i, \\
& K_{n+1}^i = \{z \in K_n^i : \|z - u_n^i\|^2 \leq \|z - x_n\|^2\}, \\
& K_{n+1} = \bigcap_{i=1}^N K_{n+1}^i, \\
& x_{n+1} = P_{K_{n+1}}x_0,
\end{aligned}
\]

where \( y_n^i \in T^i x_n \).

Using the above algorithm, they proved the following theorem:

**Theorem 1.7. (Theorem 2 [13]).** Let \( H, \{K^i\}_{i=1}^N, \{T^i\}_{i=1}^N, \{f^i\}_{i=1}^N, \{\alpha_n^i\}_{n=1}^\infty \) and \( \{r_n^i\}_{n=1}^\infty \) be as in algorithm 3. Suppose \( f^i \) satisfying (A1)-(A4) for all \( i = 1, 2, \ldots, N, \mathcal{F} = \)
Motivated by Algorithm 3 above, we introduce a combined Ishikawa and Reich-Sabach iteration scheme in a real Hilbert space $H$ for the approximation of the common solution of equilibrium problems of bifunctions and fixed point problems of multi-valued pseudocontractive-type mappings; establish close and convex property for the set of fixed points of multi-valued pseudocontractive-mapping which guarantee the application of the algorithm to this class of mappings; and prove that the iteration scheme converges strongly to a common element of the fixed point sets of a finite family of multi-valued pseudocontractive-type mappings and the set of solutions of a finite family of equilibrium problems. Furthermore, a numerical example of the computation of this algorithm is presented with concrete examples. This work is a continuation of the study on the computability of algorithms for approximating the solutions of equilibrium problems for bifunctions involving the construction of the sequences $\{K_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$, from an arbitrary $x_0 \in H$, where $K_{n+1} = \{z \in K_n : \|z - u_n\|^2 \leq \|z - x_n\|^2\}$, $x_{n+1} = P_{K_{n+1}}x_0$, while $P_{K_n}$ is the projection map and $\{u_n\}_{n=1}^{\infty}$ is the sequence of the resolvent of the bifunctions. The obtained results improve, complement and extend many results on equilibrium problems, multi-valued and single-valued mappings in the contemporary literature.

### 2. Preliminaries

**Lemma 2.1.** Let $H$ be a real Hilbert space and let $K$ be a nonempty closed convex subset of $H$. Let $P_K$ be the convex projection onto $K$, then, the convex projection is characterized by the following relations;

(i) $x^* = P_K(x) \iff \langle x - x^*, y - x^* \rangle \leq 0$, for all $y \in K$.

(ii) $\|x - P_Kx\|^2 \leq \|x - y\|^2 - \|y - P_Kx\|^2$.

(iii) $\|x - P_Ky\|^2 \leq \|x - y\|^2 - \|P_Ky - y\|^2$.

**Lemma 2.2.** ([3]). Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$ and $F : K \times K \to \mathbb{R}$ a bifunction satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in K$ such that

$$F(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \quad \forall y \in K.$$$$

**Lemma 2.3.** ([4]). Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$. Assume that $F : K \times K \to \mathbb{R}$ that satisfies (A1)-(A4). Let $r > 0$ and $x \in H$, define
\[ T_r : H \rightarrow 2^K \text{ by} \]
\[ T_r(x) = \{ z \in K : F(z, y) + \frac{1}{r}(y - z, z - x) \geq 0 \}, \quad \forall y \in K. \]

Then, the following conditions hold:

1. \( T_r \) is single valued.
2. \( T_r \) is firmly nonexpansive, that is for any \( x, y \in H \), \( \| T_r x - T_r y \|^2 \leq (T_r x - T_r y, x - y) \).
3. \( F(T_r) = EP(F) \).
4. \( EP(F) \) is closed and convex.

**Lemma 2.4.** ([6]). Let \( K \) be a nonempty closed convex subset of a real Hilbert space \( H \) and \( F : K \times K \rightarrow \mathbb{R} \) a bifunction satisfying (A1)-(A4). Let \( r > 0 \) and \( x \in H \). Then, for all \( x \in H \) and \( p \in F(T_r) \),
\[ \| p - T_r x \|^2 + \| T_r x - x \|^2 \leq \| p - x \|^2. \]

**Lemma 2.5.** ([16]). Let \( H \) be a real Hilbert space and \( T : D(T) \subseteq H \rightarrow P(H) \) be a multi-valued \( L \)-Lipschitzian mapping, then, fixed point set of \( T \) is closed.

### 3. Main Results

**Proposition 3.1.** Let \( H \) be a real Hilbert space, \( C \) a closed convex subset of \( H \) and \( T : C \subseteq H \rightarrow CC(C) \) be a multi-valued, \( L \)-Lipschitzian pseudocontractive-type mapping. If \( F_s(T) \) (the strict fixed point set) of \( T \) is nonempty, then, it is convex.

**Proof.** Let \( p_1, p_2 \in F_s(T) \), we show that \( p = \lambda p_1 + (1 - \lambda)p_2 \in F_s(T) \). For each \( x \in D(T) \), let \( T_\beta x = T[(1 - \beta)x + \beta u_x] \), where \( u_x \in Tx \) with \( d(x, Tx) = \| x - u_x \| \) and \( \beta \in \left( 0, \frac{1}{\sqrt{1 + L^2 + 1}} \right) \). Clearly, \( T_\beta x \) is well defined since \( u_x \) is unique and \( C \) is convex. Also if \( p^* \in F_s(T) \), then, \( T_\beta p^* = \{ p^* \} \). Observe that for any \( u_\beta x \in T_\beta x \), given any \( p^* \in F_s(T) \), the pseudocontractive-type condition on \( T \) implies that
\[ \| u_\beta x - p^* \|^2 \leq H^2(T_\beta x, Tp^*) \leq \| (1 - \beta)x + \beta u_x - p^* \|^2 + \| ((1 - \beta)x + \beta u_x) - u_\beta x \|^2. \]

Similarly,
\[ \| u_x - p^* \|^2 \leq H^2(Tx,Tp^*) \leq \| x - p^* \|^2 + \| x - u_x \|^2 \]
It follows that for the pair $p, (1 - \beta)p + \beta u_p$ and $u_p$, there exists $u_{\beta p} \in T_{\beta} p = T[(1 - \beta)p + \beta u_p]$ with $\|u_p - u_{\beta p}\| \leq H(Tp, T_{\beta} p)$. Now,

\[
d^2(p, T_{\beta} p) \leq \|p - u_{\beta p}\|^2 = \|\lambda p_1 + (1 - \lambda)p_2 - u_{\beta p}\|^2 \\
= \|\lambda[p_1 - u_{\beta p}] + (1 - \lambda)[p_2 - u_{\beta p}]\|^2 \\
= \lambda\|p_1 - u_{\beta p}\|^2 + (1 - \lambda)\|p_2 - u_{\beta p}\|^2 \\
- \lambda(1 - \lambda)\|p_1 - p_2\|^2.
\]

Also,

\[
d^2(p_1, T_{\beta} p) \leq \|p_1 - u_{\beta p}\|^2 \leq H^2(Tp_1, T_{\beta} p) \\
\leq \|[1 - \beta)p + \beta u_p] - p_1\|^2 + \|[1 - \beta)p + \beta u_p] - u_{\beta p}\|^2 \\
\leq \|p - p_1\|^2 + \beta\|u_p - p_1\|^2 - (1 - \beta)\|p - u_p\|^2 \\
+ (1 - \beta)\|p - u_{\beta p}\|^2 + \beta\|u_p - u_{\beta p}\|^2 - (1 - \beta)\|p - u_p\|^2 \\
\leq \|p - p_1\|^2 + \beta H^2(Tp, Tp_1) - \beta(1 - \beta)\|p - u_p\|^2 \\
+ (1 - \beta)\|p - u_{\beta p}\|^2 + \beta H^2(Tp, T_{\beta} p)^2 - \beta(1 - \beta)\|p - u_p\|^2 \\
\leq \|p - p_1\|^2 + \beta\|p - p_1\|^2 + \|p - u_p\|^2 - \beta(1 - \beta)\|p - u_p\|^2 \\
+ (1 - \beta)\|p - u_{\beta p}\|^2 + \beta L_2^2\|p - [(1 - \beta)p + \beta u_p]\|^2 \\
- \beta(1 - \beta)\|p - u_p\|^2 \\
\leq \|p - p_1\|^2 + \beta\|p - p_1\|^2 + \|p - u_p\|^2 - \beta(1 - \beta)\|p - u_p\|^2 \\
+ (1 - \beta)\|p - u_{\beta p}\|^2 + \beta L_2^2\|p - u_p\|^2 - \beta(1 - \beta)\|p - u_p\|^2 \\
\leq \|p - p_1\|^2 + (1 - \beta)\|p - u_{\beta p}\|^2
\]

Similarly,

\[
d^2(p_2, T_{\beta} p) \leq \|p_2 - u_{\beta p}\|^2 \leq \|p - p_2\|^2 + (1 - \beta)\|p - u_{\beta p}\|^2
\]

Hence,

\[
\|p - u_{\beta p}\|^2 \leq \lambda\|p - p_1\|^2 + (1 - \beta)\|p - u_{\beta p}\|^2 \\
+ (1 - \lambda)\|p - p_2\|^2 + (1 - \beta)\|p - u_{\beta p}\|^2 \\
- \lambda(1 - \lambda)\|p_1 - p_2\|^2 \\
= \|\lambda p_1 + (1 - \lambda)p_2 - p\|^2 + (1 - \beta)\|p - u_{\beta p}\|^2 \\
= +(1 - \beta)\|p - u_{\beta p}\|^2
\]
This implies that $0 \leq \beta \| p - u_{\beta p} \| \leq 0$. Since $\beta \in (0, \frac{1}{\sqrt{1 + L^2} + 1})$, we have that $\| p - u_{\beta p} \| = 0$. Observe that $d(p, T_\beta p) \leq \| p - u_{\beta p} \| = 0 \leq d(p, T_\beta p)$, therefore, $d(p, T_\beta p) = \| p - u_{\beta p} \| = 0$ and $p = u_{\beta p} \in T_\beta p$.

$$d(p, T_\beta p) \leq d(p, T_\beta p) + H(T_\beta p, T p) \leq L\|(1 - \beta)p + \beta u_p - p\| = L\beta d(p, T p).$$

Thus, $0 \leq (1 - \beta L)d(p, T_p) \leq 0$. Consequently, $d(p, T_p) = 0$ and proximinal property of $T$ guarantees the existence $u \in T p$ such that $\| u - p \| = 0$. Hence, $p \in T p$.  

We now consider the following algorithm.

Let $H$ be a real Hilbert space and $\{K^i\}_{i=1}^N$ a finite family of nonempty closed convex subsets of $H$. Let $\{F^i\}_{i=1}^N$ be a finite family of bifunctions and $\{T^i\}_{i=1}^N$ a finite family of $L^i$-Lipschitzian pseudocontractive-type mappings such that $F^i : K^i \times K^i \to \mathbb{R}$ and $T^i : K^i \to CC(K^i)$ for all $i = 1, 2, \ldots, N$ respectively. Let $\{\alpha^i_n\}_{n=1}^\infty$, $\{\beta^i_n\}_{n=1}^\infty$ be sequences in $[0,1]$ and $\{r^i_n\}_{n=1}^\infty \subset [a, \infty)$ for some $a > 0$, for all $i = 1, 2, \ldots, N$. Then, from an arbitrary $x_0 \in H$ we generate the sequence $\{x^i_n\}_{n=1}^\infty$ as follows:

Algorithm 4.

\[\begin{align*}
x_0 &\in H, \\
K^i_0 &\equiv K^i, \quad \forall i = 1, 2, \ldots, N, \\
z^i_n &\equiv (1 - \beta^i_n)x_n + \beta^i_n y^i_n, \\
y^i_n &\equiv (1 - \alpha^i_n)x_n + \alpha^i_n w^i_n, \\
u^i_n &\in K^i \text{ such that } F^i(u^i_n, y) + \frac{1}{r^i_n} \langle y - u^i_n, u^i_n - y^i_n \rangle \geq 0, \quad \forall y \in K^i, \\
K^i_{n+1} &\equiv \{ z \in K^i_n : \| z - u^i_n \|^2 \leq \| z - x_n \|^2 \}, \\
K^i_{n+1} &\equiv \bigcap_{i=1}^N K^i_{n+1}, \\
x_{n+1} &\equiv P_{K_{n+1}}x_0,
\end{align*}\]

where $u^i_n \in T^i(z^i_n) = T^i(\beta^i_n x_n + \beta^i_n y^i_n)$ with $d(\beta^i_n x_n + \beta^i_n y^i_n, T^i(1 - \beta^i_n)x_n + \beta^i_n y^i_n) = ||(1 - \beta^i_n)x_n + \beta^i_n y^i_n - w^i_n||$, $v^i_n \in T^i x_n$ with $||x_n - v^i_n|| = d(x_n, T^i x_n)$ and $||u^i_n - v^i_n|| \leq H(T^i z^i_n, T^i x_n)$.

**Theorem 3.2.** Let $H$, $\{K^i\}_{i=1}^N$, $\{F^i\}_{i=1}^N$, $\{T^i\}_{i=1}^N$, $\{\alpha^i_n\}_{n=1}^\infty$, $\{\beta^i_n\}_{n=1}^\infty$ and $\{r^i_n\}_{n=1}^\infty$ be as in Algorithm 4. Suppose $F^i$ satisfying (A1)-(A4) for all $i = 1, 2, \ldots, N$, $\mathbb{F} = \bigcap_{i=1}^N F^i \bigcap_{i=1}^N P^i \neq \emptyset$, then $\{x^i_n\}$ converges strongly to $p \in P_{\mathbb{F}}x_0$ if for each $i = 1, 2, \ldots, N$ and for all $n \geq 1$, $\{\alpha^i_n\}$ and $\{\beta^i_n\}$ are real sequences satisfying:

(i) $0 \leq \alpha^i_n \leq \beta^i_n < 1$;
(ii) $\liminf_{n \to \infty} \alpha_n^i = \alpha^i > 0$;

(iii) $\sup_{n \geq 1} \beta_n^i \leq \beta^i \leq \frac{1}{\sqrt{1 + (L)^2} + 1}$.

**Proof.** Since $K_n^i$ is closed and convex for all $n \geq 1$ and for all $i = 1, 2, \ldots, N$, $K_n = \bigcap_{i=1}^{N} K_n^i$ is closed and convex and hence $P_{K_{n+1}}x_0$ is well defined, also, $u_n^i = T_{r_n}y_n^i$. Next, we show that $F \subset K_n$, for all $n \geq 1$. $F \subset K_1^i = K_1$ for all $i = 1, 2, \ldots, N$, therefore, $F \subset \bigcap_{i=1}^{N} K_1^i = K_1$. Assume $F \subset K_n = \bigcap_{i=1}^{N} K_n^i$. Using Lemma 2.3, for all $p \in F$ we have

$$
\|p - u_n^i\|^2 = \|p - T_{r_n}y_n^i\|^2 \\
\leq \|p - y_n^i\|^2 \\
= \|(1 - \alpha_n^i)x_n + \alpha_n^i w_n^i - p\|^2 \\
= \|(1 - \alpha_n^i)(x_n - p) + \alpha_n^i (w_n^i - p)\|^2 \\
= (1 - \alpha_n^i)\|x_n - p\|^2 + \alpha_n^i \|w_n^i - p\|^2 - \alpha_n^i (1 - \alpha_n^i)\|x_n - w_n^i\|^2 \\
\leq (1 - \alpha_n^i)\|x_n - p\|^2 + \alpha_n^i H^2(T^i z_n^i, T^i p) - \alpha_n^i (1 - \alpha_n^i)\|x_n - w_n^i\|^2 \\
\leq (1 - \alpha_n^i)\|x_n - p\|^2 + \alpha_n^i \left[\|z_n^i - p\|^2 + \|z_n^i - w_n^i\|^2\right] - \alpha_n^i (1 - \alpha_n^i)\|x_n - w_n^i\|^2 \\
= (1 - \alpha_n^i)\|x_n - p\|^2 + \alpha_n^i \|z_n^i - p\|^2 + \alpha_n^i d^2(z_n^i, T^i z_n^i) - \alpha_n^i (1 - \alpha_n^i)\|x_n - w_n^i\|^2. \quad (3.1)
$$

Also,

$$
\|z_n^i - w_n^i\|^2 = \|(1 - \beta_n^i)x_n + \beta_n^i v_n^i - w_n^i\|^2 \\
= \|(1 - \beta_n^i)(x_n - w_n^i) + \beta_n^i (v_n^i n - w_n^i)\|^2 \\
= (1 - \beta_n^i)\|x_n - w_n^i\|^2 + \beta_n^i \|v_n^i - w_n^i\|^2 - \beta_n^i (1 - \beta_n^i)\|x_n - v_n^i\|^2. \quad (3.2)
$$
(3.1) and (3.2) imply that
\[
\|p - y_n^i\|^2 \leq (1 - \alpha_n^i)\|x_n - p\|^2 + \alpha_n^i (\|x_n - w_n^i\|^2 + \beta_n^i\|v_n^i - w_n^i\|^2)
\]
\[
+ \alpha_n^i [(1 - \beta_n^i)\|x_n - w_n^i\|^2 + \beta_n^i\|v_n^i - w_n^i\|^2 - \beta_n^i (1 - \beta_n^i)\|x_n - v_n^i\|^2]
\]
\[
= \alpha_n^i [(1 - \beta_n^i)\|x_n - w_n^i\|^2 + \beta_n^i\|v_n^i - w_n^i\|^2 - \beta_n^i (1 - \beta_n^i)\|x_n - v_n^i\|^2]
\]
\[
= \alpha_n^i [(1 - \beta_n^i)\|x_n - w_n^i\|^2 + \beta_n^i\|v_n^i - w_n^i\|^2 - \beta_n^i (1 - \beta_n^i)\|x_n - v_n^i\|^2]
\]
\[
\leq (1 - \alpha_n^i)\|x_n - p\|^2 + \alpha_n^i (\|x_n - p\|^2 + \|x_n - v_n^i\|^2)
\]
\[
(3.3)\]

(3.3) and (3.4) imply that
\[
\|p - y_n^i\|^2 \leq (1 - \alpha_n^i)\|x_n - p\|^2 + \alpha_n^i [(\|x_n - p\|^2 + \|x_n - v_n^i\|^2)
\]
\[
+ \alpha_n^i [(1 - \beta_n^i)\|x_n - w_n^i\|^2 + \beta_n^i\|v_n^i - w_n^i\|^2 - \beta_n^i (1 - \beta_n^i)\|x_n - v_n^i\|^2]
\]
\[
+ \alpha_n^i (\|x_n - p\|^2 + \|x_n - v_n^i\|^2)
\]
\[
= \alpha_n^i [(1 - \beta_n^i)\|x_n - w_n^i\|^2 + \beta_n^i\|v_n^i - w_n^i\|^2 - \beta_n^i (1 - \beta_n^i)\|x_n - v_n^i\|^2]
\]
\[
= \alpha_n^i [(1 - \beta_n^i)\|x_n - w_n^i\|^2 + \beta_n^i\|v_n^i - w_n^i\|^2 - \beta_n^i (1 - \beta_n^i)\|x_n - v_n^i\|^2]
\]
\[
\leq (1 - \alpha_n^i)\|x_n - p\|^2 + \alpha_n^i (\|x_n - p\|^2 + \|x_n - v_n^i\|^2)
\]
\[
\leq (1 - \alpha_n^i)\|x_n - p\|^2 + \alpha_n^i (\|x_n - p\|^2 + \|x_n - v_n^i\|^2)
\]
\[
(3.4)\]
This shows that \( p \in K_{n+1}^i \) for all \( i = 1, 2, \ldots, N \), therefore, \( p \in \bigcap_{i=1}^{N} K_{n+1}^i = K_{n+1} \) and hence \( \mathbb{F} \subseteq K_n \) for all \( n \geq 1 \). From \( x_n = P_{K_n}x_0 \) and Lemma 2.1 (i), we obtain

\[
\langle x_n - y, x_0 - x_n \rangle \geq 0, \quad \forall \ y \in K_n.
\]  

(3.6)

and

\[
\langle x_n - q, x_0 - x_n \rangle \geq 0, \quad \forall \ q \in F.
\]  

(3.7)

Using Lemma 2.1 (ii), we obtain

\[
\|x_n - x_0\|^2 = \|P_{K_n}x_0 - x_0\|^2 \leq \|x_0 - q\|^2 - \|q - x_n\|^2 \\
\leq \|x_0 - q\|^2,
\]

for each \( q \in \mathbb{F} \subset K_n \) and for all \( n \geq 1 \). Consequently, the sequences \( \{x_n\}, \{v_n^i\} \) and \( \{w_n^i\} \) \( i = 1, 2, \ldots, N \) are bounded. Furthermore, since \( x_n = P_{K_n}x_0, x_{n+1} = P_{K_{n+1}}x_0 \in K_{n+1} \subset K_n \), then from definition of \( P_K \) we have \( \|x_n - x_0\| \leq \|x_{n+1} - x_0\| \) for all \( n \geq 1 \). Therefore, the sequence \( \{\|x_n - x_0\|\}_{n=1}^{\infty} \) is nondecreasing. Thus, \( \lim_{n \to \infty} \|x_n - x_0\| \) exists.

From the construction of \( K_n \) we have that \( K_m \subset K_n \) and \( x_m = P_{K_n}x_0 \in K_n \) for any integer \( m \geq n \). Thus, from Lemma 2.1 (iii)

\[
\|x_m - x_n\|^2 = \|x_m - P_{K_n}x_0\|^2 \\
\leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2.
\]  

(3.8)

Letting \( m, n \to \infty \) in (3.8), we have \( \|x_m - x_n\| \to 0 \). Hence \( \{x_n\} \) is a Cauchy sequence. Since \( H \) is Hilbert and \( K^i \) is closed and convex for all \( i = 1, 2, \ldots, N \), we can assume that \( x_n \to P^* \in K^i \), for all \( i = 1, 2, \ldots, N \) as \( n \to \infty \). We now show that \( P^* \in F(T^i) \), for all \( i = 1, 2, \ldots, N \). From (3.5), we obtain

\[
\sum_{n=0}^{\infty} \alpha^2[1 - 2\beta - L^2\beta^2]|x_n - v_n^i|^2 \leq \sum_{n=0}^{\infty} \alpha^2 \beta_n^i[1 - 2\beta_n^i - L^2\beta_n^i]^2 |x_n - v_n^i|^2 \\
\leq \sum_{n=0}^{\infty} (|x_n - p|^2 - |x_{n+1} - x_n|^2) \\
\leq \|x_0 - p\|^2 + D < \infty.
\]

It then follows that \( \lim_{n \to \infty} |x_n - v_n^i| = 0 \). Since \( v_n^i \in T^i x_n \) we have that \( d(x_n, T^i x_n) \leq |x_n - v_n^i| \to 0 \) as \( n \to \infty \). Since \( T^i \) satisfies condition (1), \( \lim_{n \to \infty} d(x_n, F(T^i)) = 0 \).

Thus, there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \|x_{n_k}^i - p_k^i\| \leq \frac{1}{2k} \) for some \( \{p_k^i\}_{k=1}^{\infty} \subseteq F(T^i) \). We now show that \( \{p_k^i\}_{k=1}^{\infty} \) is a Cauchy sequence in \( F(T^i) \). Observe that when \( m = n + 1 \) in (3.8) we obtain

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\]  

(3.9)
Consequently, \( \lim_{n \to \infty} \|x_{n_{k+1}} - x_{n_k}\| = 0 \) for all subsequences \( \{x_{n_k}\} \) of \( \{x_n\} \). It then follows that
\[
\|p_{k+1}^i - p_k^i\| \leq \|p_{k+1}^i - x_{n_{k+1}}^i\| + \|x_{n_{k+1}}^i - x_n^i\| + \|x_n^i - p_k^i\|
\leq \frac{1}{2k+1} + \frac{1}{2k} + \|x_{n_{k+1}}^i - x_n^i\|
\leq \frac{1}{2k-1} + \|x_{n_{k+1}}^i - x_n^i\|.
\]
Therefore, \( \{p_k^i\} \) is a Cauchy sequence and converges to some \( q^i \in F(T^i) \) because \( F(T^i) \) is closed. Now,
\[
\|x_{n_k}^i - q^i\| \leq \|x_{n_k}^i - p_k^i\| + \|p_k^i - q^i\|.
\]
Hence \( x_{n_k}^i \to q^i \) as \( k \to \infty \).

It remains to show that \( p^* \) is in \( EP(F^i) \) for all \( i = 1, 2, \ldots, N \). Since \( x_n \) converges strongly to \( p^* \), uniqueness of limit of a convergent sequence guarantees that \( p^* = q^i \) for all \( i = 1, 2, \ldots, N \). Hence \( p^* \in F(T^i) \), for all \( i = 1, 2, \ldots, N \). It then follows that \( p^* \in \bigcap F(T^i) \).

Combining (3.9) and (3.10) we have
\[
\lim_{n \to \infty} \|x_{n+1} - u_n^i\| = 0.
\]

It follows from \( \lim_{n \to \infty} \|x_n - p^*\| = 0 \) and (3.11) that
\[
\lim_{n \to \infty} \|u_n^i - p^*\| = 0.
\]

Observe that
\[
\|p^* - x_n\|^2 - \|p^* - u_n^i\|^2 = \|x_n\|^2 - \|u_n^i\|^2 - 2\langle p^*, x_n - u_n^i \rangle
\leq \|x_n - u_n^i\| (\|x_n\| + \|u_n^i\|) + 2\|p^*\| \|x_n - u_n^i\|.
\]
It follows from (3.11) that
\[
\lim_{n \to \infty} \|p^* - x_n\| - \|p^* - u^i_n\| = 0. 
\tag{3.14}
\]

Now from (3.13)
\[
\|p^* - y^i_n\| \leq \|p^* - x_n\|.  
\tag{3.15}
\]

Also, using \(u^i_n = T^{r_i} y^i_n\), Lemma 2.4 and (3.15) we have
\[
\|u^i_n - y^i_n\|^2 = \|T^{r_i} y^i_n - y^i_n\|^2 \\
\leq \|p^* - y^i_n\|^2 - \|p^* - T^{r_i} y^i_n\|^2 \\
\leq \|p^* - x_n\|^2 - \|p^* - r_i y^i_n\|^2 \\
= \|p^* - x_n\|^2 - \|p^* - u^i_n\|^2. 
\tag{3.16}
\]

Therefore, from (3.14) and (3.16)
\[
\lim_{n \to \infty} \|u^i_n - y^i_n\| = 0.  
\tag{3.17}
\]

Consequently, from (3.12) and (3.17)
\[
\lim_{n \to \infty} \|y^i_n - p^*\| = 0. 
\tag{3.18}
\]

From the assumption that \(r^i_n \geq a > 0\),
\[
\lim_{n \to \infty} \frac{\|u^i_n - y^i_n\|}{r^i_n} = 0.  
\tag{3.19}
\]

Since \(u^i_n = T^{r_i} y^i_n\) implies
\[
F(u^i_n, y) + \frac{1}{r^i_n} \langle y - u^i_n, u^i_n - y^i_n \rangle \geq 0,
\]
we deduce from (A2) that
\[
\frac{\|u^i_n - y^i_n\|^2}{r^i_n} \geq \frac{1}{r^i_n} \langle y - u^i_n, u^i_n - y^i_n \rangle \geq -F(u^i_n, y) \geq F(y, u^i_n). \forall y \in K^i 
\]

By taking limit as \(n \to \infty\) of the above inequality and from (A4), (3.12) and (3.18), \(F(y, p^*) \leq 0\), for all \(y \in K^i\). Let \(t \in (0, 1)\) and for all \(y \in K^i\), since \(p^* \in K^i\), \(y_t = ty + (1-t)p^* \in K^i\). Hence \(F(y_t, p^*) \leq 0\). Therefore, from (A1),
\[
0 = F(y_t, y_t) \leq t F(y_t, y) + (1-t) F(y_t, p^*) \leq t F(y_t, y),
\]
that is, \(F(y_t, y) \geq 0\). Letting \(t \downarrow 0\), from (A3) we obtain \(F(p^*, y) \geq 0\) for all \(y \in K^i\) so that \(p^* \in EP(F^i)\) for all \(i = 1, 2, \ldots, N\). Hence \(p^* \in \mathbb{F}\).
Finally, we show that $p^* = P_Fx_0$. By taking the limits as $n \to \infty$ in (3.7) we have

$$(p^* - p^*, x_0 - p^*) \geq 0, \quad \forall q \in F.$$ 

Thus, from Lemma 2.1 (i) $p^* = P_Fx_0$. This completes the proof. ■

**Remark 3.3.** If $N = 1$ in Algorithm 4, we obtain the following algorithms considered by Isiogugu et al. in [16].

Let $H$ be a real Hilbert space and $K$ a nonempty closed convex subset of $H$. Let $F$ be a bifunction and $T$ an $L$-Lipschitzian pseudocontractive-type mapping such that $F : K \times K \to \mathbb{R}$ and $T : K \to CC(K)$ respectively. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$ and $\{r_n\}_{n=1}^{\infty} \subset [a, \infty)$ for some $a > 0$, then from an arbitrary $x_0 \in H$ we generate the sequences $\{x_n\}_{n=1}^{\infty}$ as follows.

**Algorithm 5.**

$$
\begin{align*}
x_0 & \in H, \\
K_0 & = K, \\
z_n & = (1 - \beta_n)x_n + \beta_n v_n, \\
y_n & = (1 - \alpha_n)x_n + \alpha_n w_n, \\
u_n & \in K \text{ such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in K, \\
K_{n+1} & = \{z \in K : \|z - u_n\|^2 \leq \|z - x_n\|^2\}, \\
x_{n+1} & = P_{K_{n+1}}x_0,
\end{align*}
$$

**Algorithm 6.**

$$
\begin{align*}
x_0 & \in H, \\
z_n & = (1 - \beta_n)x_n + \beta_n v_n, \\
y_n & = (1 - \alpha_n)x_n + \alpha_n w_n, \\
u_n & \in K \text{ such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in K, \\
x_{n+1} & = \frac{1}{2}(u_n + x_n),
\end{align*}
$$

where $w_n \in T(z_n) = T((1 - \beta_n)x_n + \beta_n v_n)$ with $d((1 - \beta_n)x_n + \beta_n v_n, T[(1 - \beta_n)x_n + \beta_n v_n]) = ||(1 - \beta_n)x_n + \beta_n v_n - w_n||$, $v_n \in Tx_n$ with $||x_n - v_n|| = d(x_n, Tx_n)$ and $||w_n - v_n|| \leq H(Tz_n, Tx_n)$.

4. **Numerical example of the computation**

We will apply Lemma 2.7 in the computation of the sequences $\{K_n^i\}_{n=1}^{\infty}, i = 1, 2, 3, \ldots, N$ from which we can easily determine the sequence $\{x_n\}_{n=0}^{\infty}$. 
Example 4.1. Let $H = \mathbb{R}$ (the reals with the usual norm and inner product), $i = 1, 2, \ldots, 4$ and $K_i = [-\sqrt{10i}, 1]$, for all $i$. Then for each $i$, we define:

(i) $T^i : [-\sqrt{10i}, 1] \to CC([-\sqrt{10i}, 1])$ by

\[
T^i x = \begin{cases} 
[-\sqrt{10i} x, -2x], & x \in [0, 1] 

\{ -\frac{x}{\sqrt{10i}} \}, & x \in (-\sqrt{10i}, 0).
\end{cases}
\]

Obviously, $T^i$ satisfies condition 1 since $d(x, F(T^i)) = d(x, \{0\}) = |x - 0| = |x|$, for all $i = 1, 2, 3, 4$, while

\[
d(x, T^i x) = \begin{cases} 
d(x, [-\sqrt{10i} x, -2ix]), & x \in [0, 1] 
d(x, \{-\frac{x}{\sqrt{10i}}\}), & x \in (-\sqrt{10i}, 0).
\end{cases}
\]

\[
\geq |x| = f(d(x, F(T^i)), \forall i.
\]

Where $f : [0, \infty) \to [0, \infty)$ is defined by $f(r) = r$.

Now, given any pair $x, y \in [0, 1]$,

\[
H^2(T^i x, T^i y) = |\sqrt{10i}(x - y)|^2 = 10i|x - y|^2 = |x - y|^2 + (10i - 1)|x - y|^2
\]

Also, given any $u \in T^i x, u = -\alpha x, 2i \leq \alpha \leq \sqrt{10i}$ and we can choose $v = -\alpha y \in T^i y$ so that $|u - v|^2 \leq H^2(T^i x, T^i y)$. Observe that

\[
|x - u - (y - v)|^2 = (1 + \alpha)^2 |x - y|^2.
\]

It then follows that

\[
H^2(T^i x, T^i y) = |x - y|^2 + \frac{10i - 1}{(1 + \alpha)^2} |x - u - (y - v)|^2
\leq |x - y|^2 + \frac{10i - 1}{(1 + 2i)^2} |x - u - (y - v)|^2
\leq |x - y|^2 + |x - u - (y - v)|^2, \forall i = 1, 2, 3, 4.
\]

Similarly, for any $x \in [0, 1], y \in [-\sqrt{10i}, 0]$,

\[
H^2(T^i x, T^i y) = |\sqrt{10i} x - \frac{y}{\sqrt{10i}}|^2 \leq |\sqrt{10i} x - \sqrt{10i} y|^2
\leq |x - y|^2 + |x - u - (y - v)|^2, \forall i = 1, 2, 3, 4.
\]
Furthermore, for any \( x, y \in [-\sqrt{10}, 0) \),

\[
H^2(T^i x, T^i y) = \frac{1}{\sqrt{10}i} \|x - y\|^2 \leq \|x - y\|^2 + \|x - u - (y - v)\|^2, \quad \forall i = 1, 2, 3, 4.
\]

Observe that for \( i = 1 \), any pair \( x, y \in [0, 1] \) and \( u \in T^1 x, v = 0 \). In particular for \( u = -2x \)

\[
H^2(T^1 x, T^1 y) = \|x - 0\|^2 + \frac{10 - 1}{(1 + 2)^2} \|x - (-2x)\|^2 = \|x - y\|^2 + \|x - u - (y - v)\|^2 > \|x - y\|^2 + k \|x - u - (y - v)\|^2, \quad \forall k \in (0, 1).
\]

Hence, \( T^1 \) is not \( K \)-strictly pseudocontractive-type mapping. Therefore, \( T^i \) is an \( L^i \)-Lipschitzian pseudocontractive-type mapping for each \( i = 1, 2, 3, 4 \) with \( L^i = \sqrt{10i} \).

It then follows that:

(ii) \( \psi_n^i = \begin{cases} -2ix_n, & x_n \in [0, 1] \\ -\frac{x_n}{\sqrt{10i}}, & x_n \in [-\sqrt{10i}, 0). \end{cases} \)

(iii) \( \{\alpha^i_n\}_{n=1}^\infty = \frac{10ni - (n + 1)(\sqrt{1 + 10i} + 1)}{10ni(\sqrt{1 + 10i} + 1)} \).

(iv) \( \{\beta^i_n\}_{n=1}^\infty = \frac{12ni - (n + 1)(\sqrt{1 + 10i} + 1)}{12ni(\sqrt{1 + 10i} + 1)} \).

(v) \( z_n^i = (1 - \beta^i_n)x_n + \beta^i_n \psi_n^i \).

(vi) \( w_n^i = \begin{cases} -2iz_n^i, & z_n^i \in [0, 1] \\ -\frac{z_n^i}{\sqrt{10i}}, & z_n^i \in [-\sqrt{10i}, 0). \end{cases} \)

(vii) \( y_n^i = (1 - \alpha^i_n)x_n + \alpha^i_n w_n^i \).

(viii) \( K_{n+1}^i = \begin{cases} \left[-\sqrt{10i}, \frac{1}{2}(x_n + u_n^i)\right], & x_n \in [0, 1] \\ \left[\frac{1}{2}(x_n + u_n^i), 1\right], & x_n \in [-\sqrt{10i}, 0). \end{cases} \)

(ix) \( K_{n+1} = \begin{cases} \left[-\sqrt{10i}, \min_{1 \leq i \leq 4} \frac{1}{2}(x_n + u_n^i)\right], & x_n \in [0, 1] \\ \left[\max_{1 \leq i \leq 4} \frac{1}{2}(x_n + u_n^i), 1\right], & x_n \in [-\sqrt{10i}, 0). \end{cases} \)

(x) \( x_{n+1} = \begin{cases} \min_{1 \leq i \leq 4} \frac{1}{2}(x_n + u_n^i), & x_n \in [0, 1] \\ \max_{1 \leq i \leq 4} \frac{1}{2}(x_n + u_n^i), & x_n \in [-\sqrt{10i}, 0). \end{cases} \)

We will define \( F^i : [-\sqrt{10i}, 1) \times [-\sqrt{10i}, 1] \rightarrow R, \{r_n^i\}_{n=1}^\infty \) and \( \{u_n^i\}_{n=1}^\infty \) as in [13].

That is,

(xi) \( F^i(x, y) = -ix^2 + iy^2 \).
Observe that

\[ F^i(z, y) + \frac{1}{r}(y - z, z - x) \geq 0 \Rightarrow iy^2 - iz^2 + \frac{1}{r}(y - z)(z - x) \geq 0, \]

\[ \Rightarrow iy^2 - iz^2 + \frac{1}{r}[yz - xy - z^2 + xz] \geq 0, \]

\[ \Rightarrow iyz - ixy - z^2 + xz \geq 0, \]

\[ \Rightarrow iyz + (z - x)y - irz^2 - z^2 + xz \geq 0. \]

Now \( F(y) = iyz + (z - x)y - irz^2 - z^2 + xz \) a is a quadratic function of \( y \) with coefficients \( a = ir \), \( b = z - x \) and \( c = -irz^2 - z^2 + xz \). Therefore, we can compute the discriminant \( \Delta \) of \( F \) as follows:

\[
\Delta = (z - x)^2 + 4ir(irz^2 + z^2 - xz) \\
= z^2 + x^2 - 2xz + 4i^2r^2z^2 + 4irz^2 - 4irxz \\
= (1 + 4i^2r^2 + 4ir)z^2 - 2(2ir + 1)xz + x^2 \\
= (1 + 2ir)^2z^2 - 2(1 + 2ir)xz + x^2 \\
= [(1 + 2ir)z - x]^2. \tag{4.1}
\]

Obviously, \( F(y) \geq 0 \) for all \( y \in \mathbb{R} \) if it has at most one solution in \( \mathbb{R} \). Thus \( \Delta \leq 0 \) and hence \( z = T_{ri}^i(x) = \frac{x}{1 + 2ir} \). Consequently

(xii) \( \{u^i_n\}_{n=1}^{\infty} = T_{ri}^i(y^i_n) = \left\{ \frac{y^i_n}{2ir^i_n + 1} \right\}_{n=1}^{\infty}. \)

(xiii) \( \{r^i_n\}_{n=1}^{\infty} = \left\{ \frac{ni + 1}{ni} \right\}_{n=1}^{\infty}, \)

It is easy to see that \( F_s(T^i) = \{0\} \neq \emptyset, E P(F^i) = \{0\} \) for each \( i \) and

\[
\mathbb{F} = \bigcap_{i=1}^{N} F_s(T^i) \cap \bigcap_{i=1}^{N} E P(F^i) = \{0\}.
\]

The algorithm is computed with Microsoft word Excel 97-2003 Workbook.

Table 1 shows different sequences generated from different values of \( x_0 \). In particular, we considered without loss of generality \( x_0 = 1, -1, \frac{1}{2}, -\frac{1}{2}, -\sqrt{10} \).
Approximation of the Common Solution of Equilibrium Problems

Table 1.

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Competing Interests
The Authors declare that there is no competing interest.

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