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Stochastic Process on Option Pricing Black - Scholes PDE– Financial Physics (Phynance) Approach

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Abstract

Phynance has often been perceived as a field, which incorporates both finance and physics aspects for facilitating proper understanding promotion concerning various economic aspects [1]. In this case, physics knowledge is usually applied towards explanation of experienced economic conditions. Economists realize that economic theories, which were traditionally formulated, have no capacity to offer substantial explanations regarding experienced conditions hence the need to turn to some theories found within physics, which have the capacity to bring about suitable explanations regarding experienced economic situations. This is usually essential as it makes it possible for substantial solutions to be sought regarding experienced challenges. The main objective of this study is to Construction of stochastic calculus on Option Pricing Black - Scholes PDE through phynance approach. The main goal of this study is fourfold: 1) First, we begin our approach to briefly present the financial derivatives securities (i.e. Option Contract). 2) Next we extent this approach to introduce the fundamental derivations of the Black Scholes model for the pricing of an European call option. 3) Then we reinterpreted the Black Scholes in the formalism of Quantum mechanics. 4) Finally, we construct the mathematical model for Black – Scholes equation on Heat equation by use of derivatives calculus.In addition, this paper ends with conclusion.

Key Words: Stochastic Process, Option Pricing, Black – Scholes Model, Partial Differential Equation, Financial Derivatives, Option Contract and Heat Equation.

INTRODUCTION

Frankly speaking, a stochastic process is a phenomenon that can be thought to evolve over time in a random way. Common examples are the location of a particle in a physical system, the price of the stock in a financial market, interest rates, mobile phone networks, Internet traffic, etc. A basic example is the erratic movement of pollen grains suspended in water, called Brownian motion. This movement was named after the English botanist R. Brown, who first observed it in 1827. The movement of the pollen grain is thought to be due to the impacts of the water molecules that surround it. Einstein was the first to develop a model for studying the erratic movement of pollen grains in a 1926 paper. We will give an outline of how this model was derived. It is more heuristic than mathematically.

The basic assumption for this model are the following:

1. The motion is continues, moreover, in a time – interval $[t, t+\tau]$, τ is small.
2. Particle movement in two non-overlapping time intervals of length τ are mutually independent.
3. The relative proportion of particles experiencing a displacement of size between δ and $\delta + d\delta$ is approximately $\phi_\tau(\delta)$ with three products

The probability of some displacement is 1: $\int_{-\infty}^{\infty} \phi_\tau(\delta) d\delta = 1$; the average displacement is 0: $\int_{-\infty}^{\infty} \delta\phi_\tau(\delta) d\delta = 0$; and the variation in displacement is linear in the length of the time interval: $\int_{-\infty}^{\infty} \delta^2 \phi_\tau(\delta) d\delta = D_\tau$; where $D \geq 0$ is called the diffusion coefficient.

Donated by $f(x,t)$ the density of particles at position *x*, at time *t*. Under differentiability assumption, we get by a first order Taylor expansion that

$$f(x, t+\tau) \approx f(x,t) + \tau \frac{\partial f}{\partial t}(x,t) \tag{1}$$

On the other hand, by a second order expansion

$$f(x, t+\tau) = \int_{-\infty}^{\infty} f(x-\delta, t) \phi_\tau(\delta) d\delta \tag{2}$$

$$\approx \int_{-\infty}^{\infty} \left[f(x,t) - \delta \frac{\partial f}{\partial x}(x,t) + \frac{1}{2}\delta^2 \frac{\partial^2 f}{\partial x^2}(x,t) \right] \phi_\tau(\delta) d\delta \tag{3}$$

$$\approx f(x,t) + \frac{1}{2} D_\tau \frac{\partial^2 f}{\partial x^2}(x,t) \tag{4}$$

Equating gives rise to the heat equation in one dimension:

$$\frac{\partial f}{\partial t} = \frac{1}{2} D \frac{\partial f^2}{\partial x^2} \tag{5}$$

Which has the solution

$$f(x,t)=\frac{\#\,particles}{\sqrt{4\pi Dt}}.e^{-x^2/4Dt} \tag{6}$$

So, $f(x,t)$ is the density of a $N(0,4Dt)$ distributed random variable multiplied by the number of particles.

1. OPTION CONTRACT

An options contract is an agreement between a buyer and seller that gives the purchaser of the option the right to buy or sell a particular asset at a later date at an agreed upon price. Options contracts are often used in securities, commodities, and real estate transactions. The two basic types of options are call options and put options. A call option gives the owner the right to buy the underlying security at a specified price within a specified period. A put option gives the owner the right to sell the security at a specified price within a particular period. The right, rather than the obligation, to buy or sell the underlying security is what differentiates options from futures contracts.

A European call option is a contract giving the holder the right to buy an asset, called the underlying, for a price *K* fixed in advance, known as the exercise price or strike price, at a specified future time *T*, called the exercise or expiry time. A European put option gives the right to sell the underlying asset for the strike price *K* at the exercise time *T*.

An American call or put option gives the right to buy or, respectively, to sell the underlying asset for the strike price *K* at any time between now and a specified future time *T*, called the expiry time. In other words, an American option can be exercised at any time up to and including expiry.

An option is determined by its payoff, which for a European call is

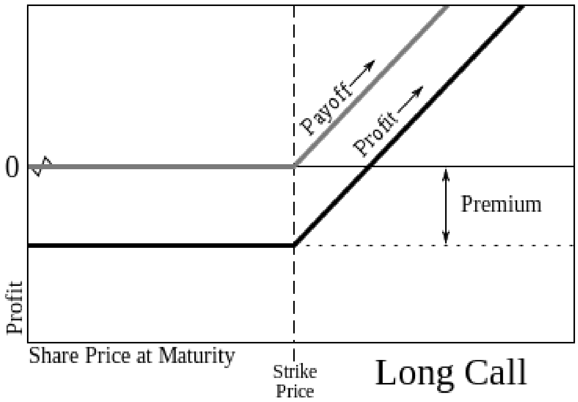
$$\begin{cases} S(T)-K & if\ S(T)>K, \\ 0 & otherwise. \end{cases} \tag{7}$$

This payoff is a random variable; contingent on the price *S(T)* of the underlying on the exercise date *T*. (This explains why options are often referred to as contingent claims.) It is convenient to use the notation

$$x=\begin{cases} x & if\ x>0, \\ 0 & otherwise. \end{cases} \tag{8}$$

for the *positive part* of a real number x. Then the payoff of a European call option can be written as $(S(T) - K)^+$. For a put option the payoff is $(K - S(T))^+$.

Since the payoffs are non-negative, a premium must be paid to buy an option. If no premium had to be paid, an investor purchasing an option could under no circumstances lose money and would in fact make a profit whenever the payoff turned out to be positive. This would be contrary to the No-Arbitrage Principle. The premium is the market price of the option.



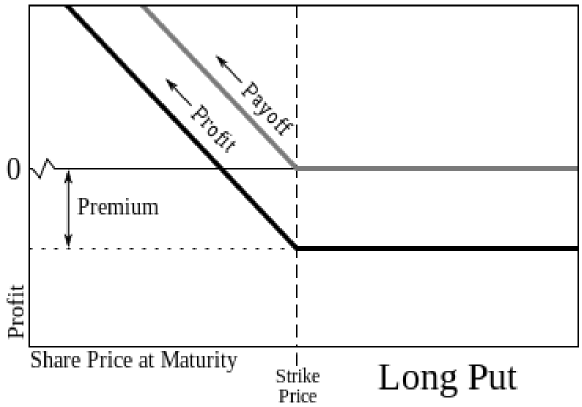


Figure 1: A: Payoff from Buying A Call. **Figure 1:** B: Payoff from Buying A Put.

Consider an underlying security S. The price of a European call option on $S(t)$ is donated by $C(t) = C(t, S(t))$, and gives the owner of the instruments the option to buy the security at some future time $T > t$ for the strike price of K.

At time $t = T$, when the option matures the value of the call option $C(t, S(t))$ is clearly given by

$$C(T, S(T)) = \begin{cases} S(T) - K, & S(T) > K \\ 0, & S(T) < K \end{cases} \tag{9}$$

$$= g(s)$$

Where $g(s)$ is the payoff function.

Consider a portfolio constructed by and writing and selling one put and buying one call option, both with the same strike price K and exercise date T. Adding the payoffs of the long position in calls and the short position in puts, we obtain the payoff of a long forward contract with forward price X and delivery time T. Indeed, if $S(T) \geq K$, then the call will pay $S(T) - K$ and the put will be worthless. If $S(T) < K$, then the call will be worth nothing and the writer of the put will need to pay $K - S(T)$. In either case, the value of the portfolio will be $S(T) - K$ at expiry.

The price of an European put option, denoted by *P(t)* is the same as above, except that the holder now has the option to sell a security *S* at a price of *K*. suppose the spot interest rate is given by *r* and is a constant. For a stock that pays no dividends the following relation holds between the price of European call and put options, both with exercise price *K* and exercise time *T*.

$C(t)-P(t)=S(t)-Ke^{-r(T-t)}; \quad t\le T$ and is called put – call parity.

The fundamental problem of option pricing is the following: given the payoff function $g(s)$ of the option maturing at time *T*. what should be the price of the option *C*, at time $T>t$, if the price of the security is *S(t).* We already discussed from $C(t)=C(t,S(t))$ with the final value of $C(T,S(T))=g(S(T))$.

If the payoff function depends only on the value of *S* at time *T*, the pricing of the option $C(t,S(t))$ is a final value since the final value of *C* at $t=T$, namely $g(s)$, has been specified, and the value of *C* at an earlier time *t* needs to be evaluated.

The present-day price of the option depends on the future value of the security; clearly, the price of the option *C* will be determined by how the security *S(t)* evolves to its future value of *S(T).* In theoretical finance it is common to model the stock price *S(t)* as a (random) stochastic process that is evolved by a stochastic differential equation [2, 3] given by

$$\frac{dS(t)}{dt}=\phi S(t)+\sigma SR(t) \tag{10}$$

where ϕ is the expected return on the security *S*, σ is its volatility and *R* is a Gaussian white noise with zero mean [4].

σ is a measure of the randomness in the evaluation of the stock price; for the special case of $\sigma=0$, the stock price evolves deterministically with its future value given by $S(t)=e^{\phi t}S(0)$.

Since, white noise is assumed independent for each time *t* and the fundamental properties of Gaussian white noise are that

$$<R(t)>=0; \quad <R(t)R(t')>=\delta(t-t') \tag{11}$$

Discretize time, namely $t=n\varepsilon$, with $R(t)\rightarrow R_n$ and , the Dirac delta function correlation is given by

$$E[R(t)]=0; \quad E[R(t)R(t')]\equiv<R(t)R(t')>=\delta(t-t') \tag{12}$$

Both the notations *E[X]* and <*X*> are used for denoting the expectation value of a random variable *X*.

The probability distribution function of the white noise is given by

$$P(R_n) = \sqrt{\frac{\varepsilon}{2\pi}} e^{-\frac{\varepsilon}{2}R_n^2} \tag{13}$$

Hence, R_n is a Gaussian random variable with zero mean and $1/\sqrt{\varepsilon}$ variance denoted by $N\left(0,\ 1/\sqrt{\varepsilon}\right)$. The following result is essential in deriving the rules of Ito calculus

$$R_n^2 = \frac{1}{\varepsilon} + \mathit{Random\ Terms\ of}\ 0(1) \tag{14}$$

Eq (14) is shown that the generating function of R_n^2 can be derived from a R_n^2 that is deterministic.

All the moments of R_n^2 can be determined from its generating function, namely

$$E\left[\left(R_n^2\right)^k\right] = \frac{d^k}{dt^k} E\left[e^{tR_n^2}\right] \Bigg|_{t=0} \tag{15}$$

This property of white noise leads to a number of important results, and goes under the name of Ito calculus in property theory.

The application of stochastic calculus to finance is discussed in great detail in [5], and a brief discussion is given to relate Ito calculus to the Langevin equation. Due to the singular nature of white noise *R(t)*, functions of white noise, such as the security *S(t)* and the option *C(t)*, have new features. In particular, the infinitesimal behaviour of such functions, as seen in their Taylor expansions.

Let f be some arbitrary function of white noise *R(t)*. From the definition of a derivative

$$\frac{df}{dt} = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon, S(t+\varepsilon)) - f(t, S(t))}{\varepsilon} \tag{16}$$

Or, using Taylors expansion

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial t}\frac{dS}{dt} + \frac{\varepsilon}{2}\frac{\partial^2 f}{\partial S^2}\left[\frac{dS}{dt}\right]^2 + 0\left(\varepsilon^{1/2}\right) \tag{17}$$

The last term in Taylors expansion is order ε for smooth functions, and goes to zero. However, due to the singular nature of white noise

$$\left[\frac{dS}{dt}\right]^2 = \sigma^2 S^2 R^2 + 0(1) = \frac{1}{\varepsilon}\sigma^2 S^2 + 0(1) \tag{18}$$

Hence, from Eqs. (10), (17) and (18), for $\varepsilon \to 0$

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial t}\frac{dS}{dt} + \frac{\sigma^2}{2}S^2\frac{\partial^2 f}{\partial S^2} \\ &= \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} + \phi S\frac{\partial f}{\partial t} + \sigma S\frac{\partial f}{\partial S}R\end{aligned} \tag{19}$$

Suppose $g(t, R(t)) \equiv g_t$ is another function of the white noise $R(t)$. The abbreviated notation $\delta g_t \equiv g_{t+\varepsilon} - g_t$ yields

$$\begin{aligned}\frac{d(fg)}{dt} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[f_{t+\varepsilon}g_{t+\varepsilon} - f_t g_t\right] \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\delta f_t g_t + f_t\delta g_t + \delta f_t g_t\right]\end{aligned} \tag{20}$$

Usually the last term $\delta f_t \delta g_t$ is of order ε^2 and goes to zero. However, due to the singular nature of white noise

$$\frac{d(fg)}{dt} = \frac{df}{dt}g + f\frac{dg}{dt} + \frac{df}{\sqrt{dt}}\frac{dg}{\sqrt{dt}}: \tag{21}$$

Ito´s chain rule

Since Eq (19) is of central important for the theory of security derivatives, a derivation is given based on Ito calculus. Rewriting Eq (10) in terms differential as

$$dS = \phi S dt + \sigma S dz \quad ; \quad dz = R dt \tag{22}$$

Where dz is a Wiener process. Since from Eq (14) $R^2(t) = 1/dt$

$$(dz)^2 = R_t^2 (dt)^2 = dt + 0\left(dt^{3/2}\right) \tag{23}$$

And hence

$$(dS)^2 = \sigma^2 S^2 dt + 0\left(dt^{3/2}\right) \tag{24}$$

From the equations for dS and $(dS)^2$ given above

$$\begin{aligned}df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 + 0\left(dt^{3/2}\right) \\ &= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}\right)dt + \sigma S\frac{\partial f}{\partial S}dz + \phi S\frac{\partial f}{\partial S}dt\end{aligned} \tag{25}$$

In addition, Eq (19) is recovered using $dz/dt = R$. Similar to Eq (21). In terms of infinitesimals, the chain rule is given by

$$d(fg) = dfg + fdg + dfdg \tag{26}$$

To explain stochastic calculus, the stochastic differential equation Eq (10) is integrated. Considered the change of variable and the subsequent integration

$$x(t) = \ln[S(t)] \quad ; \quad \Rightarrow \frac{dx}{dt} = \phi - \frac{\sigma^2}{2} + \sigma R(t) \tag{27}$$

$$\Rightarrow x(T) = x(t) + \left(\phi - \frac{\sigma^2}{2}\right)(T - t) + \sigma \int_t^T dt' R(t') \tag{28}$$

The random variable $\int_t^T dt' R(t')$ is a sum of normal variables.

Now, we consider the following integral of white noise

$$I = \int_t^T dt' R(t') \sim \varepsilon \sum_{n=0}^{M} R_n; \quad M = \left[\frac{T-t}{\varepsilon}\right] \tag{29}$$

Where ε is an infinitesimal. For Gaussian white noise

$$R_n = N\left(0, \frac{1}{\sqrt{\varepsilon}}\right) \Rightarrow \varepsilon R_n = N\left(0, \sqrt{\varepsilon}\right) \tag{30}$$

The integral of white noise is a sum of normal random variables and hence, from the Eq (13) and above, is also a Gaussian random variable given by

$$I \sim N\left(0, \sqrt{\varepsilon M}\right) \rightarrow N\left(0, \sqrt{(T-t)}\right) \tag{31}$$

In general, for

$$Z = \int_t^T dt' a(t') R(t') \Rightarrow Z = N\left(0, \sigma^2\right); \sigma^2 = \int_t^T d't' a^2(t') \tag{32}$$

From the Eq (29) to be equal to a normal $N\left(0, \sqrt{(T-t)}\right)$ random variable. Hence

$$S(T) = S(t) e^{\left(\phi - \frac{\sigma^2}{2}\right)(T-t) + \left(\sigma\sqrt{(T-t)}\right)Z} \quad with \quad Z = N(0,1) \tag{33}$$

The stock price evolves randomly from its given value of *S(t)* at time t to a whole range of possible values *S(T)* at time *T* . Since the random variable *x(T)* is a normal (Gaussian) random variable, the security *S(T)* is a lognormal random variable. Campbell et al. [6] discuss the results of empirical studies on the validity of modelling security S as a lognormal random variable.

The probability distribution of the geometric mean of the stock price can be exactly evaluated.

For $\tau = T - t$ and $m = x(t) + \frac{1}{2}\left(\phi - \frac{\sigma^2}{2}\right)\tau$, Eq (28) yields

$$S_{Geometric\,Mean} = e^G$$

$$G \equiv \frac{1}{\tau}\int_t^T dt' x(t') = m + \frac{\sigma}{\tau}\int_t^T dt' \int_t^{t'} dt' R(t'') \tag{34}$$

$$= m + \frac{\sigma}{\tau}\int_t^T dt'(T - t')R(t')$$

From (29) the integral of white noise is a Gaussian random variable, which is completely specified by its means and variance.

Hence, using $E(G) = m$ and Eq (12) for $E[R(t)R(t')] = \delta(t - t')$ yields

$$E\left[(G - m)^2\right] = \left(\frac{\sigma}{\tau}\right)^2 \int_t^T dt'(T - t')\int_t^T dt''(T - t'')E[R(t')R(t'')]$$

$$= \left(\frac{\sigma}{\tau}\right)^2 \int_t^T dt'(T - t')^2 = \frac{\sigma^2 \tau}{3} \tag{35}$$

Hence $G = N\left(m, \frac{\sigma^2 \tau}{3}\right)$ (36)

The geometric mean of the stock price is lognormal with the same mean as the stock price, but with its volatility being one-third of the stock price's volatility.

2. THE BLACK SCHOLES MODEL

In order to facilitate continuity and elucidate various concepts that will be used in the sequel, we give below two fundamental derivations of the Black Scholes model for the pricing of an European call option. The European call option is a financial contingent claim that entails a right (but not an obligation) to the holder of the option to buy one unit of the underlying asset at a future date (called the exercise date or maturity date) at a price (called the exercise price). The option contract, therefore, has a terminal payoff of $\max(S_T - E, 0) = (S_T - E)^+$ where S_T is the stock price on the exercise date and E is the exercise price.

We consider a non dividend paying stock, the price process of which follows the geometric Brownian motion with drift $S_t = e^{\left(\mu t+\sigma W_t^P\right)}$. The logarithm of the stock price $Y_t = In\ S_t$ follows the stochastic differential equation

$$dY_t = \mu dt + \sigma dW_t^P \tag{37}$$

where μ and σ are constants representing the long term drift and the noisiness (diffusion) respectively in the stock price and W_t^P is a regular Brownian motion representing Gaussian white noise with zero mean and δ correlation in time i.e. $E^P\left(dW_t dW_{t'}\right) = dtdt'\delta(t-t')$ and on some filtered probability space $\left(\Omega, \left(F_t\right), P\right)$. Application of Ito's formula yields the following SDE for the stock price process

$$dS_t = \left(\mu + \frac{1}{2}\sigma^2\right)S_t dt + \sigma S_t dW_t^P \tag{38}$$

Let $C(S_t, t)$ denote the instantaneous price of a call option with exercise price E at any time t before maturity when the price per unit of the underlying is S_t. We assume that $C(S_t, t)$ does not depend on the past price history of the underlying. Applying the Ito formula to $C(S_t, t)$ yields

$$dC_t = (\mu S_t \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S_t \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^{\ 2}\frac{\partial^2 C}{\partial S^2})dt + \frac{\partial C}{\partial S}\sigma S_t dW_t^P, \tag{39}$$

The first approach to the Black Scholes formula attempts to remove the randomness in the previously mentioned formula by constituting a hedge in the form of another random process correlated to the above price process. For this purpose, we introduce a 'bond' in our model that evolves according to the following price process

$$\frac{dB_t}{B_t} = rdt, B_T = 1, \tag{40}$$

where r is the relevant risk free interest rate.

Making use of ϕ_t units of the underlying asset and ψ_t units of the bond, where $\phi_t = \frac{\partial C(S_t, t)}{\partial S}$, $B_t\psi_t = C(S_t, t) - \phi_t S_t$, we can now construct a trading strategy that has the following properties

(a) It exactly replicates the price process of the call option i.e.

$$\phi_t S_t + \psi_t B_t = C(S_t, t), \forall t \in [0, T]. \tag{41}$$

(b) It is self-financing i.e. $\phi_t dS_t + \psi_t dB_t = dV_t, \forall t \in [0, T].$ (42)

Using eqs. (37), (39), (41) & (42) we have

$$dC = \left(\phi_t \mu S_t + \frac{1}{2}\phi_t \sigma^2 S_t + \psi_t r B_t \right) dt + \phi_t \sigma S_t dW_t^P . \tag{43}$$

Matching the diffusion terms of (39) & (43) and using (41), we get the previously mentioned expressions for ϕ_t and ψ_t respectively. Further, a comparison of the drift terms yields

$$rS_t \frac{\partial C(S_t,t)}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C(S_t,t)}{\partial S^2} + \frac{\partial C(S_t,t)}{\partial t} = rC(S_t,t), \tag{44}$$

This is the fundamental PDE for asset pricing. For the pricing of the call option, it must additionally satisfy the boundary condition $C(S_T,T) = \max(S_T - E,0)$, which leads to the following solution

$$C(x,t) = x\Phi(z) - e^{-r(T-t)}\Phi\left(z - \sigma\sqrt{(T-t)}\right),\ x = S_t \tag{45}$$

$$z = \frac{In(\frac{x}{K}) + (r+\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{46}$$

and $\Phi(z)$ denotes the cumulative density of the standard normal distribution. Eqs. (45), (47) represent the celebrated Black Scholes formula for the instantaneous price of a call option.

The second approach to the Black Scholes formula [7] is more rigorous and does not depend on the Markov property of the stock price. Applying Girsanov's theorem to the price process (38), we perform a change of measure and define a probability measure Q such that the discounted stock price process $Z_t = S_t e^{-rt}$ or equivalently

$$dZ_t = \left(\mu - r + \frac{1}{2}\sigma^2 \right) Z_t dt + \sigma Z_t dW_t^P \tag{47}$$

behaves as a martingale with respect to Q. This is performed by eliminating the drift term through the transformation

$$\frac{\left(\mu - r + \frac{1}{2}\sigma^2 \right)}{\sigma} \to \gamma_t = \gamma \text{`} \tag{48}$$

Whence $W_t^Q = W_t^P + \gamma t$ is a Brownian motion without drift with respect to the measure Q and $dZ_t = \sigma Z_t dW_t^Q$ which is drift less under the measure Q and hence, Z_t is a Q martingale. The two measures P & Q are related through the Radon Nikodym derivative i.e.

$$\xi(t)=\frac{dQ}{dP}=\exp\left(-\gamma W_t^P-\frac{1}{2}\gamma^2 t\right) \tag{49}$$

and the expectation operators under the two measures are related as

$$E^Q\left(X_t|F_s\right)=\xi^{-1}(s)E^P\left(\xi(t)X_t|F_s\right) \tag{50}$$

Our next step in martingale-based pricing is to constitute a Q martingale process that hits the discounted value of the contingent claim i.e. call option. This is formed by taking the conditional expectation of the discounted terminal payoff from the claim under the Q `measure i.e.

$$E_t=E^Q\left[e^{-rT}\left(S_T-E\right)^+|F_t\right]. \tag{51}$$

As in the PDE approach, we now constitute a self-financing portfolio of ϕ_t units of the underlying asset and ψ_t units of the bond, where $\phi_t=\frac{\partial C}{\partial S}(S_t,t)$, $B_t\psi_t=C(S_t,t)-\phi_t S_t$ which replicates the payoff of the claim at all points in time, The value of this portfolio at any time t can be shown to be equal to $V_t=e^{rt}E_t$ with E_t being given by eq.(51). It follows that the value of the replicating portfolio and hence of the call option at time t is given by

$$V_t=e^{rt}E_t=e^{-r(T-t)}E^Q\left[\left(S_T-E\right)^+|F_t\right] \tag{52}$$

The expectation value of the contingent claim $\max(S_T-E,0)=(S_T-E)^+$ under the measure Q depends only on the marginal distribution of the stock price process S_t under the measure Q, which is obtained by writing it in terms of Q Brownian motion W_t^Q. We have, from eq.(38),

$$d\left(In\ S_t\right)=\mu dt+\sigma dW_t^P=\left(r-\frac{1}{2}\sigma^2\right)dt+\sigma dW_t^Q \tag{53}$$

which on integration yields

$$S_t=S_0\exp\left[\left(r-\frac{1}{2}\sigma^2\right)t+\sigma W_t^Q\right]. \tag{54}$$

This is a S_0 scaled normal with mean and variance $\left(r-\frac{1}{2}\sigma^2\right)t$, $\sigma^2 t$ respectively.

Writing $Z = N\left(-\frac{1}{2}\sigma^2 T, \sigma^2 T\right)$, we have

$$V_t = e^{-r(T-t)} E^Q\left[\left(S_0 e^{Z+rT} - E\right)^+\right]. \tag{55}$$

This equals

$$\frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{ln(E/S_0)-r(T-t)}^{\infty} \left(S_0 e^x - E e^{-r(T-t)}\right) \exp\left(-\frac{\left(x + \frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}\right) dx \tag{56}$$

This integral can be decomposed by a change of variables to a pair of standard cumulative normal integrals, which yield the Black Scholes formula (45).

3. THE FORMALISM OF OPTION PRICING IN THE QUANTUM MECHANICS

Hamiltonian is introduced in the pricing of options. Hamiltonians occur naturally in finance; to demonstrate this the analysis of the Black–Scholes equation is recast in the formalism of quantum mechanics. It is then shown how the Hamiltonian plays a central role in the general theory of option pricing. Option pricing in finance has a mathematical description that is identical to a quantum system; hence, the key features of quantum theory are briefly reviewed. Quantum theory is a vast subject that forms the bedrock of contemporary physics, chemistry and biology [39]. Only those aspects of quantum mechanics are reviewed that are relevant for the analysis of option pricing.

The standard approach for addressing option pricing in mathematical finance is based on stochastic calculus. An independent derivation for the price of the option is given based on the formalism of quantum mechanics [2].

A stock of a company is never negative since the owner of a stock has none of the company's liabilities, and a right to dividends and pro rata ownership of a company's assets.

Hence $S = e^x \geq 0; -\infty \leq x \leq +\infty$. The real variables x is the degree of freedom for the stock price, considered as quantum system. The completeness equation for the degree of freedom is given by $\int_{-\infty}^{+\infty} dx \langle x | x \rangle = I$: Completeness Equation

(57)

Where *I* is the identify operator on (function) state space.

The option pricing is based on the following assumptions.

1. All financial instruments, including the price of the option, are elements of a state space. The stock price is given by

$$S(x) = \langle x|S\rangle = e^{x} \tag{58}$$

The option price is given by a state vector, which for the call option price and the payoff function is given by

$$C(t,x) = \langle x|C,t\rangle \tag{59}$$

$$g(x) = \langle x|g\rangle \tag{60}$$

and similarly for the put option.

The state space is not a normalizable Hilbert space since fundamental financial instruments such as the stock price *S(x)* are not normalizable.

2. The option price is evolved by a Hamiltonian operator *H,* that, due to put-call parity, evolves both the call and put options.

3. The price of the option satisfies the Schrodinger equation

$$H|C,t\rangle = \frac{\partial}{\partial t}|C,t\rangle \tag{61}$$

Eq (59) yield the following

$$|C,t\rangle = e^{tH}|C,0\rangle \tag{62}$$

And final value condition given in (9)

$$|C,T\rangle = e^{TH}|C,0\rangle = |g\rangle \tag{63}$$

$$\Rightarrow |C,0\rangle = e^{-TH}|g\rangle \tag{64}$$

Hence,

$|C,t\rangle = e^{-(T-t)H}|g\rangle$ or, more explicitly

$$C(t,x) = \langle x|C,t\rangle \tag{65}$$

$$= \langle x|e^{-(T-t)H}|g\rangle \tag{66}$$

Using completeness equation (55) yields

$$C(t,x)=\int_{-\infty}^{+\infty}dx'\langle x|e^{-(T-t)H}|x'\rangle g(x') \tag{67}$$

$$=\int_{-\infty}^{\infty}dx'\,p(x,x';\tau)g(x') \tag{68}$$

And similarly for the put option

$$P(t,x)=\int_{-\infty}^{\infty}dx'\langle x|e^{-(T-t)H}|x'\rangle h(x') \tag{69}$$

Where, for $\tau = T-t$, the pricing kernel´s is given by

$$p(x,x';\tau)=\langle x|e^{-\tau H}|x'\rangle \tag{70}$$

In terms of remaining time τ the pricing kernel is given by a decaying exponential as given in eq. (68). The pricing kernel $p(x,x';\tau)$ is the conditional probability, that given the value of the stock is e^{x} at time *t*, it will have the value of $e^{x'}$ at future time $T=t+\tau$. We see from eq. (68) that the pricing kernel is the matrix element of the differential operator $e^{-\tau H}$.

Now, we should construct the Hamiltonian driving option pricing. Assume that *H* has the following impartially general form

$$H=a+b\frac{\partial}{\partial x}-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} \tag{71}$$

Consider for starters the price of a put option. Suppose the strike price $K\to+\infty$; then the payoff function has the following limit $h(S)(K-S)^{+}\to K:$ constant.

Hence similar to Eq (64)

$$P(t,x)=\langle x|e^{-(T-t)H}|h\rangle \tag{72}$$

$$\to\langle x|e^{-(T-t)H}|K\rangle \tag{73}$$

$$e^{-a\tau}K \tag{74}$$

Since $K\to+\infty$, the option is certain to be exercised, and the holder of the option, in exchange for the stock he holds, is going to be paid an amount *K* at future time *T*. Its present-day value, from the principle of no arbitrage, must be the value of *K* discounted to the present by the risk-free spot interest rate. Hence

$$a=r \tag{75}$$

Consider the call option $C(t,x;K)$ with strike value $K = 0$, and which yields the payoff function $g(x)=(S-K)^{+}\rightarrow(S)^{+}=S$. What this means is that for the price of nothing, the holder of the call option will be able to get a stock *S*. Hence, it is certain that the holder of the call option will exercise the option, and hence be in possession of a stock at future time *T*.

Now, we have to find out the discounted value of a stock, held at time T, at present time $t<T$.

The fundamental theorem of finance states that for options to be free from arbitrage opportunities, the evolution driven by the Hamiltonian H must yield a martingale evolution. The risk-free discounted value of a stock is a martingale; in particular, we must have, from Eq (64)

$$C(t,x)=\langle x|e^{-(T-t)H}|g\rangle \tag{76}$$

$$\rightarrow\langle x|e^{-(T-t)H}|g\rangle \tag{77}$$

$$=S(x)\text{due to ,martingale condition} \tag{78}$$

$$e^{-(T-t)H}|S\rangle \quad = \quad |S\rangle \quad \Rightarrow H|S\rangle \quad =0 \tag{79}$$

The martingale evolution is expressed by the fact that the Hamiltonian annihilates the underlying security *S*; this fact is of far reaching consequences in finance since it holds for more complicated systems like the forward interest rates. The equation $H|S\rangle=0$ yields

$$b=\frac{\sigma^{2}}{2}-r \tag{80}$$

The Black–Scholes derivation is reinterpreted in the formalism of quantum mechanics. The time-evolution equation for the Black–Scholes and the Merton–Garman equations is analyzed to obtain the underlying Hamiltonians that drive the option prices.

From the Eq (37), we can rewrite, which yield from the famous Black – Scholes equation for option price with constant volatility is given by

$$\frac{\partial C}{\partial t}=-\frac{1}{2}\sigma^{2}S^{2}\frac{\partial^{2}C}{\partial S^{2}}-rS\frac{\partial C}{\partial S}+rC \tag{81}$$

Consider the change of variable $S=e^{x} \quad ; \quad -\infty\leq x\leq+\infty$

This yield the Black – Scholes - Schrodinger equation

$$\frac{\partial C}{\partial t} = H_{BS} C \tag{82}$$

With the Black – Scholes Hamiltonian given by

$$H_{BS} = -\frac{\sigma^2}{2}\frac{\partial}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial}{\partial x} + r \tag{83}$$

Viewed as a quantum mechanical system, the Black–Scholes equation has one degree of freedom, namely *x*, with volatility being the analog of the inverse of mass, the drift term a (velocity-dependent) potential, and with the price of the option *C* being the analog of the Schrodinger state function.

$$H_{BS}^{\psi} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial}{\partial x} + r \neq H_{BS} \tag{84}$$

Hence, the Black–Scholes Hamiltonian is non-Hermitian due to the drift term.

In Dirac's notation, the Black–Scholes equation is written as

$$< x|H_{BS}|C >= H_{BS}\left(\frac{\partial}{\partial x}\right) < x|C >= H_{BS}\left(\frac{\partial}{\partial x}\right)C(x) \tag{85}$$

Henceforth, it is assumed that the market price of risk has been included in the Merton–Garman equation by redefining λ. Therefore, the Merton–Garman equation [35, 77] is

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + (\lambda + \mu V)\frac{\partial C}{\partial V} + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + \rho\xi V^{\frac{1}{2}+\alpha} S\frac{\partial^2 C}{\partial S \partial V} + \xi^2 V^{2\alpha}\frac{\partial^2 C}{\partial V^2} = rC \tag{86}$$

Various stochastic processes for the volatility of a stock price have been considered. For example, Hull and White [12, 13], Heston [14] and others [15, 16, 17, 18, 19] have considered the following process

$$\frac{dV}{dt} = a + bV + \xi V^{\frac{1}{2}} Q\,;\, \sigma^2 \equiv V \tag{87}$$

Where Q is white noise. Baaquie [20], Hull and White [13] and others have considered

$$\frac{dV}{dt} = \mu V + \xi V Q \tag{88}$$

While Stein and Stein [21] consider

$$d\sigma = -\delta(\sigma - \theta)dt + kdz \tag{89}$$

where δ and θ are constants representing the mean reversion strength and the mean value of the volatility respectively.

All the processes above except for (22) are special cases of the following general form [23]

$$\frac{dV}{dt} = \lambda + \mu V + \xi V^{\alpha} Q \tag{90}$$

The choice of λ and μ is restricted by the condition that V > 0. The complete coupled process is

$$\frac{dS}{dt} = \phi S dt + S\sqrt{V} R_1 \tag{91}$$

$$\frac{dV}{dt} = \lambda + \mu V + \xi V^{\alpha} R_2 \,; \quad V \equiv \sigma^2 \tag{92}$$

Since both S and V are positive-valued random variables, define variables x and y by

$$S = e^x \quad , \quad -\infty < x < +\infty$$

$$\sigma^2 = V = e^y \quad , \quad -\infty < y < +\infty$$

In terms of these variables, the Merton–Garman equation is [20, 24]

$$\begin{aligned} &\frac{\partial C}{\partial t} + \left(r - \frac{e^y}{2} \right) \frac{\partial C}{\partial x} + \left(\lambda e^{-y} + \mu - \frac{\xi^2}{2} e^{2y(\alpha-1)} \right) \frac{\partial C}{\partial y} + \frac{e^y}{2} \frac{\partial^2 C}{\partial x^2} \\ &+ \rho \xi e^{y(\alpha - 1/2)} \frac{\partial^2 C}{\partial x \partial y} + \xi^2 e^{2y(\alpha-1)} \frac{\partial^2 C}{\partial y^2} = rC \end{aligned} \tag{93}$$

The above equation can be re-written as the Merton–Garman–Schrodinger equation given by

$$\frac{\partial C}{\partial t} = H_{MG} C \tag{94}$$

and Eq. (93) yields the Merton–Garman Hamiltonian

$$\begin{aligned} H_{MG} = &-\frac{e^y}{2} \frac{\partial^2}{\partial x^2} - \left(r - \frac{e^y}{2} \right) \frac{\partial}{\partial x} - \left(\lambda e^{-y} + \mu - \frac{\xi^2}{2} e^{2y(\alpha-1)} \right) \frac{\partial}{\partial y} \\ &- \rho \xi e^{y(\alpha - 1/2)} \frac{\partial^2}{\partial x \partial y} - \frac{\xi^2 e^{2y(\alpha-1)}}{2} \frac{\partial^2}{\partial y^2} + r \end{aligned} \tag{95}$$

The Merton–Garman Hamiltonian is a system with two degrees of freedom, and is a formidable one by any standard. The only way of solving it for general α seems to be numerical. The special case of α = 1/2 can be solved exactly using techniques of partial differential equations [14], and α = 1 will be seen to be soluble using path-integral methods [20].

4. CONVERSION OF BLACK – SCHOLES EQUATION ON HEAT EQUATION

This section is devoted to a basic discussion on the heat equation. The heat equation is a parabolic partial differential equation that describes the distribution of heat (or variation in temperature) in a given region over time.

For a function $u(x, y, z, t)$ of three spatial variables (x, y, z) (see cartesian coordinates) and the time variable t, the heat equation is

$$\frac{\partial u}{\partial t} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0 \tag{96}$$

More generally in any coordinate system:

$$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = 0 \tag{97}$$

Where α is a positive constant, and Δ or ∇^2 denotes the Laplace operator. In the physical problem of temperature variation, $u(x, y, z, t)$ is the temperature and α is the thermal diffusivity. For the mathematical treatment it is sufficient to consider the case $\alpha = 1$.

Note that the state equation, given by the first law of thermodynamics (i.e. conservation of energy), is written in the following form (assuming no mass transfer or radiation). This form is more general and particularly useful to recognize which property $e.g.\, c_p or\ \rho$ influences which term.

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) = \dot{q}_V \tag{98}$$

Where $\dot{q}_V$ is the volumetric het flux.

The heat equation is of fundamental importance in diverse scientific fields. In mathematics, it is the prototypical parabolic partial differential equation. In probability theory, the heat equation is connected with the study of Brownian motion via the Fokker–Planck equation. In financial mathematics, it is used to solve the Black–Scholes partial differential equation. The diffusion equation, a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes. Its importance resides in the remarkable fact that the Black – Scholes equation. Which is the main equation of derivatives calculus, can be reduced to this type of equation.

Let $u(\tau, x)$ denote the temperature in an infinite rod at point x and time τ. In the absence of exterior heat sources the heat diffuses accordingly to the following parabolic differential equation

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{called the heat equation} \tag{99}$$

If the initial heat distribution is known and is given by $u(0,x) = f(x)$, then we have an initial value problem for the heat equation.

One way to price options is to derive a partial differential equation (PDE) for the price of the options and then solve the equations either explicitly or numerically.

Consider an European option with the payoff $p(S(T))$. Our procedure is as follows:

1. we will make an assumption about what variables the option price depends on;

2. assume that the option can be replicated by a self-financing investment strategy and derive a PDE for the option price;

3. we will show that the assumption was justified by using a solution to the PDE for constructing a self- financing portfolio that replicates the option.

It is clear that the option price depends on time (or on how much is left until the expiration date) and on the current stock price. Therefore, the first thing to try is to assume that the option price is a function of those two variables, i.e. the price at time *t* is $v(S(t),t)$. Assume that the function v is sufficiently smooth (meaning differentiable) for using Ito's lemma. Assume also that there exists a self-financing investment strategy that replicates the option, then the price of the option at any time should be equal to the value of the portfolio at that time, $v(S(t),t) = X(t)$. Let $\eta(t)$ be the number of shares at time t that determines (with the initial value $X(0)$) the self-financing strategy.

We call an investment strategy a rule for forming *t* each *t* in a period $[t_0; T]$ a portfolio consisting of a deposit *b(t)* to a riskless bank account (if *b(t)* is negative, then it corresponds to borrowing money) and of holding $\eta(t)$ shares of the stock. Both *b(t)* and $\eta(t)$ may depend on the history up to time t (including the current value) of the stock prices but are not allowed to depend on the future values. An investment strategy is called self-financing, if the only changes in the bank account after setting up the initial portfolio are the results of accumulation of interests of the same account, cash flows coming from holding the shares of the stock (e.g dividend payments), or reflect buying or selling the shares of the stock required by changes of η, and if all cash flows that come from the changes of $\eta(t)$ are reflected in the bank account.

Let *X(t)* denote the value of a self- financing portfolio at time t. Assume that the stock pays its holders continuously dividends with the rate *D* percent (realistic if the "stock" is a foreign currency, for usual stocks *D = 0*). Then in an in finitesimally small time interval *dt* the value of a self-financing portfolio changes according to the equation

$$dX(t) = r \cdot (X(t) - \eta(t)S(t))dt + D\eta(t)S(t)dt + \eta(t)dS(t) \tag{100}$$

The first term on the right hand side corresponds to the condition that all money that is not invested in the stock, is deposited to (or borrowed from) a bank account and bears the interest with the risk free rate r, the second term takes into account dividends and the last term reflects the change in the value of the portfolio coming from the change in the stock price. The value of a self-financing portfolio at any time $t > t_0$ is determined by the initial value $X(t_0) = X_0$ and the process $\eta(t), t \in [t_0, T]$. Since nobody can borrow infinitely large sums of money, only such investment strategies, for which the value of the portfolio is almost surely bounded below by a constant, are allowed.

We can rewrite the above Eq (100)

$$dX(t) = (rX(t) - (r - D)\eta(t)S(t))dt + \eta(t)dS(t) \tag{101}$$

According to Ito´s formula we have

$$d\left(v(S(t),t) = \left(\frac{\partial v}{\partial t}(S(t),t) + S(t)^2 \frac{\sigma(S(t),t)^2}{2} \frac{\partial^2 v}{\partial s^2}(S(t),t) \right) dt + \frac{\partial v}{\partial s}(S(t),t) dS(t) \right) \tag{102}$$

As according to our assumption we have $v(S(t),t) = X(t)$, the expressions for $dX(t)$ and $d(v(S(t),t))$should also be equal. Thus, we should have

$$\eta(t) = \frac{\partial v}{\partial s}(S(t),t) \tag{103}$$

And

$$\frac{\partial v}{\partial t}(S(t),t) + \frac{S(t)^2 \sigma(S(t),t)}{2} \frac{\partial^2 v}{\partial s^2}(S(t),t) = rv(S(t),t) - (r - D)S(t)\frac{\partial v}{\partial s}(S(t),t) \tag{104}$$

The last equality is satisfy for all values of t and $S(t)$, if v is a function pf two variables satisfying the PDE

$$\frac{\partial v}{\partial t}(s,t) + \frac{s^2\sigma^2(s,t)}{2}\frac{\partial^2 v}{\partial s^2}(s,t) + (r - D)s\frac{\partial v}{\partial s}(s,t) - rv(s,t) = 0 \tag{105}$$

Now we have derived a PDE for the option pricing. It remains to show that we can indeed construct a replicating self – financing investment strategy for European options.

The Black – Scholes equation admits a closed – form solution and, hence, this solution made the founders well – known and respected. In fact, the Black – Scholes equation and rewrite the Eq (105)

$$\frac{\partial V}{\partial t}\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-\delta)S\frac{\partial V}{\partial S}-rV=0 \tag{106}$$

For a European option $V(S,t)$ is of the type of a parabolic PDE in the domain

$$D_V=\{(S,t)\ :\ S>0,\quad 0\le t\le T\} \tag{107}$$

Hence, by a suitable transformation of the variables the Black – Scholes equation is equivalent to the heat equation Eq (99)

$$\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial x^2} \tag{108}$$

For $u=u(x,t)$ for x and t in the domain

$$D_v=\left\{(x,\tau)\quad :-\infty<x<\infty,\quad 0\le\tau\le\frac{\sigma^2}{2}T\right\} \tag{109}$$

In general, the classical heat equation may be considered in a larger domain, $x\in R$ and $\tau\ge 0$. However, since the option expires at maturity *T*, and the time when the option contract is signed is assumed to be $t_0=0$, then the transformed heat equation will naturally have a bounded τ . On the other hand, although in the domain of the Black-Scholes equation the variable *S* lies on the positive real axis, the variable *x* in the domain of the heat equation lies overall real axis. These are all due to the transformations used in the sequel.

Consider the transformation of the independent variables

$$S=Ke^x,\quad and\quad t=T-\frac{\tau}{\sigma^2/2}\ , \tag{110}$$

And the dependent variables

$$v(x,\tau)=\frac{1}{K}V(S,t)=\frac{1}{K}V\left(Ke^x,T-\frac{\tau}{\sigma^2/2}\right) \tag{111}$$

In fact, the change of the independent variables ensures that the domain of the new dependent variables $v=v(x,\tau)$ is D_u . By the chain rule for functions of several variables, these changes of variables give

$$\frac{\partial V}{\partial t}=K\frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t}=-\frac{\sigma^2}{2}K\frac{\partial v}{\partial \tau} \tag{112}$$

$$\frac{\partial V}{\partial S} = K\frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = \frac{K}{S}\frac{\partial v}{\partial x} \tag{113}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{K}{S^2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) \tag{114}$$

Inserting the derivatives in the Black – Scholes equation (106) transformation it to a constant coefficient one:

$v_\tau = v_{xx} + \left(\frac{r-\delta}{\sigma^2/2} - 1\right)v_x - \frac{r}{\sigma^2/2}v$ where the subscripts represents the partial derivatives with respect to the corresponding variables. Define the following new constants,

$k = \frac{r-\delta}{\sigma^2/2}, \quad and \quad l = \frac{\delta}{\sigma^2/2},$ so that the transformed PDE turns a simpler form

$$v_\tau = v_{xx} + (k-1)v_x - (k+l)v \tag{115}$$

The coefficient of which involve the new two constant k and l . This constant coefficient PDE must be transformed further to the heat equation by some other change of the independent variables.

In order to simplify the final transformation of the dependent variables v , let us first define the following constants:

$$\gamma = \frac{1}{2}(k-1), \quad and \quad \beta = \frac{1}{2}(k+1) = \gamma + 1, \tag{116}$$

So that $\beta^2 = \gamma^2 + k$ (117)

In terms of these new constants, now the transformation can be defined by

$$v(x,\tau) = e^{-\gamma x - (\beta^2 + l)\tau}u(x,\tau) \tag{118}$$

For all (x,τ) in D_u .

Hence, the partial derivatives with respect to τ and x can be calculated as

$$v_\tau = e^{-\gamma x - (\beta^2 + l)\tau}\left\{-(\beta^2 + l)u + u_\tau\right\}, \tag{119}$$

$$v_x = e^{-\gamma x - (\beta^2 + l)\tau}\left\{-\gamma u + u_x\right\}, \tag{120}$$

$$v_{xx} = e^{-\gamma x - (\beta^2 + l)\tau}\left\{\gamma^2 u - 2\gamma u_x + u_{xx}\right\}. \tag{121}$$

Thus, substituting these derivatives into Eq (115) yield

$$u_\tau = u_{xx} + (-2\gamma + k - 1)u_x + \gamma(2\gamma - k + 1)u \tag{122}$$

After having used the fact that $\beta^2 = \gamma^2 + k$. Notice that the coefficient of the terms u_x and u in the equation vanishes by the choice of $\gamma\ as\ \frac{1}{2}(k-1)$.

Consequently, the equation that is to be satisfied by the transformed dependent variables $u = u(x,\tau)$ is the dimensional form of the heat equation, Eq (108) that is to be solved on the domain D_u.

This shows the equivalence between the Black – Scholes equation Eq (106) and the (108).

To sum up, on order to transformed the Black – Scholes equation to the classical dimensionless heat equation, the constant used above are defined to be

$$\begin{aligned} &k = \frac{r-\delta}{\sigma^2/2}, \qquad l = \frac{\delta}{\sigma^2/2}, \\ &\gamma = \frac{1}{2}(k-1), \qquad \beta\frac{1}{2}(k+1) = \gamma + 1 \end{aligned} \tag{123}$$

On the other hand, the transformation of the dependent and the independent variables that use those constants are given by

$$S = Ke^x, \qquad t = T - \frac{\tau}{\sigma^2/2}, \tag{124}$$

$$V(S,t) = Kv(x,\tau), \qquad v(x,\tau) = e^{-\gamma x - (\beta^2 + l)}u(x,\tau), \tag{125}$$

Under these changes of variables, the domain D_v is mapped to D_u.

The fundamental solution of the dimensionless heat equation $u_\tau = u_{xx}$ is given by

$$G(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}e^{x^2/4\tau}, \tau > 0, \text{ and } x \in R \tag{126}$$

We recall the most important properties of the function $G(x,\tau) = \phi_{0,\sqrt{2\tau}}(x)$, that is probability density function of the normal distribution with mean zero and variance 2τ. This can be easily shown by direct substitution into the equation. Moreover, for a given initial condition,

$$u(x,0) = u_0(x), \qquad -\infty < x < +\infty \tag{127}$$

$\tau = 0$, the solution of the heat equation can be written as a convolution integral of *G* and u_0 as

$$u(x,\tau) = \int_{-\infty}^{+\infty} G(x-\xi,\tau) u_0(\xi) d\xi \tag{128}$$

For $\tau > 0$. Write this representation, the function $G(x-\xi,\tau)$ is also called the Green's function for the diffusion equation. It is not too difficult to show that $u == u(x,\tau)$ represented by the convolution integral above is indeed a solution of the heat equation and satisfies

$$\lim_{\tau \to 0^+} u(x,\tau) = u_0(x) \tag{129}$$

Consequently, the solution of the heat equation that satisfies the initial condition Eq (125) can be represented by Eq (128) or, using Eq (126), by

$$u(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\xi)^2}{4\tau}} u_0(\xi) d\xi \tag{130}$$

Therefore, in order to solve the Black-Scholes equation we need to determine what the initial function $u_0(x) = u(x,0)$ corresponds to in the original setting. This initial function is given for $\tau = 0$, and hence, there is the corresponding given function at maturity $t = T$, the payoff function. Due to the transformations in Eq (124 & 125), the payoff function of the contingent claim stands for the terminal condition of the Black-Scholes equation.

If the terminal condition of the Black-Scholes equation is given by $V(S,T) = P(S)$ at maturity $t = T$, then it must be transformed to and the corresponding initial condition, $u_0(x) = u(x,0)$, of the heat equation. By plugging it in Eq (130) and, if possible, performing the integration, the solution to the heat equation can be found. Consequently, using the trans- formations Eq (124 & 126), the computed solution must be interpreted using the original variables *S, t* and *V* involved in the Black-Scholes PDE.

CONCLUSION

Phynance, commonly regarded as experienced relationships within physics as well as finance has expensively been discussed. It is essential due to its capacity to express comprehensive relationships experienced amid finance as well as physics. Through the highlighted relationships, it is evident that individuals get proper understanding concerning the various aspects regarding finance. Derivation of a closed-form solution to the Black-Scholes equation depends on the fundamental solution of the heat equation. Hence, it is important, at this point, to transform the Black-Scholes equation to the heat equation by change of variables. Having found the closed form solution to the heat equation, it is possible to transform it back to and the corresponding solution of the Black-Scholes PDE. The main objective of this study was to Constructed of stochastic calculus on Option Pricing Black - Scholes PDE through phynance approach. Here we discussed four goals. First, we initiated our approach to briefly present the financial derivatives securities (i.e. Option Contract). Then, we extended this approach to introduce the fundamental derivations of the Black Scholes model for the pricing of an European call option. Followed, we reinterpreted the Black Scholes in the formalism of Quantum mechanics. Finally, we constructed the mathematical model for Black – Scholes equation on Heat equation by use of derivatives calculus.In addition, this paper ends with conclusion.

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