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Approximation of the Common Solution of Equilibrium Problems and Fixed Point Problems of Multi-valued Pseudocontractive-type Mappings in Hilbert Spaces

F.O. Isiogugu[1] **P. Pillay**

School of Mathematics, Statistics and Computer Sciences,
University of KwaZulu-Natal, Westville Campus,
Durban 4000, South Africa.
Department of Mathematics,
University of Nigeria, Nsukka, Nigeria.

E. E. Chima

Department of Mathematics,
University of Nigeria, Nsukka, Nigeria.

Abstract

A combined Ishikawa and Reich-Sabach iteration scheme is introduced in a real Hilbert space H for the approximation of the common solutions of equilibrium problems of bifunctions and fixed point problems of multi-valued pseudocontractive-type mappings. It is also proved that the iteration scheme converges strongly to a common element of the sets of fixed points of a finite family of multi-valued pseudocontractive-type mappings and the sets of solutions of a finite family of equilibrium problems. A numerical example for the computation of this algorithm is presented with concrete examples. The obtained results improve, complement and extend many results on equilibrium problems, multi-valued and single-valued mappings in the contemporary literature.

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[1]Corresponding Author

1. Introduction

Let X be a normed space, K a subset of X and $T : D(T) \subseteq X \to 2^X$ a multi-valued map.

Definition 1.1. K is called proximinal if for each $x \in X$ there exists $k \in K$ such that

$$||x - k|| = \inf\{||x - y|| : y \in K\} = d(x, K). \tag{1.1}$$

It is known that every closed convex subset of a uniformly convex Banach space is proximinal. We shall denote the family of all nonempty closed and bounded subsets of X by $CB(X)$, the family of all nonempty closed and convex subsets of X by $CC(X)$, the family of all nonempty subsets of X by 2^X and the family of all proximinal subsets of X by $P(X)$, while H denotes the Hausdorff metric induced by the metric d on a normed space X, that is, for every $A, B \in CB(X)$,

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

Definition 1.2. A point $x \in D(T)$ is called a fixed point of T if $x \in Tx$. If $Tx = \{x\}$, x is called a strict fixed point of T. The set $F(T) = \{x \in D(T) : x \in Tx\}$ is called the fixed point set of the multi-valued map T while the set $F_s(T) = \{x \in D(T) : Tx = \{x\}\}$ is called the strict fixed point set of T.

Definition 1.3. T called $L - Lipschitzian$ if there exists $L \geq 0$ such that for all $x, y \in D(T)$

$$H(Tx, Ty) \leq L||x - y||. \tag{1.2}$$

In (1.2), if $L \in [0, 1)$ T is said to be a contraction while T is nonexpansive if $L = 1$. T is called quasi-nonexpansive if $F(T) = \{x \in D(T) : x \in Tx\} \neq \emptyset$ and for all $p \in F(T)$,

$$H(Tx, Tp) \leq ||x - p||. \tag{1.3}$$

Clearly, every nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive.

Definition 1.4. ([1]) T is said to be k-strictly pseudocontractive-type of Isiogugu [1] if there exists $k \in (0, 1)$ such that given any pair $x, y \in D(T)$ and $u \in Tx$, there exists $v \in Ty$ satisfying $||u - v|| \leq H(Tx, Ty)$ and

$$H^2(Tx, Ty) \leq ||x - y||^2 + k||x - u - (y - v)||^2. \tag{1.4}$$

If $k = 1$ in (1.4), T is said to be pseudocontractive-type while T is nonexpansive-type if $k = 0$.

Definition 1.5. ([15]) A multi-valued mapping $T : K \to P(K)$ is said to satisfy condition (1) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x, Tx) \geq f(d(x, F(T))), \quad \forall x \in K.$$

Let H be a real Hilbert space with an inner product $\langle ., . \rangle$ and a norm $\|.\|$, respectively and let K be a nonempty closed convex subset of H. Let $A : H \to H$ be an operator on H and $F : K \times K \to \mathbb{R}$ be a bifunction on K, where $\mathbb{R}$ is the set of real numbers. The variational inequality problem of A in K denoted by $VIP(A, K)$ is to find an $x^* \in K$ such that

$$\langle x - x^*, A(x^*) \rangle \geq 0, \quad \forall x \in K, \tag{1.5}$$

while the equilibrium problem for F is to find $x^* \in K$ such that

$$F(x^*, x) \geq 0, \quad \forall x \in K. \tag{1.6}$$

The set of solutions of (1.6) is denoted by $EP(F)$. Suppose $F(x, y) = \langle y - x, Ax \rangle$ for all $x, y \in K$, then $w \in EP(F)$ if and only if w is a solution of (1.5). Many problems in optimization, economics and physics reduce to finding a solution of (1.5), (see for examples, [3], [4] [6] and the references therein). The following conditions are assumed for solving the equilibrium problems for a bifunction $F : K \times K \to \mathbb{R}$,

(A1) $F(x, x) = 0$ for all $x \in K$.

(A2) F is monotone, that is , $F(x, y) + F(y, x) \leq 0$, for all $x, y \in K$.

(A3) For each $x, y, z \in K$, $\lim\limits_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$.

(A4) For each $x \in K$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Several algorithms have been introduced by authors for approximating solutions of equilibrium problems for a bifunction (or finite family of bifunctions) as well as a common element of the fixed point sets of finite family of multi-valued (or single-valued) mappings and the set of solutions of finite family of equilibrium problems (see for examples [7], [8], [9], [10], [11] and references therein).

In [7], Reich and Sabach proposed three algorithms for solving (common) equilibrium problems of bifunction(s) g in a general reflexive Banach spaces using the well chosen convex function f, the Bregman distance and the projection associated with it. They proposed one of the algorithms as follows.

Let X be a reflexive Banach space, $\{K_i\}_{i=1}^N$ a finite family of nonempty, closed and convex subsets of X. Let $\{\lambda^i\}_{i=1}^N$ be a finite family of positive real numbers and $\{g_i\}_{i=1}^N$ a finite family of bifunctions, with $g_i : K_i \times K_i \to \mathbb{R}$ for each $i = 1, 2, \ldots, N$. Suppose $f : X \to R$ is a coercive Legendre function which is bounded, uniformly FrÃ©chet differentiable and totally convex on bounded subsets of X, $Res^f_{\lambda^i g_i}$ is the resolvent of g_i with respect to λ^i and f for each $i = 1, 2, \ldots, N$ and D_f is the Bregman distance on X. If $E = \bigcap\limits_{i=1}^{N} EP(g_i) \neq \emptyset$, then the sequences $\{x_n\}_{n=1}^{\infty}$ were generated from an arbitrary $x_0 \in X$ as follows:

Algorithm 1 (Algorithm II [7])

$$\begin{cases} x_0 \in X, \\ K_0^i = X, \quad i = 1, 2, ..., N \\ y_n^i = Res^f_{\lambda_n^i g_i}(x_n + e_n^i), \\ K_{n+1}^i = \{z \in K_n^i : D_f(z, y_n^i) \le D_f(z, x_n + e_n^i)\}, \\ K_{n+1} = \bigcap_{i=1}^{N} K_{n+1}^i, \\ x_{n+1} = proj^f_{K_{n+1}}(x_0), \quad n = 0, 1, 2, \ldots \end{cases}$$

Furthermore, they proved that if $\liminf_{n\to\infty} \lambda_n^i > 0$ and $\lim_{n\to\infty} e_n = 0$, the sequences converge strongly to $proj^f_E(x_0)$.

Recently, Isiogugu [8] obtained a strong convergence of a hybrid algorithm to a common element of the fixed point sets of two multi-valued strictly pseudocontractive-type mappings and the set of solutions of an equilibrium problem in Hilbert spaces using a strict fixed point set condition. She proved the following theorem:

Theorem 1.6. ([8]) Let K be a nonempty closed convex subset of a real Hilbert space H, let $f : K \times K \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and let $S, T : K \to P(K)$ be two strictly pseudocontractive-type mappings with contractive coefficients λ_1 and λ_2 respectively such that $\mathbb{F} = F_s(T) \bigcap F_s(S) \bigcap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in K$ as follows:

$$\begin{cases} x_0 \in H, \\ K_1 = K, \\ x_1 = P_K x_0, \\ y_n = \alpha_n x_n + (1 - \alpha_n)[\beta_n v_n + (1 - \beta_n) z_n], \\ u_n \in K \text{ such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \ge 0, \quad \forall y \in K, \\ K_{n+1} = \{z \in K_n : \|z - u_n\|^2 \le \|z - x_n\|^2\}, \\ x_{n+1} = P_{K_{n+1}} x_0, \end{cases}$$

where $v_n \in Tx_n$, $z_n \in Sx_n$. $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are sequences in [0,1] satisfying

(i) $\alpha_n \ge \max\{\lambda_1, \lambda_2\}$.

(ii) $\liminf_{n\to\infty}(1 - \alpha_n)(1 - \beta_n)(\alpha_n - \lambda_1) > 0$, $\liminf_{n\to\infty}(1 - \alpha_n)(\alpha_n - \lambda_2)\beta_n > 0$

(iii) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$ converges strongly to $p \in P_{\mathbb{F}} x_0$.

Isiogugu et al. [13] observed that in Algorithms 1, if $T^i : K^i \to K^i$ is the identity map on K^i, $i = 1, 2, \ldots, N$, respectively and $\upsilon_n^i = \alpha_n x_n + (1-\alpha_n)T^i x_n + e_n^i = x_n + e_n^i$, then we can rewrite Algorithm 1 in the following form:

Algorithm 2.

$$\begin{cases} x_0 \in X, \\ K_0^i = X, \quad i = 1, 2, \dots, N \\ \upsilon_n^i = \alpha_n x_n + (1-\alpha_n)T^i x_n + e_n^i, \\ y_n^i = Res^f_{\lambda_n^i g_i}(\upsilon_n^i), \\ K_{n+1}^i = \{z \in K_n^i : D_f(z, y_n^i) \leq D_f(z, \upsilon_n^i)\}, \\ K_{n+1} = \bigcap_{i=1}^N K_{n+1}^i, \\ x_{n+1} = proj^f_{K_{n+1}}(x_0), \quad n = 0, 1, 2, \dots \end{cases}$$

Motivated by the above observations in Algorithms 2 and the iteration scheme in Theorem 1.1 considered by Isiogugu in [8], Isiogugu et al. [13] constructed a hybrid algorithm for approximating a common element of the fixed point sets of a finite family of multi-valued nonexpansive mappings and the set of solutions of a finite family of equilibrium problems in Hilbert spaces without error terms. They studied the following iteration scheme: Let H be a real Hilbert space and $\{K^i\}_{i=1}^N$ a finite family of nonempty closed convex subsets of H. Let $\{f^i\}_{i=1}^N$ be a finite family of bifunctions and $\{T^i\}_{i=1}^N$ a finite family of nonexpansive mappings such that $f^i : K^i \times K^i \to \mathbb{R}$ and $T^i : K^i \to P(K^i)$ for all $i = 1, 2, \dots, N$, respectively. Let $\{\alpha_n^i\}_{n=1}^\infty$ be sequences in [0,1] and $\{r_n^i\}_{n=1}^\infty \subset [a, \infty)$ for some $a > 0$ for all i=1, 2, ..., N, then from an arbitrary $x_0 \in H$ the sequence $\{x_n\}_{n=1}^\infty$ is generated as follows:

Algorithm 3 (Algorithm 7 [13]).

$$\begin{cases} x_0 \in H, \\ K_1^i = K^i, \quad \forall\, i = 1, 2, ..., N, \\ K_1 = \bigcap_{i=1}^N K_1^i, \\ x_1 = P_{K_1} x_0, \\ y_n^i = \alpha_n^i x_n + (1-\alpha_n^i) v_n^i, \\ u_n^i \in K^i \text{ such that } f^i(u_n^i, y) + \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - y_n^i\rangle \geq 0, \quad \forall y \in K^i, \\ K_{n+1}^i = \{z \in K_n^i : \|z - u_n^i\|^2 \leq \|z - x_n\|^2\}, \\ K_{n+1} = \bigcap_{i=1}^N K_{n+1}^i, \\ x_{n+1} = P_{K_{n+1}} x_0, \end{cases}$$

where $v_n^i \in T^i x_n$.

Using the above algorithm, they proved the following theorem:

Theorem 1.7. (Theorem 2 [13]). Let $H, \{K^i\}_{i=1}^N, \{T^i\}_{i=1}^N, \{f^i\}_{i=1}^N, \{\alpha_n^i\}_{n=1}^\infty$ and $\{r_n^i\}_{n=1}^\infty$ be as in algorithm 3. Suppose f^i satisfying (A1)-(A4) for all $i = 1, 2, \dots, N$, $\mathbb{F} =$

$\bigcap_{i=1}^{N} F_s T^i \bigcap_{i=1}^{N} EP(f^i) \neq \emptyset$, then $\{x_n\}$ converges strongly to $p \in P_{\mathbb{F}} x_0$ if for each $i = 1, 2, \ldots, N$ and for all $n \geq 1$, $\liminf_{n\to\infty} \alpha_n^i(1 - \alpha_n^i) > 0$.

Motivated by Algorithm 3 above, we introduce a combined Ishikawa and Reich-Sabach iteration scheme in a real Hilbert space H for the approximation of the common solution of equilibrium problems of bifunctions and fixed point problems of multi-valued pseudocontractive-type mappings; establish close and convex property for the set of fixed points of multi-valued pseudocontractive-mapping which guarantee the application of the algorithm to this class of mappings; and prove that the iteration scheme converges strongly to a common element of the fixed point sets of a finite family of multi-valued pseudocontractive-type mappings and the set of solutions of a finite family of equilibrium problems. Furthermore, a numerical example of the computation of this algorithm is presented with concrete examples. This work is a continuation of the study on the computability of algorithms for approximating the solutions of equilibrium problems for bifunctions involving the construction of the sequences $\{K_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$, from an arbitrary $x_0 \in H$, where $K_{n+1} = \{z \in K_n : \|z - u_n\|^2 \leq \|z - x_n\|^2\}$, $x_{n+1} = P_{K_{n+1}} x_0$, while P_{K_n} is the projection map and $\{u_n\}_{n=1}^{\infty}$ is the sequence of the resolvent of the bifunctions. The obtained results improve, complement and extend many results on equilibrium problems, multi-valued and single-valued mappings in the contemporary literature.

2. Preliminaries

Lemma 2.1. Let H be a real Hilbert space and let K be a nonempty closed convex subset of H. Let P_K be the convex projection onto K, then, the convex projection is characterized by the following relations;

(i) $x^* = P_K(x) \Leftrightarrow \langle x - x^*, y - x^* \rangle \leq 0$, for all $y \in K$.

(ii) $||x - P_K x||^2 \leq ||x - y||^2 - ||y - P_K x||^2$.

(iii) $||x - P_K y||^2 \leq ||x - y||^2 - ||P_K y - y||^2$.

Lemma 2.2. ([3]). Let K be a nonempty closed convex subset of a real Hilbert space H and $F : K \times K \to \mathbb{R}$ a bifunction satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in K$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in K.$$

Lemma 2.3. ([4]). Let K be a nonempty closed convex subset of a real Hilbert space H. Assume that $F : K \times K \to \mathbb{R}$ that satisfies (A1)-(A4). Let $r > 0$ and $x \in H$, define

$T_r : H \to 2^K$ by

$$T_r(x) = \{z \in K : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0\}, \quad \forall y \in K.$$

Then, the following conditions hold:

(1) T_r is single valued.

(2) T_r is firmly nonexpansive, that is for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$.

(3) $F(T_r) = EP(F)$.

(4) $EP(F)$ is closed and convex.

Lemma 2.4. ([6]). Let K be a nonempty closed convex subset of a real Hilbert space H and $F : K \times K \to \mathbb{R}$ a bifunction satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, for all $x \in H$ and $p \in F(T_r)$,

$$\|p - T_r x\|^2 + \|T_r x - x\|^2 \leq \|p - x\|^2.$$

Lemma 2.5. ([16]). Let H be a real Hilbert space and $T : D(T) \subseteq H \to P(H)$ be a multi-valued L-Lipschtizian mapping, then, fixed point set of T is closed.

3. Main Results

Proposition 3.1. Let H be a real Hilbert space, C a closed convex subset of H and $T : C \subseteq H \to CC(C)$ be a multi-valued, $L-$Lipschtizian pseudocontractive-type mapping. If $F_s(T)$ (the strict fixed point set) of T is nonempty, then, it is convex.

Proof. Let $p_1, p_2 \in F_s(T)$, we show that $p = \lambda p_1 + (1 - \lambda)p_2 \in F_s(T)$. For each $x \in D(T)$, let $T_\beta x = T[(1 - \beta)x + \beta u_x]$, where $u_x \in Tx$ with $d(x, Tx) = \|x - u_x\|$ and $\beta \in (0, \frac{1}{\sqrt{1 + L^2} + 1})$. Clearly, $T_\beta x$ is well defined since u_x is unique and C is convex. Also if $p^* \in F_s(T)$, then, $T_\beta p^* = \{p^*\}$. Observe that for any $u_{\beta x} \in T_\beta x$, given any $p^* \in F_s(T)$, the pseudocontractive-type condition on T implies that

$$\begin{aligned}\|u_{\beta x} - p^*\|^2 &\leq H^2(T_\beta x, Tp^*)\\ &\leq \|(1 - \beta)x + \beta u_x - p^*\|^2 + \|((1 - \beta)x + \beta u_x) - u_{\beta x}\|^2.\end{aligned}$$

Similarly,

$$\begin{aligned}\|u_x - p^*\|^2 &\leq H^2(Tx, Tp^*)\\ &\leq \|x - p^*\|^2 + \|x - u_x\|^2\end{aligned}$$

It follows that for the pair $p, (1-\beta)p+\beta u_p$ and u_p, there exists $u_{\beta p} \in T_\beta p = T[(1-\beta)p+\beta u_p]$ with $\|u_p - u_{\beta p}\| \leq H(Tp, T_\beta p)$. Now,

$$\begin{aligned} d^2(p, T_\beta p) \leq \|p-u_{\beta p}\|^2 &= \|\lambda p_1 + (1-\lambda)p_2 - u_{\beta p}\|^2 \\ &= \|\lambda[p_1 - u_{\beta p}] + (1-\lambda)[p_2 - u_{\beta p}]\|^2 \\ &= \lambda\|p_1 - u_{\beta p}\|^2 + (1-\lambda)\|p_2 - u_{\beta p}\|^2 \\ &\quad -\lambda(1-\lambda)\|p_1 - p_2\|^2. \end{aligned}$$

Also,

$$\begin{aligned} d^2(p_1, T_\beta p) &\leq \|p_1 - u_{\beta p}\|^2 \leq H^2(Tp_1, T_\beta p) \\ &\leq \|[(1-\beta)p+\beta u_p] - p_1\|^2 + \|[(1-\beta)p+\beta u_p] - u_{\beta p}\|^2 \\ &= \|(1-\beta)[p-p_1] + \beta[u_p - p_1]\|^2 \\ &\quad +\|(1-\beta)[p-u_{\beta p}] + \beta[u_p - u_{\beta p}]\|^2 \\ &= (1-\beta)\|p-p_1\|^2 + \beta\|u_p - p_1\|^2 - \beta(1-\beta)\|p-u_p\|^2 \\ &\quad +(1-\beta)\|p-u_{\beta p}\|^2 + \beta\|u_p - u_{\beta p}\|^2 - \beta(1-\beta)\|p-u_p\|^2 \\ &\leq (1-\beta)\|p-p_1\|^2 + \beta H^2(Tp, Tp_1) - \beta(1-\beta)\|p-u_p\|^2 \\ &\quad +(1-\beta)\|p-u_{\beta p}\|^2 + \beta H^2(Tp, T_\beta p\|^2 - \beta(1-\beta)\|p-u_p\|^2 \\ &\leq (1-\beta)\|p-p_1\|^2 + \beta[\|p-p_1\|^2 + \|p-u_p\|^2] - \beta(1-\beta)\|p-u_p\|^2. \\ &\quad +(1-\beta)\|p-u_{\beta p}\|^2 + \beta L^2\|p-[(1-\beta)p+\beta u_p]\|^2 \\ &\quad -\beta(1-\beta)\|p-u_p\|^2 \\ &\leq (1-\beta)\|p-p_1\|^2 + \beta[\|p-p_1\|^2 + \|p-u_p\|^2] - \beta(1-\beta)\|p-u_p\|^2 \\ &\quad +(1-\beta)\|p-u_{\beta p}\|^2 + \beta L^2\beta^2\|p-u_p\|^2 - \beta(1-\beta)\|p-u_p\|^2 \\ &= \|p-p_1\|^2 - \beta[1-2\beta-L^2\beta^2]\|p-u_p\|^2 + (1-\beta)\|p-u_{\beta p}\|^2 \\ &\leq \|p-p_1\|^2 + (1-\beta)\|p-u_{\beta p}\|^2 \end{aligned}$$

Similarly,

$$d^2(p_2, T_\beta p) \leq \|p_2 - u_{\beta p}\|^2 \leq \|p-p_2\|^2 + (1-\beta)\|p-u_{\beta p}\|^2$$

Hence,

$$\begin{aligned} \|p-u_{\beta p}\|^2 &\leq \lambda[\|p-p_1\|^2 + (1-\beta)\|p-u_{\beta p}\|^2] \\ &\quad +(1-\lambda)[\|p-p_2\|^2 + (1-\beta)\|p-u_{\beta p}\|^2] \\ &\quad -\lambda(1-\lambda)\|p_1-p_2\|^2 \\ &= \|\lambda p_1 + (1-\lambda)p_2 - p\|^2 + (1-\beta)\|p-u_{\beta p}\|^2 \\ &= +(1-\beta)\|p-u_{\beta p}\|^2 \end{aligned}$$

This implies that $0 \leq \beta\|p - u_{\beta p}\| \leq 0$. Since $\beta \in (0, \frac{1}{\sqrt{1+L^2}+1})$, we have that $\|p - u_{\beta p}\| = 0$. Observe that $d(p, T_\beta p) \leq \|p - u_{\beta p}\| = 0 \leq d(p, T_\beta p)$, therefore, $d(p, T_\beta p) = \|p - u_{\beta p}\| = 0$ and $p = u_{\beta p} \in T_\beta p$.

$$\begin{aligned} d(p, Tp) \leq d(p, T_\beta p) + H(T_\beta p, Tp) &\leq L\|(1-\beta)p + \beta u_p - p\| \\ &= L\beta d(p, Tp). \end{aligned}$$

Thus, $0 \leq (1-\beta L)d(p, T_P) \leq 0$. Consequently, $d(p, Tp) = 0$ and proximinal property of T guarantees the existence $u \in Tp$ such that $\|u - p\| = 0$. Hence, $p \in Tp$. ■

We now consider the following algorithm.

Let H be a real Hilbert space and $\{K^i\}_{i=1}^N$ a finite family of nonempty closed convex subsets of H. Let $\{F^i\}_{i=1}^N$ be a finite family of bifunctions and $\{T^i\}_{i=1}^N$ a finite family of L^i-Lipschitzian pseudocontractive-type mappings such that $F^i : K^i \times K^i \to \mathbb{R}$ and $T^i : K^i \to CC(K^i)$ for all $i = 1, 2, \ldots, N$ respectively. Let $\{\alpha_n^i\}_{n=1}^\infty$, $\{\beta_n^i\}_{n=1}^\infty$be sequences in [0,1] and $\{r_n^i\}_{n=1}^\infty \subset [a, \infty)$ for some $a > 0$, for all i=1, 2, …, N. Then, from an arbitrary $x_0 \in H$ we generate the sequence $\{x_n\}_{n=1}^\infty$ as follows:

Algorithm 4.

$$\begin{cases} x_0 \in H, \\ K_0^i = K^i, \quad \forall\, i = 1, 2, \ldots, N, \\ z_n^i = (1-\beta_n^i)x_n + \beta_n^i v_n^i, \\ y_n^i = (1-\alpha_n^i)x_n + \alpha_n^i w_n^i, \\ u_n^i \in K^i \text{ such that } F^i(u_n^i, y) + \dfrac{1}{r_n^i}\langle y - u_n^i, u_n^i - y_n^i\rangle \geq 0, \quad \forall y \in K^i, \\ K_{n+1}^i = \{z \in K_n^i : \|z - u_n^i\|^2 \leq \|z - x_n\|^2\}, \\ K_{n+1} = \displaystyle\bigcap_{i=1}^N K_{n+1}^i, \\ x_{n+1} = P_{K_{n+1}}x_0, \end{cases}$$

where $w_n^i \in T^i(z_n^i) = T^i((1-\beta_n^i)x_n + \beta_n^i v_n^i)$ with $d((1-\beta_n^i)x_n + \beta_n^i v_n^i, T^i[(1-\beta_n^i)x_n + \beta_n^i v_n^i]) = ||(1-\beta_n^i)x_n + \beta_n^i v_n^i - w_n^i||$, $v_n^i \in T^i x_n$ with $||x_n - v_n^i|| = d(x_n, T^i x_n)$ and $||w_n^i - v_n^i|| \leq H(T^i z_n^i, T^i x_n)$.

Theorem 3.2. Let H, $\{K^i\}_{i=1}^N$, $\{T^i\}_{i=1}^N$, $\{F^i\}_{i=1}^N$, $\{\alpha_n^i\}_{n=1}^\infty$, $\{\beta_n^i\}_{n=1}^\infty$ and $\{r_n^i\}_{n=1}^\infty$ be as in Algorithm 4. Suppose F^i satisfying (A1)-(A4) for all $i = 1, 2, \ldots, N$, $\mathbb{F} = \bigcap_{i=1}^N F_s T^i \bigcap \bigcap_{i=1}^N EP(F^i) \neq \emptyset$, then $\{x_n\}$ converges strongly to $p \in P_{\mathbb{F}}x_0$ if for each $i = 1, 2, \ldots, N$ and for all $n \geq 1$, $\{\alpha_n^i\}$ and $\{\beta_n^i\}$ are real sequences satisfying:

(i) $0 \leq \alpha_n^i \leq \beta_n^i < 1$;

(ii) $\liminf_{n\to\infty} \alpha_n^i = \alpha^i > 0$;

(iii) $\sup_{n\geq 1} \beta_n^i \leq \beta^i \leq \dfrac{1}{\sqrt{1+(L^i)^2}+1}$.

Proof. Since K_n^i is closed and convex for all $n \geq 1$ and for all $i = 1, 2, \ldots, N$, $K_n = \bigcap_{i=1}^{N} K_n^i$ is closed and convex and hence $P_{K_{n+1}}x_0$ is well defined, also, $u_n^i = T_{r_n^i} y_n^i$. Next, we show that $\mathbb{F} \subset K_n$, for all $n \geq 1$. $\mathbb{F} \subset K_1^i = K^i$ for all $i = 1, 2, \ldots, N$, therefore, $\mathbb{F} \subset \bigcap_{i=1}^{N} K_1^i = K_1$. Assume $\mathbb{F} \subset K_n = \bigcap_{i=1}^{N} K_n^i$. Using Lemma 2.3, for all $p \in \mathbb{F}$ we have

$$\begin{aligned}
\|p - u_n^i\|^2 &= \|p - T_{r_n^i} y_n^i\|^2 \\
&\leq \|p - y_n^i\|^2 \\
&= ||(1-\alpha_n^i)x_n + \alpha_n^i w_n^i - p||^2 \\
&= ||(1-\alpha_n^i)(x_n - p) + \alpha_n^i(w_n^i - p)||^2 \\
&= (1-\alpha_n^i)||x_n - p||^2 + \alpha_n^i||w_n^i - p||^2 \\
&\quad - \alpha_n^i(1-\alpha_n^i)||x_n - w_n^i||^2 \\
&\leq (1-\alpha_n^i)||x_n - p||^2 + \alpha_n^i H^2(T^i z_n^i, T^i p) \\
&\quad - \alpha_n^i(1-\alpha_n^i)||x_n - w_n^i||^2 \\
&\leq (1-\alpha_n^i)||x_n - p||^2 + \alpha_n^i\Big[||z_n^i - p||^2 \\
&\quad + ||z_n^i - w_n^i||^2\Big] - \alpha_n^i(1-\alpha_n^i)||x_n - w_n^i||^2 \\
&= (1-\alpha_n^i)||x_n - p||^2 + \alpha_n^i||z_n^i - p||^2 + \alpha_n^i d^2(z_n^i, T^i z_n^i) \\
&\quad - \alpha_n^i(1-\alpha_n^i)||x_n - w_n^i||^2. \qquad (3.1)
\end{aligned}$$

Also,

$$\begin{aligned}
||z_n^i - w_n^i||^2 &= ||(1-\beta_n^i)x_n + \beta_n^i v_n^i - w_n^i||^2 \\
&= ||(1-\beta_n^i)(x_n - w_n^i) + \beta_n^i(v_n^i n - w_n^i)||^2 \\
&= (1-\beta_n^i)||x_n - w_n^i||^2 + \beta_n^i||v_n^i - w_n^i||^2 \\
&\quad - \beta_n^i(1-\beta_n^i)||x_n - v_n^i||^2. \qquad (3.2)
\end{aligned}$$

(3.1) and (3.2) imply that

$$\begin{aligned}\|p - y_n^i\|^2 \le\ & (1-\alpha_n^i)||x_n - p||^2 + \alpha_n^i||z_n^i - p||^2 \\ & + \alpha_n^i\Big[(1-\beta_n^i)||x_n - w_n^i||^2 + \beta_n^i||v_n^i - w_n^i||^2 \\ & - \beta_n^i(1-\beta_n^i)||x_n - v_n^i||^2\Big] \\ & - \alpha_n^i(1-\alpha_n^i)||x_n - w_n^i||^2. \end{aligned} \tag{3.3}$$

$$\begin{aligned}||z_n^i - p||^2 &= ||(1-\beta_n^i)x_n + \beta_n^i v_n^i - p||^2 \\ &= ||(1-\beta_n^i)(x_n - p) + \beta_n^i(v_n^i - p)||^2 \\ &= (1-\beta_n^i)||x_n - p||^2 + \beta_n^i||v_n^i - p||^2 \\ &\quad - \beta_n^i(1-\beta_n^i)||x_n - v_n^i||^2 \\ &\le (1-\beta_n^i)||x_n - p||^2 + \beta_n^i H^2(T^i x_n, T^i p) \\ &\quad - \beta_n^i(1-\beta_n^i)||x_n - v_n^i||^2 \\ &\le (1-\beta_n^i)||x_n - p||^2 + \beta_n^i\Big[||x_n - p||^2 + ||x_n - v_n^i||^2\Big] \\ &\quad - \beta_n^i(1-\beta_n^i)||x_n - v_n^i||^2 \\ &= ||x_n - p||^2 + {\beta_n^i}^2||x_n - v_n^i||^2. \end{aligned} \tag{3.4}$$

(3.3) and (3.4) imply that

$$\begin{aligned}\|p - y_n^i\|^2 &\le (1-\alpha_n^i)||x_n - p||^2 + \alpha_n^i\Big[||x_n - p||^2 + {\beta_n^i}^2||x_n - v_n^i||^2\Big] \\ &\quad + \alpha_n^i\Big[(1-\beta_n^i)||x_n - w_n^i||^2 + \beta_n^i||v_n^i - w_n^i||^2 \\ &\quad - \beta_n^i(1-\beta_n^i)||x_n - v_n^i||^2\Big] - \alpha_n^i(1-\alpha_n^i)||x_n - w_n^i||^2 \\ &= (1-\alpha_n^i)||x_n - p||^2 + \alpha_n^i||x_n - p||^2 + \alpha_n^i{\beta_n^i}^2||x_n - v_n^i||^2 \\ &\quad + \alpha_n^i(1-\beta_n^i)||x_n - w_n^i||^2 + \alpha_n^i\beta_n^i||v_n^i - w_n^i||^2 \\ &\quad - \alpha_n^i\beta_n^i(1-\beta_n^i)||x_n - v_n^i||^2 - \alpha_n^i(1-\alpha_n^i)||x_n - w_n^i||^2 \\ &\le ||x_n - p||^2 + \alpha_n^i{\beta_n^i}^2||x_n - v_n^i||^2 + \alpha_n^i\beta_n^i H^2(T^i x_n, T^i z_n^i) \\ &\quad - \alpha_n^i(\beta_n^i - \alpha_n^i)||x_n - w_n^i||^2 - \alpha_n^i\beta_n^i(1-\beta_n^i)||x_n - v_n^i||^2 \\ &\le ||x_n - p||^2 + \alpha_n^i{\beta_n^i}^2||x_n - v_n^i||^2 + \alpha_n^i{\beta_n^i}^3{L^i}^2\|x_n - v_n^i\|^2 \\ &\quad - \alpha_n^i\beta_n^i(1-\beta_n^i)||x_n - v_n^i||^2 - \alpha_n^i(\beta_n^i - \alpha_n^i)||x_n - w_n^i||^2 \\ &= ||x_n - p||^2 - \alpha_n^i\beta_n^i[1 - 2\beta_n^i - {L^i}^2{\beta_n^i}^2]||x_n - v_n^i||^2 \\ &\quad - \alpha_n^i(\beta_n^i - \alpha_n^i)||x_n - w_n^i||^2 \\ &\le ||x_n - p||^2 - \alpha_n^i\beta_n^i[1 - 2\beta_n^i - {L^i}^2{\beta_n^i}^2]||x_n - v_n^i||^2 \end{aligned} \tag{3.5}$$

This shows that $p \in K_{n+1}^i$ for all $i = 1, 2, \ldots, N$, therefore, $p \in \bigcap_{i=1}^{N} K_{n+1}^i = K_{n+1}$ and hence $\mathbb{F} \subseteq K_n$ for all $n \geq 1$. From $x_n = P_{K_n}x_0$ and Lemma 2.1 (i), we obtain

$$\langle x_n - y, x_0 - x_n \rangle \geq 0, \quad \forall\, y \in K_n. \tag{3.6}$$

and

$$\langle x_n - q, x_0 - x_n \rangle \geq 0, \quad \forall\, q \in F. \tag{3.7}$$

Using Lemma 2.1 (ii), we obtain

$$\begin{aligned} \|x_n - x_0\|^2 &= \|P_{K_n}x_0 - x_0\|^2 \leq \|x_0 - q\|^2 - \|q - x_n\|^2 \\ &\leq \|x_0 - q\|^2, \end{aligned}$$

for each $q \in \mathbb{F} \subset K_n$ and for all $n \geq 1$. Consequently, the sequences $\{x_n\}$, $\{v_n^i\}$ and $\{w_n^i\}$ $i = 1, 2, \ldots, N$ are bounded. Furthermore, since $x_n = P_{K_n}x_0$, $x_{n+1} = P_{K_{n+1}}x_0 \in K_{n+1} \subset K_n$, then from definition of P_K we have $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$ for all $n \geq 1$. Therefore, the sequence $\{\|x_n - x_0\|\}_{n=1}^{\infty}$ is nondecreasing. Thus, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. From the construction of K_n we have that $K_m \subset K_n$ and $x_m = P_{K_m}x_0 \in K_n$ for any integer $m \geq n$. Thus, from Lemma 2.1 (iii)

$$\begin{aligned} \|x_m - x_n\|^2 &= \|x_m - P_{K_n}x_0\|^2 \\ &\leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2. \end{aligned} \tag{3.8}$$

Letting $m, n \to \infty$ in (3.8), we have $\|x_m - x_n\| \to 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since H is Hilbert and K^i is closed and convex for all $i = 1, 2, \ldots, N$, we can assume that $x_n \to p^* \in K^i$, for all $i = 1, 2, \ldots, N$ as $n \to \infty$. We now show that $p^* \in F(T^i)$, for all $i = 1, 2, \ldots, N$. From (3.5), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \alpha^2[1 - 2\beta - {L^i}^2\beta^2]||x_n - v_n^i||^2 &\leq \sum_{n=0}^{\infty} \alpha_n^i\beta_n^i[1 - 2\beta_n^i - {L^i}^2{\beta_n^i}^2]||x_n - v_n^i||^2 \\ &\leq \sum_{n=0}^{\infty}[||x_n - p||^2 - ||x_{n+1} - p||^2] \\ &\leq ||x_0 - p||^2 + D < \infty. \end{aligned}$$

It then follows that $\lim_{n\to\infty} ||x_n - v_n^i|| = 0$. Since $v_n^i \in T^i x_n$ we have that $d(x_n, T^i x_n) \leq ||x_n - v_n^i|| \to 0$ as $n \to \infty$. Since T^i satisfies condition (1), $\lim_{n\to\infty} d(x_n, F(T^i)) = 0$. Thus, there exists a subsequence $\{x_{n_k}^i\}$ of $\{x_n\}$ such that $\|x_{n_k}^i - p_k^i\| \leq \frac{1}{2^k}$ for some $\{p_k^i\}_{k=1}^{\infty} \subseteq F(T^i)$. We now show that $\{p_k^i\}_{n=1}^{\infty}$ is a Cauchy sequence in $F(T^i)$. Observe that when $m = n + 1$ in (3.8) we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.9}$$

Consequently, $\lim_{n\to\infty} \|x_{n_{k+1}} - x_{n_k}\| = 0$ for all subsequences $\{x_{n_k}\}$ of $\{x_n\}$. It then follows that

$$\begin{aligned} \|p^i_{k+1} - p^i_k\| &\le \|p^i_{k+1} - x^i_{n_{k+1}}\| + \|x^i_{n_{k+1}} - x^i_{n_k}\| + \|x^i_{n_k} - p^i_k\| \\ &\le \frac{1}{2^{k+1}} + \frac{1}{2^k} + \|x^i_{n_{k+1}} - x^i_{n_k}\| \\ &\le \frac{1}{2^{k-1}} + \|x^i_{n_{k+1}} - x^i_{n_k}\|. \end{aligned}$$

Therefore, $\{p^i_k\}$ is a Cauchy sequence and converges to some $q^i \in F(T^i)$ because $F(T^i)$ is closed. Now,

$$\|x^i_{n_k} - q^i\| \le \|x^i_{n_k} - p^i_k\| + \|p^i_k - q^i\|.$$

Hence $x^i_{n_k} \to q^i$ as $k \to \infty$.

$$\begin{aligned} d(q^i, T^i q^i) &\le \|q^i - p^i_k\| + \|p^i_k - x^i_{n_k}\| + d(x^i_{n_k}, T^i x^i_{n_k}) + H(T^i x^i_{n_k}, T^i q^i) \\ &\le \|q^i - p^i_k\| + \|p^i_k - x^i_{n_k}\| + d(x^i_{n_k}, T^i x^i_{n_k}) + L^i \|x^i_{n_k} - q^i\|. \end{aligned}$$

Hence, $q^i \in T^i q^i$ and $\{x^i_{n_k}\}$ converges strongly to q^i. Since x_n converges strongly to p^*, uniqueness of limit of a convergent sequence guarantees that $p^* = q^i$ for all $i = 1, 2, \ldots, N$. Hence $p^* \in F(T^i)$, for all $i = 1, 2, \ldots, N$. It then follows that $p^* \in \bigcap F(T^i)$. ■

It remains to show that p^* is in $EP(F^i)$ for all $i = 1, 2, \ldots, N$. Since $x_{n+1} \in K_{n+1}$, $\|x_{n+1} - u^i_n\| \le \|x_{n+1} - x_n\|$, using (3.9),

$$\lim_{n\to\infty} \|x_{n+1} - u^i_n\| = 0. \tag{3.10}$$

Combining (3.9) and (3.10) we have

$$\lim_{n\to\infty} \|x_n - u^i_n\| = 0. \tag{3.11}$$

It follows from $\lim_{n\to\infty} \|x_n - p^*\| = 0$ and (3.11) that

$$\lim_{n\to\infty} \|u^i_n - p^*\| = 0. \tag{3.12}$$

$$\|y^i_n - p^*\|^2 \le ||x_n - p^*||^2 - \alpha^i_n \beta^i_n [1 - 2\beta^i_n - {L^i}^2 {\beta^i_n}^2]||x_n - v^i_n||^2 \tag{3.13}$$

Observe that

$$\begin{aligned} \|p^* - x_n\|^2 - \|p^* - u^i_n\|^2 &= \|x_n\|^2 - \|u^i_n\|^2 - 2\langle p^*, x_n - u^i_n\rangle \\ &\le \|x_n - u^i_n\|(\|x_n\| + \|u^i_n\|) + 2\|p^*\|\|x_n - u^i_n\|. \end{aligned}$$

It follows from (3.11) that

$$\lim_{n\to\infty} \|p^* - x_n\| - \|p^* - u_n^i\| = 0. \tag{3.14}$$

Now from (3.13)

$$\|p^* - y_n^i\| \leq \|p^* - x_n\|. \tag{3.15}$$

Also, using $u_n^i = T_{r_n^i} y_n^i$, Lemma 2.4 and (3.15) we have

$$\begin{aligned} \|u_n^i - y_n^i\|^2 &= \|T_{r_n^i} y_n^i - y_n^i\|^2 \\ &\leq \|p^* - y_n^i\|^2 - \|p^* - T_{r_n^i} y_n^i\|^2 \\ &\leq \|p^* - x_n\|^2 - \|p^* - T_{r_n^i} y_n^i\|^2 \\ &= \|p^* - x_n^i\|^2 - \|p^* - u_n^i\|^2. \end{aligned} \tag{3.16}$$

Therefore, from (3.14) and (3.16)

$$\lim_{n\to\infty} \|u_n^i - y_n^i\| = 0. \tag{3.17}$$

Consequently, from (3.12) and (3.17)

$$\lim_{n\to\infty} \|y_n^i - p^*\| = 0. \tag{3.18}$$

From the assumption that $r_n^i \geq a > 0$,

$$\lim_{n\to\infty} \frac{\|u_n^i - y_n^i\|}{r_n^i} = 0. \tag{3.19}$$

Since $u_n^i = T_{r_n^i} y_n^i$ implies

$$F(u_n^i, y) + \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - y_n^i\rangle \geq 0,$$

we deduce from (A2) that

$$\frac{\|u_n^i - y_n^i\|^2}{r_n^i} \geq \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - y_n^i\rangle \geq -F(u_n^i, y) \geq F(y, u_n^i).\ \forall y \in K^i$$

By taking limit as $n \to \infty$ of the above inequality and from (A4), (3.12) and (3.18), $F(y, p^*) \leq 0$, for all $y \in K^i$. Let $t \in (0, 1)$ and for all $y \in K^i$, since $p^* \in K^i$, $y_t = ty + (1-t)p^* \in K^i$. Hence $F(y_t, p^*) \leq 0$. Therefore, from (A1),

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, p^*) \leq tF(y_t, y),$$

that is, $F(y_t, y) \geq 0$. Letting $t \downarrow 0$, from (A3) we obtain $F(p^*, y) \geq 0$ for all $y \in K^i$ so that $p^* \in EP(F^i)$ for all $i = 1, 2, \ldots, N$. Hence $p^* \in \mathbb{F}$.

Finally, we show that $p^* = P_{\mathbb{F}}x_0$. By taking the limits as $n \to \infty$ in (3.7) we have

$$\langle p^* - p^*, x_0 - p^* \rangle \geq 0, \quad \forall q \in \mathbb{F}.$$

Thus, from Lemma 2.1 (i) $p^* = P_{\mathbb{F}}x_0$. This completes the proof. ■

Remark 3.3. If $N = 1$ in Algorithm 4, we obtain the following algorithms considered by Isiogugu et al. in [16].

Let H be a real Hilbert space and K a nonempty closed convex subset of H. Let F be a bifunction and T an L-Lipschitzian pseudocontractive-type mapping such that $F : K \times K \to \mathbb{R}$ and $T : K \to CC(K)$ respectively. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$be sequences in [0,1] and $\{r_n\}_{n=1}^{\infty} \subset [a, \infty)$ for some $a > 0$, then from an arbitrary $x_0 \in H$ we generate the sequences $\{x_n\}_{n=1}^{\infty}$ as follows.

Algorithm 5.

$$\begin{cases} x_0 \in H, \\ K_0 = K, \\ z_n = (1 - \beta_n)x_n + \beta_n v_n, \\ y_n = (1 - \alpha_n)x_n + \alpha_n w_n, \\ u_n \in K \text{ such that } F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in K, \\ K_{n+1} = \{z \in K_n : \|z - u_n\|^2 \leq \|z - x_n\|^2\}, \\ x_{n+1} = P_{K_{n+1}}x_0, \end{cases}$$

Algorithm 6.

$$\begin{cases} x_0 \in H, \\ z_n = (1 - \beta_n)x_n + \beta_n v_n, \\ y_n = (1 - \alpha_n)x_n + \alpha_n w_n, \\ u_n \in K \text{ such that } F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in K, \\ x_{n+1} = \dfrac{1}{2}(u_n + x_n), \end{cases}$$

where $w_n \in T(z_n) = T((1-\beta_n)x_n + \beta_n v_n)$ with $d((1-\beta_n)x_n + \beta_n v_n, T[(1-\beta_n)x_n + \beta_n v_n]) = ||(1 - \beta_n)x_n + \beta_n v_n - w_n||$, $v_n \in Tx_n$ with $||x_n - v_n|| = d(x_n, Tx_n)$ and $||w_n - v_n|| \leq H(Tz_n, Tx_n)$.

4. Numerical example of the computation

We will apply Lemma 2.7 in the computation of the sequences $\{K_n^i\}_{n=1}^{\infty}, i = 1, 2, 3, \ldots, N$ from which we can easily determine the sequence $\{x_n\}_{n=0}^{\infty}$.

Example 4.1. Let $H = \mathbb{R}$ (the reals with the usual norm and inner product), $i = 1, 2, \ldots, 4$ and $K^i = [-\sqrt{10i}, 1]$, for all i. Then for each i, we define:
(i) $T^i : [-\sqrt{10i}, 1] \to CC([-\sqrt{10i}, 1])$ by

$$T^i x = \begin{cases} [-\sqrt{10i}x, -2x], & x \in [0, 1] \\ \{-\frac{x}{\sqrt{10i}}\}, & x \in (-\sqrt{10i}, 0). \end{cases}$$

Obviously, T^i satisfies condition 1 since $d(x, F(T^i)) = d(x, \{0\}) = |x - 0| = |x|$, for all $i = 1, 2, 3, 4$, while

$$\begin{aligned} d(x, T^i x) &= \begin{cases} d(x, [-\sqrt{10i}x, -2ix]), & x \in [0, 1] \\ d(x, \{-\frac{x}{\sqrt{10i}}\}), & x \in [-\sqrt{10i}, 0). \end{cases} \\ &= \begin{cases} |x - (-2ix)|, & x \in [0, 1] \\ |x - (-\frac{x}{\sqrt{10i}})|, & x \in [-\sqrt{10i}, 0). \end{cases} \\ &\geq |x| = f(d(x, F(T^i)), \ \forall\, i. \end{aligned}$$

Where $f : [0.\infty) \to [0, \infty)$ is defined by $f(r) = r$.

Now, given any pair $x, y \in [0, 1]$,

$$H^2(T^i x, T^i y) = |\sqrt{10i}(x - y)|^2 = 10i|x - y|^2 = |x - y|^2 + (10i - 1)|x - y|^2$$

Also, given any $u \in T^i x, u = -\alpha x, 2i \leq \alpha \leq \sqrt{10i}$ and we can choose $v = -\alpha y \in T^i y$ so that $|u - v|^2 \leq H^2(T^i x, T^i y)$. Observe that

$$|x - u - (y - v)|^2 = (1 + \alpha)^2 |x - y|^2.$$

It then follows that

$$\begin{aligned} H^2(T^i x, T^i y) &= |x - y|^2 + \frac{10i - 1}{(1 + \alpha)^2}|x - u - (y - v)|^2 \\ &\leq |x - y|^2 + \frac{10i - 1}{(1 + 2i)^2}|x - u - (y - v)|^2 \\ &\leq |x - y|^2 + |x - u - (y - v)|^2, \forall i = 1, 2, 3, 4. \end{aligned}$$

Similarly, for any $x \in [0, 1]$, $y \in [-\sqrt{10i}, 0)$,

$$\begin{aligned} H^2(T^i x, T^i y) = |\sqrt{10i}x - \frac{y}{\sqrt{10i}}|^2 &\leq |\sqrt{10i}x - \sqrt{10i}y|^2 \\ &\leq |x - y|^2 + |x - u - (y - v)|^2, \ \forall i = 1, 2, 3, 4. \end{aligned}$$

Furthermore, for any $x, y \in [-\sqrt{10i}, 0)$,

$$H^2(T^i x, T^i y) = \frac{1}{\sqrt{10i}}|x-y|^2 \leq |x-y|^2 + |x-u-(y-v)|^2, \ \forall i = 1,2,3,4.$$

Observe that for $i = 1$, any pair $x, y = 0 \in [0,1]$ and $u \in T^1 x, v = 0$. In particular for $u = -2x$

$$\begin{aligned} H^2(T^1 x, T^1 y) &= |x-0|^2 + \frac{10-1}{(1+2)^2}|x-(-2x)|^2 \\ &= |x-y|^2 + |x-u-(y-v)|^2 \\ &> |x-y|^2 + k|x-u-(y-v)|^2, \forall k \in [0,1). \end{aligned}$$

Hence, T^1 is not $K-$strictly pseudocontractive-type mapping. Therefore, T^i is an L^i-Lipschitzian pseudocontractive-type mapping for each $i = 1,2,3,4$ with $L^i = \sqrt{10i}$. It then follows that:

(ii) $v_n^i = \begin{cases} -2ix_n, & x_n \in [0,1] \\ -\dfrac{x_n}{\sqrt{10i}}, & x_n \in [-\sqrt{10i}, 0). \end{cases}$

(iii) $\{\alpha_n^i\}_{n=1}^{\infty} = \dfrac{10ni - (n+1)(\sqrt{1+10i}+1)}{10ni(\sqrt{1+10i}+1)}$.

(iv) $\{\beta_n^i\}_{n=1}^{\infty} = \dfrac{12ni - (n+1)(\sqrt{1+10i}+1)}{12ni(\sqrt{1+10i}+1)}$.

(v) $z_n^i = (1-\beta_n^i)x_n + \beta_n^i v_n^i$.

(vi) $w_n^i = \begin{cases} -2iz_n^i, & z_n^i \in [0,1] \\ -\dfrac{z_n^i}{\sqrt{10i}}, & z_n^i \in [-\sqrt{10i}, 0). \end{cases}$

(vii) $y_n^i = (1-\alpha_n^i)x_n + \alpha_n^i w_n^i$.

(viii) $K_{n+1}^i = \begin{cases} [-\sqrt{10i}, \frac{1}{2}(x_n + u_n^i)], & x_n \in [0,1] \\ [\frac{1}{2}(x_n + u_n^i), 1], & x_n \in [-\sqrt{10i}, 0). \end{cases}$

(ix) $K_{n+1} = \begin{cases} [-\sqrt{10}, \min\limits_{1\leq i\leq 4}\{\frac{1}{2}(x_n + u_n^i)\}], & x_n \in [0,1] \\ [\max\limits_{1\leq i\leq 4}\{\frac{1}{2}(x_n + u_n^i)\}, 1], & x_n \in [-\sqrt{10i}, 0). \end{cases}$

(x) $x_{n+1} = \begin{cases} \min\limits_{1\leq i\leq 4}\{\frac{1}{2}(x_n + u_n^i)\}, & x_n \in [0,1] \\ \max\limits_{1\leq i\leq 4}\{\frac{1}{2}(x_n + u_n^i)\}, & x_n \in [-\sqrt{10i}, 0). \end{cases}$

We will define $F^i : [-\sqrt{10i}, 1]) \times [-\sqrt{10i}, 1] \to R$, $\{r_n^i\}_{n=1}^{\infty}$ and $\{u_n^i\}_{n=1}^{\infty}$ as in [13]. That is,

(xi) $F^i(x,y) = -ix^2 + iy^2$,

Observe that

$$\begin{aligned} F^i(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0 \;&\Rightarrow\; iy^2-iz^2+\frac{1}{r}(y-z)(z-x)\geq 0,\\ &\Rightarrow\; iy^2-iz^2+\frac{1}{r}[yz-xy-z^2+xz]\geq 0,\\ &\Rightarrow\; iry^2-irz^2+yz-xy-z^2+xz\geq 0,\\ &\Rightarrow\; iry^2+(z-x)y-irz^2-z^2+xz\geq 0. \end{aligned}$$

Now $F(y)=iry^2+(z-x)y-irz^2-z^2+xz$ a is a quadratic function of y with coefficients $a=ir$, $b=z-x$ and $c=-irz^2-z^2+xz$. Therefore, we can compute the discriminant Δ of F as follows:

$$\begin{aligned} \Delta &= (z-x)^2+4ir(irz^2+z^2-xz)\\ &= z^2+x^2-2xz+4i^2r^2z^2+4irz^2-4irxz\\ &= (1+4i^2r^2+4ir)z^2-2(2ir+1)xz+x^2\\ &= (1+2ir)^2z^2-2(1+2ir)xz+x^2\\ &= [(1+2ir)z-x]^2. \end{aligned} \tag{4.1}$$

Obviously, $F(y)\geq 0$ for all $y\in\mathbb{R}$ if it has at most one solution in $\mathbb{R}$. Thus $\Delta\leq 0$ and hence $z=T_{r_n^i}(x)=\dfrac{x}{1+2ir}$. Consequently

(xii) $\{u_n^i\}_{n=1}^{\infty}=T_{r_n^i}(y_n^i)=\{\dfrac{y_n^i}{2ir_n^i+1}\}_{n=1}^{\infty}$.

(xiii) $\{r_n^i\}_{n=1}^{\infty}=\{\dfrac{ni+1}{ni}\}_{n=1}^{\infty}$,

It is easy to see that $F_s(T^i)=\{0\}\neq\emptyset$, $EP(F^i)=\{0\}$ for each i and
$\mathbb{F}=\bigcap_{i=1}^{N}F_s(T^i)\bigcap\bigcap_{i=1}^{N}EP(F^i)=\{0\}$.
The algorithm is computed with Microsoft word Excel 97-2003 Workbook.

Table 1 shows different sequences generated from different values of x_0. In particular, we considered without loss of generality $x_0=1,-1,\frac{1}{2},-\frac{1}{2},-\sqrt{10}$.

Table1.

n	X_n	X_n	X_n	X_n	X_n
0	1	-1	0.5	-0.5	-3.16227766
1	0.536659107	-0.54104205	0.268329553	-0.270521025	-1.71092519
2	0.29117122	-0.294566249	0.145585609	-0.147283124	-0.93150027
3	0.158693418	-0.160754457	0.079346708	-0.080377228	-0.508350231
4	0.086702252	-0.087838166	0.043351125	-0.043919083	-0.277768673
5	0.047442303	-0.048032645	0.023721151	-0.024016322	-0.151892562
6	0.025986962	-0.026279408	0.01299348	-0.013139703	-0.083102786
7	0.014243777	-0.014383278	0.007121888	-0.007191638	-0.04548392
8	0.007809537	-0.007874512	0.003904768	-0.003937255	-0.024901395
9	0.004282798	-0.004312073	0.002141398	-0.002156036	-0.013635973
10	0.002349157	-0.002361708	0.001174578	-0.001180854	-0.007468379
11	0.001288736	-0.00129369	0.000644367	-0.000646845	-0.004091009
12	0.000707086	-0.000708741	0.000353542	-0.00035437	-0.002241237
13	0.000387997	-0.00038832	0.000193998	-0.000194159	-0.001227978
14	0.000212924	-0.00021278	0.000106462	-0.000106389	-0.000672871
15	0.000116857	-0.000116602	0.000058428	-0.0000583	-0.000368729
16	0.000064138	-0.000063901	0.000032069	-0.00003195	-0.000202075
17	0.000035205	-0.000035021	0.000017602	-0.00001751	-0.00011075
18	0.000019325	-0.000019194	0.000009662	-0.000009597	-0.000060701
19	0.000010608	-0.00001052	0.000005304	-0.00000526	-0.000033271
20	0.000005823	-0.000005766	0.000002911	-0.000002883	-0.000018237
21	0.000003196	-0.00000316	0.000001598	-0.00000158	-0.000009996
22	0.000001754	-0.000001732	0.000000877	-0.000000866	-0.000005479
23	0.000000963	-0.000000949	0.000000481	-0.000000474	-0.000003003
24	0.000000528	-0.00000052	0.000000264	-0.000000259	-0.000001646
25	0.000000289	-0.000000285	0.000000144	-0.000000141	-0.000000902
26	0.000000158	-0.000000156	0.000000079	-0.000000077	-0.000000494
27	0.000000086	-0.000000085	0.000000043	-0.000000042	-0.00000027
28	0.000000047	-0.000000046	0.000000023	-0.000000022	-0.000000148
29	0.000000025	-0.000000025	0.000000012	-0.000000012	-0.000000081
30	0.000000013	-0.000000013	0.000000006	-0.000000006	-0.000000044
31	0.000000007	-0.000000007	0.000000003	-0.000000003	-0.000000024
32	0.000000003	-0.000000003	0.000000001	-0.000000001	-0.000000013
33	0.000000001	-0.000000001	0	0	-0.000000007
34	0	0	0	0	-0.000000003
35	0	0	0	0	-0.000000001
36	0	0	0	0	0

Competing Interests

The Authors declare that there is no competing interest.

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References

[1] F. O. Isiogugu, Demiclosedness principle and approximation theorems for certain classes of multivalued mappings in Hilbert spaces, *Fixed Point Theory and Applications* 2013 **2013**:61.

[2] F. O. Isiogugu, A. Udomene and M. O. Osilike, Approximation of common solutions of equilibrium and fixed point problems for certain class of mappings in Hilbert spaces, *J. Nigerian Math. Soc.*, **30**, (2011), 179–194.

[3] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Mathematic Students*, **63**, (1994), 123–145.

[4] P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, *J. of Nonlinear Convex Anal.* , **6**, (2005), 117–136.

[6] W. Takahashi and K. Zembayashi, Strong and weak convergence theorems for equilibruim problems and relatively nonexpansive mappings in Banach spaces, *Nonlinear Anal.*, doi:10.1016/j.na.2007.11.031, (2008).

[6] A. Moudafi, Second-order differencial proximal methods for equlibrium problems, *J. Inequal. Pure Appl. Math.*, **4 (art. 18)**, (2003).

[7] S. Reich and S. Sabach, Three strong convergence theorems regarding iterative methods for solving equilibrium problems in reflexive Banach Spaces, *Contemporary Mathematics*, **568**, (2012), 225–240.

[8] F. O. Isiogugu, Approximation of a common element of the fixed point sets of multivalued strictly pseudocontractive-Type mappings and the set of solutions of an equilibrium problem in hilbert spaces, *Abstract and Applied Analysis*, **2016**:3094838.

[9] A. Tada, W. Takahashi, Strong convergence theorem for an equilib- rium problem and a nonexpansive mapping, in: W.Takahashi, T. Tanaka (Eds.), *Nonlinear Analysis and Convex Analysis*, (2006), 609–617.

[10] Li-Juan Qin and Lin Wang, An iteration method for solving equilibrium problems, common fixed point problems of strictly pseudocontractive mappings of rowder-Petryshyn type in Hilbert spaces, *International Mathematical Forum*, **6**(2), (2011), 63–74

[11] C. Jaiboon, P. Kumam, Strong convergence theorems for solving equilib- rium problems and fixed point problems of ξ-strictly pseudocontraction mappingsby two hybrid projection methods, *J. Computational and Applied Math.*, (2010), doi:10.1016/j.cam.2010.01.012

[12] F. O. Isiogugu, P. Pillay and D. Baboolal, Approximation of a common element of the set of fixed points of multi-valued type-one demicontractive-type mappings

and the set of solutions of an equilibrium problem in Hilbert spaces, *Journal of Nonlinear and Convex Analysis*, **17**(6), (2016), 1181–1197.

[13] F. O. Isiogugu, P. Pillay C. C. Okeke, and F. U. Ogbuisi, Modified Reich-Sabach Iteration Scheme for Approximating a Common Element of the Solutions of Equilibrium Problems and Fixed Point Problems in Hilbert Spaces, *Global Journal of Pure and Applied Mathematics*, **12**(6)(2016), 5185–5204.

[14] F. O. Isiogugu, P. Pillay and P. U. Nwokoro, On Computability and Applicability of Mann-Reich-Sabach-type algorithms for Approximating the Solutions of Equilibrium Problems in Hilbert Spaces, **submitted for publication**.

[15] Y. Song and H. Wang, Erratum to Mann and Ishikawa iterative processes for multivalued mappings in Banach Spaces, [*Comput. Math. Appl.*, **54** (2007), 872–877], *Comput. Math. Appl.*, **55**, (2008), 2999–3002.

[16] F. O. Isiogugu, P. Pillay and O. Uzoma-Oguguo, Strong convergence of a selection of Ishikawa-Reich-Sabach-type algorithm **submitted for publication**.