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On Relationship between Some Types of Subsemigroups and Generalized Bipolar Fuzzy Subsemigroups

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Abstract

In this paper, a generalization of bipolar fuzzy subsemigroups are introduced. Some properties of the concept are discussed such as ideal, generalized bi-ideal, bi-ideal and quasi-ideal. The main purpose of this paper is to characterize regular and inter-regular semigroup by using generalized bipolar fuzzy semigroups.

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1. Introduction

Application of bipolar information has been frequently used in daily life, for example using in an organization, economic, performance, development, evaluation, risk management etc. Bipolar information versus a two dimension of positive/negative affect dimension. Therefore, any events should consider separate positive and negative information response, which positive information represents are possible, while negative information represents are forbidden or surely false [2]. If we consider that they are both positively and negatively information, we should consider information both advantages

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and disadvantages relevant to lead to better decision making. Thus, the concepts of bipolar fuzzy sets are more interested in mathematics.

In 1965, Zadeh [14] introduced the fuzzy set theory which can be applied to many areas such as mathematics, computer, engineering, etc. The fuzzy set was used to establish the mathematical method for dealing with imprecise and uncertainty environment. In 1971, Rosenfeld [12] applied the fuzzy sets to group structures. Then, the fuzzy set was used in the theory of semigroups in 1979. Kuroki [4] initiated fuzzy semigroup based on the notion of fuzzy ideals in semigroups and introduced some properties of fuzzy ideals and fuzzy bi-ideals of semigroups. Davvaz et al. [3] studied and characterized regular ordered semigroups in terms of $(\alpha; \beta)$-fuzzy generalized bi-ideals.

The fundamental concepts of bipolar fuzzy sets initiated by Zhang [15] in 1994. He innovated the bipolar fuzzy set as bipolar fuzzy logic which has been widely applied to solve many real world problems. In 2000, Lee [9] studied the notion of bipolar valued fuzzy sets which is an extension of fuzzy sets. The concepts of bipolar fuzzy subalgebra and ideal in BCK/BCI-algebras presented by Lee [10]. Jun and Park [7] proposed a bipolar fuzzy regularity, regular subalgebra, filter, and closed quasi filter in BCH-algebras. Kim et al. [8] investigated the notions of bipolar fuzzy subsemigroups, bipolar fuzzy left (right) ideals and bipolar fuzzy bi-ideals. He provided some necessary and sufficient conditions for a bipolar fuzzy subsemigroup and a bipolar fuzzy left (right, bi-) ideals of semigroups. Ban et al. [1] proposed bipolar fuzzy ideals with operators in semigroups by using a set Ω. In 2014, Zhou and Li [16] introduced the concept of bipolar fuzzy h-ideals. They obtained the relation of bipolar fuzzy h-ideals and h-ideals of hermrings. In 2016, Yaqoob et al. [13] studied bipolar (λ, δ)-fuzzy sets in Γ-semihypergroups.

In this paper, a generalizations of bipolar fuzzy semigroups are introduced. Some properties of $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-bipolar fuzzy left(right, generalized bi-, bi-, quasi-) ideal are obtained. Finally, we characterize a regular and intra-regular semigroup by using generalized bipolar fuzzy semigroups.

2. Preliminaries

In this section, we give definitions and examples which are used in this paper.

Definition 2.1. [11] A semigroup S is called **regular** if for all $a \in S$ there exists $x \in S$ such that $a = axa$.

Theorem 2.2. [11] For a semigroup S, the following conditions are equivalent.

1. S is intra-regular.

2. $R \cap L \subseteq RL$ for every right ideal R and every left ideal L of S.

Definition 2.3. [14] Let X be a set, a **fuzzy set** (or **fuzzy subset**) f on X is mapping $f : X \to [0, 1]$ where [0,1] is the usual interval of real numbers.

The symbols $f \wedge g$ and $f \vee g$ will mean the following fuzzy sets on S $(f \wedge g)(x) = f(x) \wedge g(x)$, $(f \vee g)(x) = f(x) \vee g(x)$ for all $x \in S$.

Definition 2.4. [8] Let S be a non-empty set. A BF-set f on S is an object having form $f := \{(x, f_p(x), f_n(x)) : x \in S\}$, where $f_p : S \to [0, 1]$ and $f_n : S \to [-1, 0]$.

Remark 2.5. For the sake of simplicity we shall use the symbol $f = (S; f_p, f_n)$ for the BF set $f = \{(x, f_p(x), f_n(x)) : x \in S\}$.

We shall give the generalization of BF-subsemigroup which is defined by Kim et al. (2011).

Definition 2.6. [5] A BF-set $f = (S; f_p, f_n)$ on S is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF-subsemigroup* on S where $\alpha_1, \alpha_2 \in [0, 1]$, $\beta_1, \beta_2 \in [-1, 0]$ if it satisfies the following conditions:

1. $f_p(xy) \vee \alpha_1 \geq f_p(x) \wedge f_p(y) \wedge \alpha_2$,
2. $f_n(xy) \wedge \beta_2 \leq f_n(x) \vee f_n(y) \vee \beta_1$

for all $x, y \in S$.

Note that every BF-subsemigroup is $(0, 1, -1, 0)$-BF-subsemigroup.

The following examples show that $f = (S; f_p, f_n)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-subsemigroup on S but $f = (S; f_p, f_n)$ is not a BF-subsemigroup on S.

Example 2.7. Let $S = \{a, b, c\}$ be a semigroup with following cayley table:

$\cdot$	a	b	c
a	a	a	a
b	a	b	a
c	a	a	c

Define a BF-set $f = (S; f_p, f_n)$ of S as follows: $f_p(a) = 0.3$, $f_b(a) = 0.5$, $f_p(c) = 0.5$, $f_n(a) = -0.7$, $f_n(b) = -0.5$ and $f_n(c) = -0.3$. It is easy to show that $f = (S; f_p, f_n)$ is a $(0.5, 0.7; -0.3, -0.2)$-BF-bi-ideal on S. But we see that f is not a BF-ideal on S since $f_p(bc) = f_p(a) = 0.3 < 0.5 = f_p(b) \wedge f_p(c)$.

Definition 2.8. [5] A BF-set $f = (S; f_p, f_n)$ on S is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF-left (right) ideal* on S where $\alpha_1, \alpha_2 \in [0, 1]$, $\beta_1, \beta_2 \in [-1, 0]$ if it satisfies the following conditions:

1. $f_p(xy) \vee \alpha_1 \geq f_p(y) \wedge \alpha_2$ $(f_p(xy) \vee \alpha_1 \geq f_p(x) \wedge \alpha_2)$,
2. $f_n(xy) \wedge \beta_2 \leq f_n(y) \vee \beta_1$ $(f_n(xy) \wedge \beta_2 \leq f_n(x) \vee \beta_1)$

for all $x, y \in S$.

A BF-set $f = (S; f_p, f_n)$ on S is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*$BF$-ideal* on S, $(\alpha_1, \alpha_2 \in [0, 1], \beta_1, \beta_2 \in [-1, 0])$, if it is both a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-left ideal and a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-right ideal on S.

Definition 2.9. [5] A $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-subsemigroup $f = (S; f_p, f_n)$ on a semigroup S is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*$BF$-bi-ideal* on S where $\alpha_1, \alpha_2 \in [0, 1], \beta_1, \beta_2 \in [-1, 0]$ if it satisfies the following conditions:

1. $f_p(xay) \vee \alpha_1 \geq f_p(x) \wedge f_p(y) \wedge \alpha_2,$
2. $f_n(xay) \wedge \beta_2 \leq f_n(x) \vee f_n(y) \vee \beta_1$

for all $x, y \in S$.

A product of BF-sets and characterize a regular semigroup by generalized bipolar fuzzy subsemigroups.

Let $f = (S : f_p, f_n)$ and $g = (S : g_p, g_n)$ be two BF-sets on a semigroup S and $\alpha_1, \alpha_2 \in [0, 1], \beta_1, \beta_2 \in [-1, 0]$. Define two fuzzy sets $f_p^{(\alpha_1,\alpha_2)}$and $f_n^{(\beta_1,\beta_2)}$ on S as follows: $f_p^{(\alpha_1,\alpha_2)}(x) = (f_p(x) \wedge \alpha_1) \vee \alpha_2$ and $f_n^{(\beta_1,\beta_2)}(x) = (f_n(x) \vee \beta_2) \wedge \beta_1$ for all $x \in S$ [6].

Define two operations $\overset{(\alpha_1,\alpha_2)}{\wedge}$ and $\underset{(\beta_1,\beta_2)}{\vee}$ on S as follows:

$$(f_p \overset{(\alpha_1,\alpha_2)}{\wedge} g_p)(x) = ((f_p \wedge g_p)(x) \wedge \alpha_1) \vee \alpha_2,$$

$$(f_n \underset{(\beta_1,\beta_2)}{\wedge} g_n)(x) = ((f_n \vee g_n)(x) \vee \alpha_2) \wedge \alpha_1$$

for all $x \in S$,
and define products $f_p \overset{(\alpha_1,\alpha_2)}{\circ} g_p$ and $f_n \underset{(\beta_1,\beta_2)}{\circ} g_n$ as follows: For all $x \in S$

$$(f_p \overset{(\alpha_1,\alpha_2)}{\circ} g_p)(x) = ((f_p \bar{\circ} g_p)(x) \wedge \alpha_1) \vee \alpha_2,$$

$$(f_n \underset{(\beta_1,\beta_2)}{\circ} g_n)(x) = ((f_n \underline{\circ} g_n)(x) \vee \alpha_2) \wedge \alpha_1$$

where

$$(f_p \bar{\circ} g_p)(x) = \begin{cases} \bigvee_{x=yz} \{f_p(y) \wedge g_p(z)\} & \text{if } x = yz \text{ for some } y, z \in S; \\ 0 & \text{if otherwise.} \end{cases}$$

$$(f_n \underline{\circ} g_n)(x) = \begin{cases} \bigwedge_{y=yz} \{f_n(y) \vee g_n(z)\} & \text{if } x = yz \text{ for some } y, z \in S; \\ 0 & \text{if otherwise.} \end{cases}$$

Set

$$f \underset{(\beta_1,\beta_2)}{\overset{(\alpha_1,\alpha_2)}{\circ}} g := (S; f_p \overset{(\alpha_1,\alpha_2)}{\circ} g_p, f_n \underset{(\beta_1,\beta_2)}{\circ} g_n).$$

Then it is a BF-set.
Note that

1. $f = (S; f_p, f_n) = (S; f_p^{(1,0)}, f_n^{(0,-1)})$,

2. $(f \underset{(\beta_1,\beta_2)}{\overset{(\alpha_1,\alpha_2)}{\circ}} g)_p = f_p \overset{(\alpha_1,\alpha_2)}{\circ} g_p$ and $(f \underset{(\beta_1,\beta_2)}{\overset{(\alpha_1,\alpha_2)}{\circ}} g)_n = f_n \underset{(\beta_1,\beta_2)}{\circ} g_n$.

Definition 2.10. [6] A BF-set $f = (S; f_p, f_n)$ on S is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF-generalized bi-ideal* on S where $\alpha_1, \alpha_2 \in [0, 1], \beta_1, \beta_2 \in [-1, 0]$ if it satisfies the following conditions:

1. $f_p(xay) \vee \alpha_1 \geq f_p(x) \wedge f_p(y) \wedge \alpha_2$,

2. $f_n(xay) \wedge \beta_2 \leq f_n(x) \vee f_n(y) \vee \beta_1$

for all $x, y, a \in S$.

Definition 2.11. [6] A BF-set $f = (S; f_p, f_n)$ on S is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*BF-quasi-ideal* on S where $\alpha_1, \alpha_2 \in [0, 1], \beta_1, \beta_2 \in [-1, 0]$ if it satisfies the following conditions:

1. $f_p(x) \vee \alpha_1 \geq (f_p \bar{\circ} \mathcal{S}_p)(x) \wedge (\mathcal{S}_p \bar{\circ} f_p)(x) \wedge \alpha_2$,

2. $f_n(x) \wedge \beta_2 \leq (f_n \underline{\circ} \mathcal{S}_n)(x) \vee (\mathcal{S}_n \underline{\circ} f_n)(x) \vee \beta_1$

for all $x, y \in S$.

Theorem 2.12. [5] Let $f = (S; f_p, f_n)$ be a BF-set on a semigroup S. Then f is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-subsemigroup on S if and only if $f_p \overset{(\alpha_1,\alpha_2)}{\circ} f_p \leq f_p^{(\alpha_1,\alpha_2)}$ and $f_n \underset{(\beta_1,\beta_2)}{\circ} f_n \geq f_n^{(\beta_1,\beta_2)}$.

Lemma 2.13. [6] If $f = (S; f_p, f_n)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-left ideal and $g = (S; g_p, g_n)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-right ideal on a semigroup S, then $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \leq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \geq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$.

Theorem 2.14. [6] Let $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-subsemigroup $f = (S; f_p, f_n)$ on a semigroup S. Then f is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi-ideal on S if and only if $f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \leq f_p^{(\alpha_2,\alpha_1)}$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} \mathcal{S}_n \underset{(\beta_2,\beta_1)}{\circ} f_n \geq f_n^{(\beta_2,\beta_1)}$.

Theorem 2.15. [6] Let $f = (S; f_p, f_n)$ be a BF-set on a semigroup S. Then f is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-genralized bi-ideal on S if and only if $f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \leq f_p^{(\alpha_2,\alpha_1)}$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} \mathcal{S}_n \underset{(\beta_2,\beta_1)}{\circ} f_n \geq f_n^{(\beta_2,\beta_1)}$.

Theorem 2.16. [6] Let S be a semigroup. Then a non-empty subset I is a left (right, generalized bi-, bi-, quasi-) ideal of S if and only if the bipolar characteristic function $C_I = (S; C_I^p, C_I^n)$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-left (right, generalized bi-, bi-, quasi-) ideal on S for all $\alpha_1, \alpha_2 \in [0, 1]$, $\beta_1, \beta_2 \in [-1, 0]$.

Theorem 2.17. [5] Every f is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-left (right) ideal is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi-ideal.

Theorem 2.18. [6] Every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-left (right) ideal on a semigroup S is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-quasi-ideal on S.

Lemma 2.19. [6] Every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-quasi-ideal on a semigroup S is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi-ideal on S.

Lemma 2.20. [6] Let A and B be non-empty subsets of a semigroup S. Then the following holds.

1. $(C_A)_p \overset{(\alpha_2,\alpha_1)}{\wedge} (C_B)_p = (C_{A\cap B})_p^{(\alpha_2,\alpha_1)}$.
2. $(C_A)_n \overset{(\beta_2,\beta_1)}{\vee} (C_B)_n = (C_{A\cup B})_n^{(\beta_2,\beta_1)}$.
3. $(C_A)_p \overset{(\alpha_2,\alpha_1)}{\circ} (C_B)_p = (C_{AB})_p^{(\alpha_2,\alpha_1)}$.
4. $(C_A)_n \overset{(\beta_2,\beta_1)}{\circ} (C_B)_n = (C_{AB})_n^{(\beta_2,\beta_1)}$.

Theorem 2.21. [6] For a semigroup S, the followings are equivalent.

1. S is regular.
2. $f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p = f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\vee} g_n = f_n \underset{(\beta_2,\beta_1)}{\circ} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-right ideal $f = (S; f_p, f_n)$ and every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-left $g = (S; g_p, g_n)$ ideal on S.

Theorem 2.22. [6] For a semigroup S, the followings are equivalent.

1. S is regular.
2. $f_p^{(\alpha_2,\alpha_1)} = f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p$ and $f_n^{(\beta_2,\beta_1)} = f_n \underset{(\beta_2,\beta_1)}{\circ} \mathcal{S}_n \underset{(\beta_2,\beta_1)}{\circ} f_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-generalized bi-ideal $f = (S; f_p, f_n)$ on S.

3. $f_p^{(\alpha_2,\alpha_1)} = f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p$ and $f_n^{(\beta_2,\beta_1)} = f_n \underset{(\beta_2,\beta_1)}{\circ} \mathcal{S}_n \underset{(\beta_2,\beta_1)}{\circ} f_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi-ideal $f = (S; f_p, f_n)$ on S.

4. $f_p^{(\alpha_2,\alpha_1)} = f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p$ and $f_n^{(\beta_2,\beta_1)} = f_n \underset{(\beta_2,\beta_1)}{\circ} \mathcal{S}_n \underset{(\beta_2,\beta_1)}{\circ} f_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-quasi-ideal $f = (S; f_p, f_n)$ on S.

3. Regular and Intra-regular Semigroups

Definition 3.1. [5] A BF-subsemigroup $f = (S; f_n, f_p)$ on S is called a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-*$BF$-interior ideal* on S where $\alpha_1, \alpha_2 \in [-1, 0]$, $\beta_1, \beta_2 \in [0, 1]$ if it satisfies the following conditions:

1. $f_p(xay) \vee \beta_1 \geq f_p(a) \wedge \beta_2$,
2. $f_n(xay) \wedge \alpha_2 \leq f_n(a) \vee \alpha_1$

for all $x, y, a \in S$.

Theorem 3.2. For a semigroup S, the followings are equivalent.

1. S is regular.
2. $f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p = f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p$ and $f_n \underset{(\beta_2,\beta_1)}{\vee} g_n = f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \underset{(\beta_2,\beta_1)}{\circ} f_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-generalized bi-ideal $f = (S; f_p, f_n)$ and every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-interior ideal $g = (S; g_p, g_n)$ on S.
3. $f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p = f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p$ and $f_n \underset{(\beta_2,\beta_1)}{\vee} g_n = f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \underset{(\beta_2,\beta_1)}{\circ} f_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi-ideal $f = (S; f_p, f_n)$ and every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-interior ideal $g = (S; g_p, g_n)$ on S.
4. $f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p = f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p$ and $f_n \underset{(\beta_2,\beta_1)}{\vee} g_n = f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \underset{(\beta_2,\beta_1)}{\circ} f_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-quasi-ideal $f = (S; f_p, f_n)$ and every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-interior ideal$g = (S; g_p, g_n)$ on S.
5. $f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p = f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p$ and $f_n \underset{(\beta_2,\beta_1)}{\vee} g_n = f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \underset{(\beta_2,\beta_1)}{\circ} f_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-generalized bi-ideal $f = (S; f_p, f_n)$ and every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-two sided ideal $g = (S; g_p, g_n)$ on S.
6. $f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p = f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p$ and $f_n \underset{(\beta_2,\beta_1)}{\vee} g_n = f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \underset{(\beta_2,\beta_1)}{\circ} f_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi-ideal $f = (S; f_p, f_n)$ and every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-two sided ideal $g = (S; g_p, g_n)$ on S.

7. $f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p = f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p$ and $f_n \underset{(\beta_2,\beta_1)}{\vee} g_n = f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \underset{(\beta_2,\beta_1)}{\circ} f_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-quasi-ideal $f = (S; f_p, f_n)$ and every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-two sided ideal $g = (S; g_p, g_n)$ on S.

Proof. (1) $\Rightarrow$ (2) Let f be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-generalized bi-ideal and g is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-interior-ideal on S. By Theorem 2.15, $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \le f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \le f_p^{(\alpha_2,\alpha_1)}$, $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \underset{(\beta_2,\beta_1)}{\circ} f_n \ge f_n \underset{(\beta_2,\beta_1)}{\circ} \mathcal{S}_n \underset{(\beta_2,\beta_1)}{\circ} f_n \ge f_n^{(\beta_2,\beta_1)}$ and $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \le \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \le g_p^{(\alpha_2,\alpha_1)}$, $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \underset{(\beta_2,\beta_1)}{\circ} f_n \ge \mathcal{S}_n \underset{(\beta_2,\beta_1)}{\circ} g_n \underset{(\beta_2,\beta_1)}{\circ} \mathcal{S}_n \ge f_n^{(\beta_2,\beta_1)}$. Thus $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \le f_p^{(\alpha_2,\alpha_1)} \wedge g_p^{(\alpha_2,\alpha_1)} = f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \underset{(\beta_2,\beta_1)}{\circ} f_n \ge f_n^{(\beta_2,\beta_1)} \vee g_n^{(\beta_2,\beta_1)} = f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$. Let $a \in S$, then there exists $x \in S$ such that $a = axa = axaxa$. Since g is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-interior-ideal on S, we have

$$
\begin{aligned}
(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p)(a) &= ((f_p \bar{\circ} g_p \bar{\circ} f_p)(a) \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{a=yz} \{f_p(y) \wedge (g_p \bar{\circ} f_p)(z) \wedge \alpha_2\}) \vee \alpha_1 \\
&\ge (f_p(a) \wedge (g_p \bar{\circ} f_p)(xaxa) \wedge \alpha_2) \vee \alpha_1 \\
&= (f_p(a) \wedge (\bigvee_{xaxa=rs} \{g_p(r) \wedge f_p(s)\}) \wedge \alpha_2) \vee \alpha_1 \\
&\ge (f_p(a) \wedge g_p(xax) \wedge f_p(a) \wedge \alpha_2) \vee \alpha_1 \\
&\ge (f_p(a) \vee \alpha_1) \wedge (g_p(xax) \vee \alpha_1) \wedge (f_p(a) \vee \alpha_1) \wedge (\alpha_2 \vee \alpha_1) \\
&\ge (f_p(a) \vee \alpha_1) \wedge ((g_p(a) \wedge \alpha_2) \vee \alpha_1) \wedge (f_p(a) \vee \alpha_1) \wedge (\alpha_2 \vee \alpha_1) \\
&= (f_p(a) \wedge g_p(a) \wedge \alpha_2) \vee \alpha_1 \\
&= (f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p)(a).
\end{aligned}
$$

Thus $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \ge f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$. Similarly, we can show that $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \underset{(\beta_2,\beta_1)}{\circ} f_n \le f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$. Hence $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p = f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \underset{(\beta_2,\beta_1)}{\circ} f_n = f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (7) and (2) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are clear.

(7) $\Rightarrow$ (1) Let f be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-quasi-ideal on S. Then f be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-ideal on S. Since $\mathcal{S}$ is a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$ ideal on S, we have $((f_p \overset{(\alpha_2,\alpha_1)}{\circ} \mathcal{S}_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p)(a) = f_p^{(\alpha_2,\alpha_1)}(a) = (f_p(a) \wedge \alpha_2) \vee \alpha_1 = ((f_p \wedge \mathcal{S}_p)(a) \wedge \alpha_2) \vee \alpha_1 =$

$((f_p \overset{(\alpha_2,\alpha_1)}{\wedge} \mathcal{S}_p)(a)$. Similarly, we can show that $f_n \underset{(\beta_2,\beta_1)}{\vee} g_n = f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \underset{(\beta_2,\beta_1)}{\circ} f_n$. Thus it follows from Theorem 2.22, S is regular. ■

Definition 3.3. [11] A semigroup S is called **intra-regular** if for all $a \in S$ there exist $x, y \in S$ such that $a = xa^2y$.

Theorem 3.4. [11] For a semigroup S, the following conditions are equivalent.

1. S is regular and intra-regular.
2. Every quasi-ideal of S is idempotent.
3. Every bi-ideal of S is idempotent.

Theorem 3.5. For a semigroup S, the followings are equivalent.

1. S is intra regular.
2. $f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p \leq f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\vee} g_n \geq f_n \underset{(\beta_2,\beta_1)}{\circ} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-left ideal $f = (S; f_p, f_n)$ and every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-right ideal $g = (S; g_p, g_n)$ on S.

Proof. ($\Rightarrow$) Let f be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-left ideal and g be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-right ideal on intra-regular semigroup S. Let $a \in S$. Then there exist $x, y \in S$ such that $a = xa^2y$. Thus

$$\begin{aligned}
(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p)(a) &= ((f_p \bar{\circ} g_p)(a) \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{a=bc} \{f_p(b) \wedge g_p(c)\} \wedge \alpha_2) \vee \alpha_1 \\
&\geq (f_p(ax) \wedge g_p(ay)) \wedge \alpha_2) \vee \alpha_1 \\
&= (f_p(ax) \vee \alpha_1) \wedge (g_p(ay) \vee \alpha_1) \wedge (\alpha_2 \vee \alpha_1) \\
&\geq ((f_p(a) \wedge \alpha_2) \vee \alpha_1) \wedge ((g_p(a) \wedge \alpha_2) \vee \alpha_1) \wedge (\alpha_2 \vee \alpha_1) \\
&= (f_p(a) \wedge g_p(a) \wedge \alpha_2) \vee \alpha_1 \\
&= ((f_p \wedge g_p)(a) \wedge \alpha_2) \vee \alpha_1 \\
&= (f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p)(a).
\end{aligned}$$

Hence $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \geq (f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p)$. Similarly, we can show that $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$.

($\Leftarrow$) Conversely, let R be a right ideal and L be a left ideal of S. Then, by Theorem 2.16 and Lemma 2.20, we have $(C_{RL})_p^{(\alpha_2,\alpha_1)} = (C_R)_p \overset{(\alpha_2,\alpha_1)}{\circ} (C_L)_p \geq (C_R)_p \overset{(\alpha_2,\alpha_1)}{\wedge} (C_L)_p = (C_{R\cap L})_p^{(\alpha_2,\alpha_1)}$. Thus, $R \cap L \subseteq RL$. Therefor it follows from Theorem 2.2 that S is intra-regular. ∎

Theorem 3.6. For a semigroup S, the followings are equivalent.

1. S is both regular and intra-regular.
2. $f_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p = f_p^{(\alpha_2,\alpha_1)}$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} f_n = f_n^{(\beta_2,\beta_1)}$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-quasi-ideal $f = (S; f_p, f_n)$ a on S.
3. $f_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p = f_p^{(\alpha_2,\alpha_1)}$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} f_n = f_n^{(\beta_2,\beta_1)}$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi ideal $f = (S; f_p, f_n)$ a on S.
4. $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for all $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-quasi-ideal $f = (S; f_p, f_n)$ and $g = (S; g_p, g_n)$ on S.
5. $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for all $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-quasi-ideal $f = (S; f_p, f_n)$ and for all $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi ideal $g = (S; g_p, g_n)$ on S.
6. $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for all $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi ideal $f = (S; f_p, f_n)$ and $g = (S; g_p, g_n)$ on S.

Proof. ($1 \Rightarrow 6$). Let f and g be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi ideal $f = (S; f_p, f_n)$ and $g = (S; g_p, g_n)$ on S and $a \in S$. Then there exist $x, y, z \in S$ such that $a = axa$ and $a = ya^2z$. Thus $a = axa = axaxa = ax(ya^2z)xa = (axya)(azxa)$. Thus

$$\begin{aligned}
(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p)(a) &= ((f_p \bar{\circ} g_p)(a) \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{a=bc} \{f_p(b) \wedge g_p(c)\} \wedge \alpha_2) \vee \alpha_1 \\
&\geq (f_p(axya) \wedge g_p(azxa)) \wedge \alpha_2) \vee \alpha_1 \\
&= (f_p(axya) \vee \alpha_1) \wedge (g_p(azxa) \vee \alpha_1) \wedge (\alpha_2 \vee \alpha_1) \\
&\geq ((f_p(a) \wedge \alpha_2) \vee \alpha_1) \wedge ((g_p(a) \wedge \alpha_2) \vee \alpha_1) \wedge (\alpha_2 \vee \alpha_1) \\
&= (f_p(a) \wedge g_p(a) \wedge \alpha_2) \vee \alpha_1 \\
&= ((f_p \wedge g_p)(a) \wedge \alpha_2) \vee \alpha_1 \\
&= (f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p)(a).
\end{aligned}$$

Hence $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$. Similarly, we can show that $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$.

$(6 \Rightarrow 5 \Rightarrow 4)$. Obvious.

$(4 \Rightarrow 2)$. Take $f = g$ in (4), we get $f_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} f_p = f_p^{(\alpha_2,\alpha_1)}$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} f_n \leq f_n \underset{(\beta_2,\beta_1)}{\vee} f_n = f_n^{(\beta_2,\beta_1)}$. Since every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-quasi-ideal on S is $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-subsemigroup on S and by Theorem 2.12, we have $f_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \leq f_p^{(\alpha_2,\alpha_1)}$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} f_n \geq f_n^{(\beta_2,\beta_1)}$. Therefore $f_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p = f_p^{(\alpha_2,\alpha_1)}$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} f_n = f_n^{(\beta_2,\beta_1)}$.

$(6 \Rightarrow 3)$. Take $f = g$ in (6), we get $f_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} f_p = f_p^{(\alpha_2,\alpha_1)}$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} f_n \leq f_n \underset{(\beta_2,\beta_1)}{\vee} f_n = f_n^{(\beta_2,\beta_1)}$. Since every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi-ideal on S is $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-subsemigroup on S and by Theorem 2.12, we have $f_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \leq f_p^{(\alpha_2,\alpha_1)}$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} f_n \geq f_n^{(\beta_2,\beta_1)}$. Therefore $f_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p = f_p^{(\alpha_2,\alpha_1)}$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} f_n = f_n^{(\beta_2,\beta_1)}$.

$(6 \Rightarrow 3)$. Obvious.

$(2 \Rightarrow 1)$. Let Q be a quasi-ideal of S. Then, by Theorem 2.16 and Lemma 2.20, we have $(C_Q)_p \overset{(\alpha_2,\alpha_1)}{\circ} (C_Q)_p = (C_Q)_p^{(\alpha_2,\alpha_1)}$. Thus, $QQ = Q$. Therefore it follows from Theorem 3.4 that S is regular and intra-regular. ■

Theorem 3.7. For a semigroup S, the followings are equivalent.

1. S is both regular and intra regular.
2. $(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p) \wedge (g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p) \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $(f_n \underset{(\beta_2,\beta_1)}{\circ} g_n) \vee (g_n \underset{(\beta_2,\beta_1)}{\circ} f_n) \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-right ideal $f = (S; f_p, f_n)$ and for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-left ideal $g = (S; g_p, g_n)$ on S.
3. $(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p) \wedge (g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p) \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $(f_n \underset{(\beta_2,\beta_1)}{\circ} g_n) \vee (g_n \underset{(\beta_2,\beta_1)}{\circ} f_n) \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-right ideal $f = (S; f_p, f_n)$ and for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-quasi-ideal $g = (S; g_p, g_n)$ on S.
4. $(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p) \wedge (g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p) \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $(f_n \underset{(\beta_2,\beta_1)}{\circ} g_n) \vee (g_n \underset{(\beta_2,\beta_1)}{\circ} f_n) \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-right ideal $f = (S; f_p, f_n)$ and for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi-ideal $g = (S; g_p, g_n)$ on S.

5. $(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p) \wedge (g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p) \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $(f_n \underset{(\beta_2,\beta_1)}{\circ} g_n) \vee (g_n \underset{(\beta_2,\beta_1)}{\circ} f_n) \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-right ideal $f = (S; f_p, f_n)$ and for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-generalized bi-ideal $g = (S; g_p, g_n)$ on S.

6. $(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p) \wedge (g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p) \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $(f_n \underset{(\beta_2,\beta_1)}{\circ} g_n) \vee (g_n \underset{(\beta_2,\beta_1)}{\circ} f_n) \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-left ideal $f = (S; f_p, f_n)$ and for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-quasi-ideal $g = (S; g_p, g_n)$ on S.

7. $(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p) \wedge (g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p) \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $(f_n \underset{(\beta_2,\beta_1)}{\circ} g_n) \vee (g_n \underset{(\beta_2,\beta_1)}{\circ} f_n) \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-left ideal $f = (S; f_p, f_n)$ and for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi-ideal $g = (S; g_p, g_n)$ on S.

8. $(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p) \wedge (g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p) \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $(f_n \underset{(\beta_2,\beta_1)}{\circ} g_n) \vee (g_n \underset{(\beta_2,\beta_1)}{\circ} f_n) \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-left ideal $f = (S; f_p, f_n)$ and for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-generalized bi-ideal $g = (S; g_p, g_n)$ on S.

9. $(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p) \wedge (g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p) \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $(f_n \underset{(\beta_2,\beta_1)}{\circ} g_n) \vee (g_n \underset{(\beta_2,\beta_1)}{\circ} f_n) \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for all $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-quasi-ideal $f = (S; f_p, f_n)$ and $g = (S; g_p, g_n)$ on S.

10. $(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p) \wedge (g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p) \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $(f_n \underset{(\beta_2,\beta_1)}{\circ} g_n) \vee (g_n \underset{(\beta_2,\beta_1)}{\circ} f_n) \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-quasi-ideal $f = (S; f_p, f_n)$ and for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi-ideal $g = (S; g_p, g_n)$ on S.

11. $(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p) \wedge (g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p) \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $(f_n \underset{(\beta_2,\beta_1)}{\circ} g_n) \vee (g_n \underset{(\beta_2,\beta_1)}{\circ} f_n) \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-quasi-ideal $f = (S; f_p, f_n)$ and for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-generalized bi-ideal $g = (S; g_p, g_n)$ on S.

12. $(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p) \wedge (g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p) \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $(f_n \underset{(\beta_2,\beta_1)}{\circ} g_n) \vee (g_n \underset{(\beta_2,\beta_1)}{\circ} f_n) \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for all $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi-ideal $f = (S; f_p, f_n)$ and $g = (S; g_p, g_n)$ on S.

13. $(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p) \wedge (g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p) \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $(f_n \underset{(\beta_2,\beta_1)}{\circ} g_n) \vee (g_n \underset{(\beta_2,\beta_1)}{\circ} f_n) \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi-ideal $f = (S; f_p, f_n)$ and for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-generalized bi-ideal $g = (S; g_p, g_n)$ on S.

14. $(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p) \wedge (g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p) \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $(f_n \underset{(\beta_2,\beta_1)}{\circ} g_n) \vee (g_n \underset{(\beta_2,\beta_1)}{\circ} f_n) \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for all $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-generalized bi-ideal $f = (S; f_p, f_n)$ and $g = (S; g_p, g_n)$ on S.

Proof. (1 ⇒ 14). Let f and g be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-bi ideal $f = (S; f_p, f_n)$ and $g = (S; g_p, g_n)$ on S and $a \in S$. Then there exist $x, y, z \in S$ such that $a = axa$ and $a = ya^2z$. Thus $a = axa = axaxa = ax(ya^2z)xa = (axya)(azxa)$. Thus

$$
\begin{aligned}
(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p)(a) &= ((f_p \bar{\circ} g_p)(a) \wedge \alpha_2) \vee \alpha_1 \\
&= (\bigvee_{a=bc} \{f_p(b) \wedge g_p(c)\} \wedge \alpha_2) \vee \alpha_1 \\
&\geq (f_p(axya) \wedge g_p(azxa)) \wedge \alpha_2) \vee \alpha_1 \\
&= (f_p(axya) \vee \alpha_1) \wedge (g_p(azxa) \vee \alpha_1) \wedge (\alpha_2 \vee \alpha_1) \\
&\geq ((f_p(a) \wedge \alpha_2) \vee \alpha_1) \wedge ((g_p(a) \wedge \alpha_2) \vee \alpha_1) \wedge (\alpha_2 \vee \alpha_1) \\
&= (f_p(a) \wedge g_p(a) \wedge \alpha_2) \vee \alpha_1 \\
&= ((f_p \wedge g_p)(a) \wedge \alpha_2) \vee \alpha_1 \\
&= (f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p)(a).
\end{aligned}
$$

Hence $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$. Similarly, we can show that $g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$, $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ and $g_n \underset{(\beta_2,\beta_1)}{\circ} f_n \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$. Therefore $(f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p) \wedge (g_p \overset{(\alpha_2,\alpha_1)}{\circ} f_p) \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $(f_n \underset{(\beta_2,\beta_1)}{\circ} g_n) \vee (g_n \underset{(\beta_2,\beta_1)}{\circ} f_n) \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$.

(14 ⇒ 13 ⇒ 12 ⇒ 9 ⇒ 6 ⇒ 2), (14 ⇒ 11 ⇒ 10 ⇒ 9) , (14 ⇒ 8 ⇒ 7 ⇒ 6) and (14 ⇒ 5 ⇒ 4 ⇒ 3 ⇒ 2) are obvious.

(2 ⇒ 1). Let f be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-rigt ideal and g be a $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-left ideal on S. By Lemma 2.13, $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \geq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \leq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ and hypothesis, $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \leq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \geq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$. Hence $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p = f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n = f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$, so by Theorem 2.21, S is

regular. Agian by hypothesis $f_p \overset{(\alpha_2,\alpha_1)}{\circ} g_p \leq f_p \overset{(\alpha_2,\alpha_1)}{\wedge} g_p$ and $f_n \underset{(\beta_2,\beta_1)}{\circ} g_n \geq f_n \underset{(\beta_2,\beta_1)}{\vee} g_n$ for every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-right ideal f and every $(\alpha_1, \alpha_2; \beta_1, \beta_2)$-$BF$-left ideal g, so by Theorem 3.5, S is intra regular. ■

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