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The Strong Version of the Countably Lipschitz Integral

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Abstract

A countably Lipschitz condition is more general than Lipschitz condition. However, the countably Lipschitz function may not be continuous. It will be defined a more general Lipschitz condition which is more general than Lipschitz condition but it is still continuous. The condition is called by strongly countably Lipschitz condition. Based on the condition, it will be constructed a type of integral. It is called the CL^*-integral. The new integral is stronger than the CL-integral and the Henstock-Kurzweil integral on $[a, b]$.

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1. Introduction

In [2, 3], Indrati and Aryati defined a countably Lipschitz condition. A Lipschitz function is countably Lipschitz condition, however not every countably Lipschitz condition is Lipschitz. It is also well-known that every Lipschitz function is continuous [1]. This fact is not always true for countably Lipschitz functions. It is a fact that every function with countably Lipschitz condition is generalized absolutely continuous.

Based on the countably Lipschitz condition, it has been defined an integral, which is called by the CL-integral [3]. The CL-integral on $[a, b]$ includes the Lebesgue integral on $[a, b]$. The CL-integral on $[a, b]$ is stronger than the Denjoy in the wide sense on $[a, b]$.

In this paper, it will be developed a new version of countably Lipschitz condition that still preserves continuity. The new version is called by a strongly countably Lipschitz condition. Based on the strongly countably Lipschitz condition, it will be developed an integral, which is called by the CL^*-integral. From the characterization of the integrals, it will be shown that the CL^*-integral on $[a, b]$ is stronger than the CL-integral on $[a, b]$ and the Henstock-Kurzweil integral. The relationship of the CL^*-integral and the Henstock-Kurzweil integral will be done by using the equivalency between the Henstock-Kurzweil integral and the Denjoy integral in the restricted sense.

2. Preliminary Notes

We recall some results of characteristics of the countably Lipschitz condition and its integral without any proof. The proofs could be found in [3].

A function F is said to be absolutely continuous on X, written $F \in AC(X)$, if for every $\epsilon > 0$, there exists an $\eta > 0$, such that for every finite or infinite collection intervals $\{I_n = [a_n, b_n]\}$, $a_n, b_n \in X$, where $I_n^o \cap I_m^o = \emptyset$, $m \neq n$, $\sum |b_n - a_n| < \eta$, we have

$$\sum |F(b_n) - F(a_n)| < \epsilon.$$

Furthermore, a function F is said to be generalized absolutely continuous on X, $F \in ACG(X)$. if there exists a countable collection of sets $\{X_n\}$, with $\cup X_n = X$, and $F \in AC(X_n)$, for every n [4, 5].

In this paper, we use strongly absolutely continuous concepts. We cite the concepts as in [4, 5]. A function F is said to be strongly absolutely continuous on X, written $F \in AC^*(X)$, if for every $\epsilon > 0$, there exists an $\eta > 0$, such that for any finite or infinite collection of intervals $\{I_n = [a_n, b_n]\}$, $a_n, b_n \in X$, where $I_n^o \cap I_m^o = \emptyset$, $m \neq n$, $\sum |b_n - a_n| < \eta$ we have

$$\sum \omega(F; I_n) < \epsilon.$$

Furthermore, a function F is said to be generalized strongly absolutely continuous on $[a, b]$, if there exists a countable collection of sets $\{X_n\}$, with $\cup_n X_n = [a, b]$, and $F \in AC^*(X_n)$, for every n.

In the definition of the $AC^*(X)$, the end points a_n and b_n of I_n in the collection belong to X. In Theorem 2.1, the condition can be replaced by at least one of the end points belongs to X.

Theorem 2.1. Let $F : [a, b] \to \mathbb{R}$ be a continuous function. If $X \subseteq [a, b]$ is closed then $F \in AC^*(X)$ if and only if for every $\epsilon > 0$, there exists an $\eta > 0$, such that for any finite or infinite collection of intervals $\{I_n = [a_n, b_n]\}$, $a_n \in X$ or $b_n \in X$, where

$I_n^o \cap I_m^o = \emptyset, m \neq n, \sum |b_n - a_n| < \eta$ we have

$$\sum \omega(F; I_n) < \epsilon.$$

We rephrase the definition of countable Lipschitz condition in [3] to have strongly countably Lipschitz condition.

A function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is said to have countably Lipschitz condition on X if and only if there exists a countable collection $\{X_n\}$ with $\cup_n X_n = X$ and f satisfies Lipschitz condition on X_n for every n.

A countably Lipschitz function may not be continuous. However, the number of discontinuity points of a countably Lipschitz function is at most countable.

Let $CLC(X)$ denote a collection of all functions that satisfy countably Lipschitz condition on $X \subseteq \mathbb{R}$. It is a fact that if $F \in CLC[a, b]$ then $F \in ACG[a, b]$.

Based on the countably condition, it has been constructed an integral, which is called the CL-integral as in Definition 2.2.

Definition 2.2. A function $f : [a, b] \to \mathbb{R}$ is said to be CL-integrable on $[a, b]$, if there exists a continuous function $F : [a, b] \to \mathbb{R}$ such that $F' = f$ almost everywhere in $[a, b]$ and $F \in CLC[a, b]$.

The function F in Definition 2.2 is called the CL-primitive of f on $[a, b]$.

The collection of all CL-integrable functions on $[a, b]$ is a linear space. Based on the properties of derivative and CLC, we have a fact that if a function f is CL-integrable on $[a, c]$ and on $[c, b]$, then f is CL-integrable on $[a, b]$. Furthermore, if f is CL-integrable on $[a, b]$ with its primitive is F, we have F is continuous, $F' = f$ almost everywhere in $[a, b]$, and $F \in ACG[a, b]$. That means, every CL-integrable function on $[a, b]$ is Denjoy integrable in the wide sense [3].

3. Main Results

The strong version of the countably Lipschitz condition will be given in this section. We develop some properties that will be given to define the correlated integral on $[a, b] \subseteq \mathbb{R}$, which is called the CL^*-integral. The CL^*-integral on $[a, b]$ is stronger than the CL-integral on $[a, b]$. It will be given also the relationship between the constructed integral with the Henstock-Kurzweil integral on $[a, b]$.

3.1. Strongly Countably Lipschitz Condition

Definition 3.1. A function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ is said to have strongly countably Lipschitz condition on X, written $f \in CLC^*(X)$, if there is a countable collection $\{X_n\}$, where $X = \cup_n X_n$ and for every n, there exists a constant M_n, such that for every $x \in X$ or $y \in X$, we have

$$|f(x) - f(y)| \leq M_n |x - y|.$$

In the next discussion, $CLC^*(X)$ denotes a collection of all functions that satisfy strongly countably Lipschitz condition on $X \subseteq \mathbb{R}$.

In Theorem 3.2, it will be proved the continuity of $f \in CLC^*(X)$. Furthermore, Theorem 3.3 provides a relationship between strongly countably Lipschitz fuction and strongly generalized absolutely continuous function.

Theorem 3.2. Let $f : X \to \mathbb{R}$ be a function. If $f \in CLC^*(X)$, then f is continuous on X.

Proof. Since $f \in CLC^*(X)$, there exists a countable collection $\{X_n\}$ with $\cup_n X_n = X$ and for every n, there exists $M_n \geq 0$ such that for every u, v, where $u \in X_n$ or $v \in X_n$, we have

$$|f(u) - f(v)| \leq M_n|u - v|.$$

Let $x \in X$ arbitrarily and let $\epsilon > 0$ be given.

There is a positive integer n such that $x \in X_n$. Put $\delta = \dfrac{\epsilon}{1 + M_n}$. Therefore, for any $y \in X$, $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq M_n|x - y| < M_n\delta < \epsilon.$$

■

Theorem 3.3. If $F \in CLC^*[a, b]$ then $F \in ACG^*[a, b]$.

Proof. Since $F \in CLC^*[a, b]$, there exists a countable collection $\{X_n\}$ with $\cup_n X_n = [a, b]$ such that for every n, there exists $M_n \geq 0$ such that for every u, v, where $u \in X_n$ or $v \in X_n$, we have

$$|f(u) - f(v)| \leq M_n|u - v|.$$

For every n, put $E_n = \overline{X}_n$, the closure of X_n. Let fix n. For every $x \in E_n$ or $y \in E_n$, there exists a sequence $\{x_k\} \subseteq E_n$ converges to x. Therefore, by continuity of F, we have

$$|F(x) - F(y)| = |F(\lim_{k\to\infty} x_n) - F(y)| = \lim_{k\to\infty} |F(x) - F(y)| \leq M_n|x - y|.$$

Let $\epsilon > 0$ be given. Put $\delta_n = \dfrac{\epsilon}{1 + M_n}$. As corollary, for every collection of intervals $\{I_{nk} = [x_{nk}, y_{nk}]\}$, with $x_{nk} \in E_n$ or $y_{nk} \in E_n$,

$$\sum_k |I_k| = \sum_k |x_{nk} - y_{nk}| < \delta_n,$$

we have

$$\begin{aligned} |\sum_k F(I_{nk})| &\leq \sum_k |F(x_{nk}) - F(y_{nk})| \\ &\leq \sum_k M_n|x_{nk} - y_{nk}| < M_n\delta_n < \epsilon. \end{aligned}$$

By Theorem 2.1 and Theorem 3.2, $F \in ACG^*[a,b]$. ■

Although every Lipschitz function is strongly continuous function, the converse is not always true. For example, a function $h : \mathbb{R} \to \mathbb{R}$, where $h(x) = x^2$. The function h is a strongly continuous function on $\mathbb{R}$, but it is not a Lipschitz function on $\mathbb{R}$.

The converse of Theorem 3.2 is not always true. Let consider Example 3.4.

Example 3.4. Let consider a continuous function F that is constructed as the following. Let $K \subseteq [0,1]$ be a Cantor set. The complement of K is a union of open intervals

$$\left(\frac{1}{3},\frac{2}{3}\right),\left(\frac{1}{3^2},\frac{2}{3^2}\right),\left(\frac{2}{3}+\frac{1}{3^2},\frac{2}{3}+\frac{2}{3^2}\right),\left(\frac{1}{3^3},\frac{2}{3^3}\right),\left(\frac{2}{3^2}+\frac{1}{3^3},\frac{2}{3^2}+\frac{2}{3^3}\right),$$
$$\left(\frac{2}{3}+\frac{1}{3^3},\frac{2}{3}+\frac{2}{3^3}\right),\left(\frac{2}{3}+\frac{2}{3^2}+\frac{1}{3^3},\frac{2}{3}+\frac{2}{3^2}+\frac{2}{3^3}\right),\ldots.$$

For simplicity, those intervals will be denoted respectively by

$$I\left(\frac{1}{2}\right),I\left(\frac{1}{3}\right),I\left(\frac{3}{4}\right),I\left(\frac{1}{8}\right),I\left(\frac{3}{8}\right),I\left(\frac{5}{8}\right),I\left(\frac{7}{8}\right),\ldots.$$

We define a continuous function F_1 on $[0,1]$, with $F_1(0) = 0$, $F_1(1) = 1$, and $F_1(x) = \frac{1}{2}$ for $x \in I\left(\frac{1}{2}\right)$, and F_1 is line segment on two components of $[0,1] \setminus I\left(\frac{1}{2}\right)$ such that F_1 is continuous on $[0,1]$, i.e.

$$F_1(x) = \begin{cases} 0, & x = 0 \\ \frac{3}{2}x, & x \in \left(0,\frac{1}{3}\right) \\ \frac{1}{2}, & x \in I\left(\frac{1}{2}\right) = \left(\frac{1}{3},\frac{2}{3}\right) \\ \frac{3}{2}x - \frac{1}{2}, & x \in \left(\frac{2}{3},1\right) \\ 1, & x = 1. \end{cases}$$

We continue by defining a continuous function F_2 on $[0,1]$ as the following: $F_2(x) = F_1(x)$ for $x \in \{0,1\} \cup I\left(\frac{1}{2}\right)$, $F_2(x) = \frac{1}{4}$ for $x \in I\left(\frac{1}{4}\right)$, $F_2(x) = \frac{3}{4}$ for $x \in I\left(\frac{3}{4}\right)$, and linear on four components $[0,1] \setminus \left(I\left(\frac{1}{2}\right) \cup I\left(\frac{1}{4}\right) \cup I\left(\frac{3}{4}\right)\right)$.

We continue the process so we have F_n, $n = 1,2,3,\ldots$. We have $F_n \in AC[0,1]$ for every n. From the construction, the sequence $\{F_n\}$ is uniformly convergent on $[0,1]$, so there is a continuous function F on $[0,1]$, where $F_n \to F$ uniformly on $[0,1]$.

Since the measure of Cantor set K is zero, then $F(K) = \{F(x) : x \in K\}$ has measure 1. Therefore, $F \notin ACG[0,1]$ [4]. As corollary, $F \notin ACG^*[0,1]$. By Theorem 3.3, $F \notin CLC^*[0,1]$.

It is clear that every strongly countably Lipschitz condition function on X is countably Lipschitz condition on X. However, not every $F \in CLC(X)$ satisfies strongly countably Lipschitz condition function on X. The function $f : [0, 2] \to \mathbb{R}$, where

$$f(x) = \begin{cases} 2x, & x \in [-1, 1] \\ x + 3, & x \in (1, 4], \end{cases} \tag{3.1}$$

satisfies countably Lipschitz condition on $[-1, 4]$ but it does not satisfy strongly countably Lipschitz condition on $[-1, 4]$, since f is not continuous on $[-1, 4]$.

3.2. The Strong Version of the Countably Lipschitz Integral

The idea of the definition comes from the definition of the CL-integral on $[a, b]$. Since every $F \in CLC^*[a, b]$ is continuous on $[a, b]$, we do not need to state the continuity of the primitive.

Definition 3.5. A function $f : [a, b] \to \mathbb{R}$ is said to be countably Lipschitz integrable in the strong version on $[a, b]$, if there exists a function $F : [a, b] \to \mathbb{R}$ such that $F' = f$ almost everywhere in $[a, b]$ and $F \in CLC^*[a, b]$.

For the next discussion, the strong version of countably Lipschitz integral will be written by CL^*-integral.

Example 3.6. A continuous function on $[a, b]$ is CL^*-integrable on $[a, b]$.

Proof. Let f be an arbitrary continuous function on $[a, b]$. The function f is Riemann integrable on $[a, b]$. By Mean Value Theorem, the primitive F of f on $[a, b]$ satisfies Lipschitz condition on $[a, b]$. ■

The function F in Definition 3.5 is called the CL^*-primitive of f on $[a, b]$.

The collection of all CL^*-integrable function is a vector space.

Based on the properties of derivative and CLC^*, the integrability of f on $[a, b]$ implies the integrability of f on each sub-interval $[c, d] \subseteq [a, b]$. Furthermore, by the definition of the strongly countably Lipschitz function, if the function f is CL^*-integrable on $[a, c]$ and on $[c, b]$, then f is CL^*-integrable on $[a, b]$.

Theorem 3.7. If f is CL^*-integrable on $[a, b]$, then f is CL-integrable on $[a, b]$.

Proof. Let F is the CL^*-primitive of f on $[a, b]$. Since $F \in ACG^*[a, b]$, then $F \in ACG[a, b]$. The rest of the proof comes from Theorem 3.2. ■

Corollary 3.8. Every CL^*-integrable function on $[a, b]$ is Denjoy integrable on $[a, b]$ in the wide sense.

The Denjoy integrable in the wide sense is more general than the Denjoy integral in the restricted sense. The Denjoy integral in the restricted sense is equivalent with the Henstock-Kurzweil integral [4]. In Theorem 3.9, it is proved that every CL^*-integrable function on $[a, b]$ is Denjoy integrable on $[a, b]$ in the restricted sense.

Theorem 3.9. Every CL^*-integrable function on $[a, b]$ is Henstock-Kurzweil integrable on $[a, b]$.

Proof. The proof is a consequence of Theorem 3.3 and the equivalence between the Denjoy integral in the restricted sense on $[a, b]$ and the Henstock-Kurzweil integral on $[a, b]$. ■

In Theorem 3.12, it will be proved the CL^*-integrability of the limit of a sequence of continuous function on $[a, b]$. We will prove the theorem by using Theorem 3.10 and Theorem 3.11.

Theorem 3.10. Let J be a bounded interval in $\mathbb{R}$ and (g_n) be a sequence of real-valued functions on J. If there is an element $x_o \in J$ such that $g_n(x_o))$ is convergence and the sequence (g_n') exist on J and converges uniformly on J to a function h on J, then (g_n) converges uniformly to a differentiable function g on J, where $g' = h$ on J.

Proof. Theorema 8.2.3 in Bartle and Sherbert (2011). ■

Theorem 3.11. If a sequence $\{F_k\} \subseteq CLC^*(X)$ is uniformly convergent to F on X, then $F \in CLC^*(X)$.

Proof. Let $\epsilon > 0$ be given. There exists a positive integer k_o such that for every $k \geq k_o$, we have

$$|F_k(x) - F(x)| < \frac{\epsilon}{2},$$

for every $x \in X$.

Since $F_{k_o} \in CLC^*(X)$, there exists a collection of sets $\{E_{k_o n}\}$, where $X = \cup_n E_{k_o n}$ and for every n, there exists $M_{k_o n}$ such that for every x and y, where $x \in E_{k_o n}$ or $y \in E_{k_o n}$, we have

$$|F_{k_o}(x) - F_{k_o}(y)| \leq M_{k_o n}|x - y|.$$

For every n, put $A_n = E_{k_o n}$ and $M_n = M_{k_o n}$. As corollary, for every x and y, where $x \in A_n$ or $y \in A_n$, we have

$$|F(x)-F(y)| \leq |F(x)-F_{k_o}(x)|+|F_{k_o}(x)-F_{k_o}(y)|+|F_{k_o}(y)-F(y)| < M_n|x-y|+\epsilon.$$

That means, $F \in CLC^*[a, b]$. ■

Theorem 3.12. Let (f_k) be a sequence of continuous function on $[a, b]$. If (f_k) converges uniformly to f on $[a, b]$, then f is CL^*-integrable on $[a, b]$.

Proof. For every k, f_k is CL^*-integrable on $[a, b]$. Let F_k be the CL^*-primitive of each f_k, $k = 1, 2, 3, \ldots$. The sequence $(F_k(a))$ converges. For every k, $F_k \in CLC^*[a, b]$ and $F_k' = f_k$ on $[a, b]$. Since (f_k) converges uniformly to f on $[a, b]$, by Theorem 3.10, then (F_k) converges uniformly to a function F with $F' = f$. Therefore, the function F is continuous on $[a, b]$. Furthermore, by Theorem 3.11, $F \in CLC^*[a, b]$. That means, f is CL^*-integrable on $[a, b]$. ■

4. Concluding Remarks

From the discussion it can be derived that every strongly countably Lipschitz function is continuous. If $F \in CLC^*(X)$ then $F \in CLC$. The collection of all CL^*-integrable functions is a linear space. Every CL^*-integrable function on $[a, b]$ is CL-integrable on $[a, b]$. From Theorem 3.9 we have a fact that every CL^*-integrable function on $[a, b]$ is Henstock-Kurzweil integrable on $[a, b]$.

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