Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 11 (2017), pp. 8093–8102

http://www.ripublication.com/gjpam.htm

The Class of Absolute Square K-Paranormal Operator

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Abstract

In this paper we introduce and study a new class of operator called absolute square k-paranormal operator. We give a characterization of such an operator and investigate some relations between other known classes of operators.

AMS subject classification: Primary 47A53; Secondary 47B20.
Keywords: Hilbert space, paranormal, class A, absolute k-paranormal operator, normaloid.

1. Introduction

Let H be an infinite dimensional complex Hilbert space and $L(H)$ denote the algebra of bounded linear operators on H. In this paper we introduce and study a new class of operator. We call it absolute square k-paranormal operator and define as $\|Tx\|^{2(k+1)} \leq \||T^2|^k T^2 x\|$ for some $k \geq 0$, $x \in H$ with $\|x\| = 1$. We prove this class of operator is an intermediate class between paranormal and normaloid classes. We show that every paranormal operator is absolute square k-paranormal operator for $k > 1$ and every absolute square k-paranormal operator is normaloid for $k \geq 1$.

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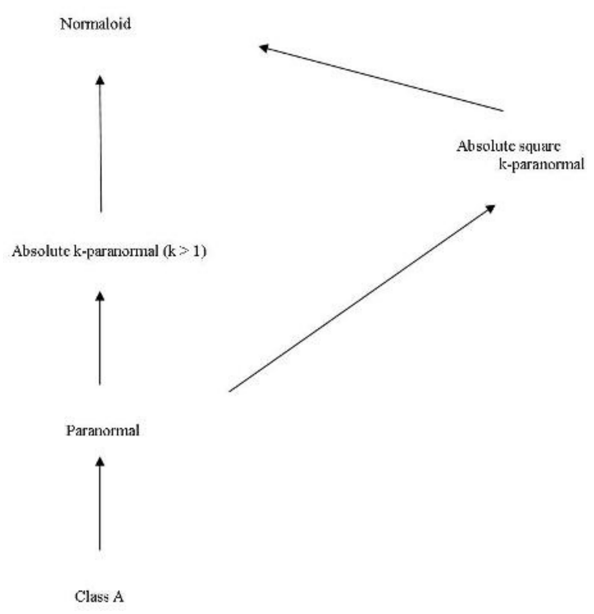
2. Preliminaries

An operator $T \in H$ is said to be a paranormal if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$, and class A if $|T|^2 \leq |T^2|$. For each $k > 0$, an operator T is absolute k-paranormal operator if $\||T|^k Tx\| \geq \|Tx\|^{k+1}$ for every unit vector $x \in H$ [3]. An operator T is said to be normaloid if $\|T\| = r(T)$ where $r(T)$ is the spectral radius of T [5]. Also we have an equivalent condition: If T is normaloid then $\|T^n\| = \|T\|^n$ for all natural number n [3]. The following inclusion relations hold among these classes.

Class $A \subseteq$ paranormal $\subseteq$ absolute k-paranormal operator $\subseteq$ normaloid [2].

In this paper a new inclusion relation
Class $A \subseteq$ paranormal $\subseteq$ absolute square k-paranormal operator $\subseteq$ normaloid is obtained. Also we obtain absolute k-paranormal and absolute square k-paranormal operators are not related to each other. This motivates to study this new class of operator.

Remark 2.1. The following diagram shows the inclusion relations among the classes of operators.



Theorem 2.2. (Holder - McCarthy inequality) [3, 8] Let A be a positive linear operator on a Hilbert space H. Then the following properties (i), (ii) and (iii) hold.

(i) $(A^\lambda x, x) \geq (Ax, x)^\lambda$ for any $\lambda > 1$ and any unit vector x.

(ii) $(A^\lambda x, x) \geq (Ax, x)^\lambda \|x\|^{2(1-\lambda)}$ for any $\lambda > 1$ and any vector x.

(iii) $\|A^\lambda x\| \geq \|Ax\|^\lambda \|x\|^{1-\lambda}$ for any $\lambda > 1$ and any vector x.

Lemma 2.3. [7] Let a and b be positive real numbers. Then $a^{\lambda}b^{\mu} \leq \lambda a + \mu b$ holds for $\lambda, \mu > 0$ such that $\lambda + \mu = 1$.

3. Absolute Square K-Paranormal Operator

Definition 3.1. An operator $T \in L(H)$ is said to be a absolute square k-paranormal operator if for some $k \geq 0$
$\|Tx\|^{2(k+1)} \leq \||T^2|^k T^2 x\|$ for every $x \in H$, with $\|x\| = 1$.

Theorem 3.2. An operator $T \in L(H)$ is absolute square k-paranormal operator if and only if
$T^{*2}|T^2|^{2k}T^2 - 2(k+1)\lambda^{2k+1}|T|^2 + (2k+1)\lambda^{2(k+1)}I \geq 0$ holds for all $\lambda > 0$.

Proof. Assume that $T \in L(H)$ is a absolute square k-paranormal operator. Then for some $k > 0$ we have
$\||T^2|^k T^2 x\| \geq \|Tx\|^{2(k+1)}$ for every unit vector $x \in H$.
$\Rightarrow \||T^2|^k T^2 x\|^{\frac{1}{(k+1)}} \|x\|^{\frac{2k+1}{(k+1)}} \geq \|Tx\|^2$ for every $x \in H$.
$\Rightarrow (T^{*2}|T^2|^{2k}T^2 x, x)^{\frac{1}{2(k+1)}} (x, x)^{\frac{2k+1}{2(k+1)}} \geq (|T|^2 x, x)$ for every $x \in H$. (A)
$\Rightarrow [\frac{1}{\lambda^{2k+1}}(T^{*2}|T^2|^{2k}T^2 x, x)]^{\frac{1}{2(k+1)}} [\lambda(x, x)]^{\frac{2k+1}{2(k+1)}} \geq (|T|^2 x, x)$ for all $x \in H$ and $\lambda > 0$.
$\Rightarrow \frac{1}{2(k+1)} \frac{1}{\lambda^{2k+1}}(T^{*2}|T^2|^{2k}T^2 x, x) + \frac{2k+1}{2(k+1)}\lambda(x, x) \geq (|T|^2 x, x)$. (B)
where we have used the lemma 2.3
$\Rightarrow (T^{*2}|T^2|^{2k}T^2 x, x) + (2k+1)\lambda^{2k+2}(x, x) \geq (|T|^2 x, x)2(k+1)\lambda^{2k+1}$.
$\Rightarrow T^{*2}|T^2|^{2k}T^2 + (2k+1)\lambda^{2(k+1)} \geq 2(k+1)\lambda^{2k+1}|T|^2$.
$\Rightarrow T^{*2}|T^2|^{2k}T^2 - 2(k+1)\lambda^{2k+1}|T|^2 + (2k+1)\lambda^{2(k+1)}I \geq 0$ for all $x \in H$ and $\lambda > 0$.
Conversely (B) implies (A) by putting λ=$[\frac{(T^{*2}|T^2|^{2k}T^2 x, x)}{(x, x)}]^{\frac{1}{2(k+1)}}$.
In case $(T^{*2}|T^2|^{2k}T^2 x, x) = 0$ and let $\lambda \to 0$. Hence (B) holds if and only if
$T^{*2}|T^2|^{2k}T^2 - 2(k+1)\lambda^{2k+1}|T|^2 + (2k+1)\lambda^{2(k+1)}I \geq 0$ holds for all $\lambda > 0$. ■

Corollary 3.3. Every absolute square 0-paranormal operator is paranormal. For T is paranormal if and only if $T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0$ for all $\lambda > 0$ [2].

Theorem 3.4. Every paranormal operator T is absolute square k-paranormal operator for $k > 1$.

Proof. Assume that $T \in L(H)$ is a paranormal operator. Then $\|Tx\|^2 \leq \|T^2 x\|$ and T^n

is paranormal for any $n \in N$ [4]. Consider,

$$\begin{aligned} \||T^2|^k T^2 x\|^2 &= (|T^2|^k T^2 x, |T^2|^k T^2 x) \\ &= (|T^2|^{2k} T^2 x, T^2 x) \\ &= (|T^2|^2 T^2 x, T^2 x)^k \|T^2 x\|^{2(1-k)} \text{ [By (ii) of Theorem 2.1]} \\ &= (T^{*2} T^2 T^2 x, T^2 x)^k \|T^2 x\|^{2(1-k)} \\ &= (T^4 x, T^4 x)^k \|T^2 x\|^{2(1-k)} \\ &= \|T^4 x\|^{2k} \|T^2 x\|^{2(1-k)} \\ &\geq \|T^2 x\|^{4k} \|T^2 x\|^{2(1-k)} \text{ [Since } T^2 \text{ is paranormal]} \\ &\geq \|Tx\|^{4(1+k)} \text{ [Since } T \text{ is paranormal].} \end{aligned}$$

Showing that $\|Tx\|^{2(k+1)} \leq \||T^2|^k T^2 x\|$. ■

We recall the following proposition from [2].

Proposition 3.5. Let $K = \oplus_{n=-\infty}^{\infty}$, where $H_n \cong H$. For given positive operators A and B on H, define the operator $T_{A,B}$ on K as follows:

$$T_{A,B} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & B & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & B & [0] & 0 & 0 & \dots \\ \dots & 0 & 0 & B & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & A & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & A & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Where [0] shows the place of the (0,0) matrix element. Then the following assertions hold.

(i) For each $k > 0$, $T_{A,B}$ is absolute k-paranormal if and only if $BA^{2k}B - (k+1)\lambda^k B^2 + k\lambda^{k+1} \geq 0$ for all $\lambda > 0$.

(ii) $T_{A,B}$ is paranormal if and only if $BA^2B - 2\lambda B^2 + \lambda^2 \geq 0$ for all $\lambda > 0$.

Example 3.6. A non paranormal and absolute square 1-paranormal operator.
Take A and B as

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

$T_{A,B}$ is paranormal if and only if
$BA^2B - 2\lambda B^2 + \lambda^2 \geq 0$ for all $\lambda > 0$ [By (ii) of proposition 3.5] (C).
Put $\lambda = 4$ in (C) we get

$$BA^2B - 8B^2 + 16 = \begin{pmatrix} -8 & 0 \\ 0 & 16 \end{pmatrix}.$$

Here the eigen values are -8,16, so that $BA^2B - 2\lambda B^2 + \lambda^2 < 0$
Hence $T_{A,B}$ is non paranormal by (C),
$T_{A,B}$ is absolute square 1- paranormal operator if and only if
$B^2A^4B^2 - 4\lambda^3B^2 + 3\lambda^4 \geq 0$ for all $\lambda > 0$ (D).
Then for $\lambda > 0$

$$B^2A^4B^2 - 4\lambda^3B^2 + 3\lambda^4 = \begin{pmatrix} 320 - 16\lambda^3 + 3\lambda^4 & 0 \\ 0 & 3\lambda^4 \end{pmatrix}.$$

We get $320 - 16\lambda^3 + 3\lambda^4 > 0$ for all $\lambda > 0$
So, $B^2A^4B^2 - 4\lambda^3B^2 + 3\lambda^4 \geq 0$ for all $\lambda > 0$
Hence $T_{A,B}$ is absolute square 1-paranormal by (D).
Hence every absolute square k-paranormal need not be a paranormal operator.

Theorem 3.7. If an operator T is absolute square k- paranormal operator for some $k \geq 1$, then T is normaloid.

Proof. T is absolute square k-paranormal operator for some $k \geq 1$.
$\|Tx\|^{2(k+1)} \leq \||T^2|^kT^2x\|$ for every unit vector $x \in H$ and we may assume that $\|T\| = 1$ without loss of generality. We remark that $\|T^n\| \leq \|T\|^n = 1$ for all natural number n. Then

$$\begin{aligned}\|Tx\|^{2(k+1)} &\leq \||T^2|^kT^2x\| \\ &\leq \||T^2|^{k-1}\|\||T^2|T^2x\| \\ &\leq \|T^4x\| \\ &\leq \|T\|^4 \text{ for every unit vector } x \in H \\ &\leq 1\end{aligned}$$

That is

$$\frac{\|Tx\|^{2(k+1)}}{\|x\|^{2k+1}} \leq \|T^4x\| \leq \|x\| \text{ for all } x \in H \qquad \text{(a)}$$

Let x_j be a sequence of unit vectors such that $\|Tx_j\| \to 1$ (b)
Put $x = x_j$ in (a), then we have $\frac{\|Tx_j\|^{2(k+1)}}{\|x_j\|^{2k+1}} \leq \|T^4x_j\| \leq \|x_j\| = 1$ (c)
So $\|T^4x_j\| \to 1$ by (b) and (c), that is $\|T^4\| = 1 = \|T\|^4$

Now suppose that $\|T^{n-4}x_j\| \to 1$ and $\|T^{n-3}x_j\| \to 1$ for $n \geq 4$ (d)
Put $x = T^{n-4}x_j$ in (a), then we have
$\frac{\|T^{n-3}x_j\|^{2(k+1)}}{\|T^{n-4}x_j\|^{2k+1}} \leq \|T^n x_j\| \leq \|T^{n-4}x_j\|$ (e)
So $\|T^n x_j\| \to 1$ by (d) and (e), that is $\|T^n\| = 1 = \|T\|^n$
Consequently $\|T^n\| = 1 = \|T\|^n$ for all natural number n by induction.
Hence T is normaloid. ∎

Example 3.8. A non absolute square k-paranormal for any $k > 0$ and normaloid operator. Let

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then $\|T^n\| = \|T\|^n$ for all natural number n, so that T is normaloid [3].
T is absolute square k-paranormal operator if $\|Tx\|^{2(k+1)} \leq \||T^2|^k T^2 x\|$.
When $k = 2$ we have $\|Tx\|^6 \leq \||T^2|^2 T^2 x\|$.
$\|Tx\|^6 = (x^2 + yz)^6$ (E)
$\||T^2|^2 T^2 x\| = x^2$ (F)
From (E) and (F) we get $\|Tx\|^6 > \||T^2|^2 T^2 x\|$.
Hence every normaloid need not be absolute square k-paranormal for any $k > 0$.

Example 3.9. A non absolute 2-paranormal and absolute square 2-paranormal operator. Take A and B as

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}^{1/2}$$

and

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

$T_{A,B}$ is absolute 2-paranormal if and only if
$BA^4B - 3\lambda^2 B^2 + 2\lambda^3 \geq 0$ for all $\lambda > 0$ [By (i) of proposition 3.5] (G).
Put $\lambda = 2$ in (G) we get

$$BA^4B - 12B^2 + 16 = \begin{pmatrix} -24 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here the eigen value is -24, so that $BA^4B - 3\lambda^2 B^2 + 2\lambda^3 < 0$
Hence $T_{A,B}$ is non absolute 2-paranormal by (G)
$T_{A,B}$ is absolute square 2- paranormal if and only if
$B^2A^8B^2 - 6\lambda^5 B^2 + 5\lambda^6 \geq 0$ for all $\lambda > 0$ (H).
Then for $\lambda > 0$

$$B^2A^8B^2 - 6\lambda^5 B^2 + 5\lambda^6 = \begin{pmatrix} 320 - 24\lambda^5 + 5\lambda^6 & 0 \\ 0 & 5\lambda^6 \end{pmatrix}.$$

We get $320 - 24\lambda^5 + 5\lambda^6 > 0$ for all $\lambda > 0$
So, $B^2A^8B^2 - 6\lambda^5B^2 + 5\lambda^6 \geq 0$ for all $\lambda > 0$
Hence $T_{A,B}$ is absolute square 2-paranormal operator by (H).
Hence every absolute square k-paranormal need not be absolute k-paranormal operator.

Example 3.10. A non absolute square 2-paranormal and absolute 2-paranormal operator.
Take A and B as

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 20 \end{pmatrix}^{\frac{1}{4}}$$

and

$$B = \frac{1}{2}\begin{pmatrix} 1+\sqrt{3} & 1-\sqrt{3} \\ 1-\sqrt{3} & 1+\sqrt{3} \end{pmatrix}.$$

$T_{A,B}$ is absolute square 2-paranormal if and only if
$B^2A^8B^2 - 6\lambda^5B^2 + 5\lambda^6 \geq 0$ for all $\lambda > 0$ (I).
Put $\lambda = 2$ in (I) we get

$$B^2A^8B^2 - 192B^2 + 320 = \begin{pmatrix} -398.14 & -637.9 \\ 191.61 & 1549.12 \end{pmatrix}.$$

Here the eigen values are 1484.18 and -333.205, so that $B^2A^8B^2 - 6\lambda^5B^2 + 5\lambda^6 < 0$
Hence $T_{A,B}$ is non absolute square 2-paranormal by (I)
$T_{A,B}$ is absolute 2- paranormal if and only if
$BA^4B - 3\lambda^2B^2 + 2\lambda^3 \geq 0$ for all $\lambda > 0$ [By (i) of Proposition 3.5]————(J).
Then for $\lambda > 0$, define $X_1(\lambda)$ as

$$X_1(\lambda) = BA^4B - 3\lambda^2B^2 + 2\lambda^3 = \begin{pmatrix} 24 - 8\sqrt{3} - 6\lambda^2 + 2\lambda^3 & -12 + 3\lambda^2 \\ -12 + 3\lambda^2 & 24 + 8\sqrt{3} - 6\lambda^2 + 2\lambda^3 \end{pmatrix}.$$

Put $p_1(\lambda) = trX_1(\lambda)$ and $q_1(\lambda) = detX_1(\lambda)$ Then

$$p_1(\lambda) = 4\lambda^3 - 12\lambda^2 + 48 > 0 \text{ for all } \lambda > 0$$

$$q_1(\lambda) = 4\lambda^6 - 24\lambda^5 + 27\lambda^4 + 96\lambda^3 - 216\lambda^2 + 240$$

$$q_1^1(\lambda) = 24\lambda^5 - 120\lambda^4 + 108\lambda^3 + 288\lambda^2 - 432\lambda$$

$$= 12\lambda(\lambda - 2)(2\lambda^3 - 6\lambda^2 - 3\lambda + 18)$$

$q_1^1(\lambda) = 0$ if and only if $\lambda = 0, 2$, since $2\lambda^3 - 6\lambda^2 - 3\lambda + 18 > 0$ for all $\lambda > 0$
$q_1(\lambda) > q_1(2) = 64 > 0$ for all $\lambda > 0$
Hence $X_1(\lambda) \geq 0$ for all $\lambda > 0$ since$trX_1(\lambda) = p_1(\lambda) > 0$ and $detX_1(\lambda) = q_1(\lambda) > 0$ for all $\lambda > 0$.
Hence $T_{A,B}$ is absolute 2-paranormal operator by (J).
Hence every absolute k-paranormal need not be absolute square k-paranormal operator.

By example 3.9 and 3.10, it follows that absolute k-paranormal and absolute square k-paranormal operators are not related to each other.

Theorem 3.11. If T^2 is absolute k-paranormal operator then T is a absolute square k-paranormal operator for $k > 1$.

Proof. Assume that $T^2 \in L(H)$ is a absolute k-paranormal operator.
Then $\|T^2x\|^{(k+1)} \leq \||T^2|^k T^2 x\|$
Consider, $\||T^2|^k T^2 x\| \geq \|T^2 x\|^{(k+1)}$ [Since T^2 is absolute k-paranormal operator]
$\geq (\|Tx\|^2\|x\|^{1-2})^{k+1}$ [By (iii) of Theorem 2.1] ■

Example 3.12. A non T^2 absolute 2-paranormal operator and T is absolute square 2-paranormal operator.
Take A and B as

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}^{\frac{1}{2}}$$

and

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

$T_{A,B}^2$ is absolute 2-paranormal if and only if
$B^2A^8B^2 - 3\lambda^2B^4 + 2\lambda^3 > 0$ for all $\lambda > 0$ (K).
Put $\lambda = 4$ in (K) we get

$$B^2A^8B^2 - 3\lambda^2B^4 + 2\lambda^3 = \begin{pmatrix} -320 & 0 \\ 0 & 16 \end{pmatrix}.$$

Here the eigen values are -320, 16 so that $B^2A^8B^2 - 3\lambda^2B^4 + 2\lambda^3 < 0$
Hence $T_{A,B}^2$ is non absolute 2-paranormal operator by (K)
$T_{A,B}$ is absolute square 2- paranormal operator if and only if
$B^2A^8B^2 - 6\lambda^5B^2 + 5\lambda^6 \geq 0$ for all $\lambda > 0$ ————(L).
Then for $\lambda > 0$

$$B^2A^8B^2 - 6\lambda^5B^2 + 5\lambda^6 = \begin{pmatrix} 320 - 24\lambda^5 + 5\lambda^6 & 0 \\ 0 & 5\lambda^6 \end{pmatrix}.$$

We get $320 - 24\lambda^5 + 5\lambda^6 > 0$ for all $\lambda > 0$
So $B^2A^8B^2 - 6\lambda^5B^2 + 5\lambda^6 > 0$ for all $\lambda > 0$
Hence $T_{A,B}$ is absolute square 2-paranormal operator by (L).
Hence every absolute square 2-paranormal need not be T^2 absolute 2-paranormal operator.

Theorem 3.13. Let T be a absolute square k-paranormal operator for $k > 0$. Then for every unit vector $x \in H$, $F(l)$=$\||T^2|^l T^2 x\|^{\frac{1}{(l+1)}}$ is increasing for $l \geq k > 0$ and the inequality $F(l) \geq \|Tx\|^2$ holds for $l \geq k > 0$. (ie) T is absolute square l-paranormal operator.

Proof. Assume that $T \in L(H)$ is absolute square k-paranormal for $k > 0$.
$\|Tx\|^{2(k+1)} \leq \||T^2|^k T^2 x\|$ for every $x \in H$ ————————-(M)
(M) holds if and only if $F(k)=\||T^2|^k T^2 x\|^{\frac{1}{(k+1)}} \geq \|Tx\|^2$ for every unit vector $x \in H$.
Then for any unit vector $x \in H$ and any l such that $l \geq k > 0$, we have

$$\begin{aligned}
F(l) &= \||T^2|^l T^2 x\|^{\frac{1}{(l+1)}} \\
&= (|T^2|^{2l} T^2 x, T^2 x)^{\frac{1}{2(l+1)}} \\
&\geq \{(|T^2|^{2k} T^2 x, T^2 x)^{\frac{l}{k}} \|T^2 x\|^{2(1-\frac{l}{k})}\}^{\frac{1}{2(l+1)}} \text{ [By (ii) of Theorem 2.1]} \\
&= [\||T^2|^k T^2 x)\|^{\frac{2l}{k}} \|T^2 x\|^{2(1-\frac{l}{k})}]^{\frac{1}{2(l+1)}} \\
&\geq [\|Tx\|^{\frac{4l(k+1)}{k}} \|T^2 x\|^{2(1-\frac{l}{k})}]^{\frac{1}{2(l+1)}} \text{ By equation (M)]} \\
&\geq [\|Tx\|^{\frac{4l(k+1)}{k}} (\|Tx\|^2 \|x\|^{1-2})^{2(1-\frac{l}{k})}]^{\frac{1}{2(l+1)}} \text{ [By (iii) of Theorem 2.1]} \\
&= \|Tx\|^2
\end{aligned}$$

Hence

$$F(l) = \||T^2|^l T^2 x\|^{\frac{1}{(l+1)}} \geq \|Tx\|^2$$

for every unit vector $x \in H$ and $l \geq k$, so that T is absolute square l-paranormal for $l \geq k > 0$.

Next we show that $F(l)$ is increasing for $l \geq k > 0$. For every unit vector $x \in H$ and any m and l such that for $m \geq l \geq k > 0$.

$$\begin{aligned}
F(m) &= \||T^2|^m T^2 x\|^{\frac{1}{(m+1)}} \\
&= (|T^2|^{2m} T^2 x, T^2 x)^{\frac{1}{2(m+1)}} \\
&\geq \{(|T^2|^{2l} T^2 x, T^2 x)^{\frac{m}{l}} \|T^2 x\|^{2(1-\frac{m}{l})}\}^{\frac{1}{2(m+1)}} \text{ [By (ii) of Theorem 2.1]} \\
&= [\||T^2|^l T^2 x)\|^{\frac{2m}{l}} \|T^2 x\|^{2(1-\frac{m}{l})}]^{\frac{1}{2(m+1)}} \\
&\geq [\||T^2|^l T^2 x)\|^{\frac{2m}{l}} (\|Tx\|^2 \|x\|^{1-2})^{2(1-\frac{m}{l})}]^{\frac{1}{2(m+1)}} \text{ [By (iii) of Theorem 2.1]} \\
&= [\||T^2|^l T^2 x)\|^{\frac{2m}{l}} \|Tx\|^{4(1-\frac{m}{l})}]^{\frac{1}{2(m+1)}} \text{ for every unit vector } x \in H. \\
&= [\||T^2|^l T^2 x)\|^{\frac{2m}{l}} \||T^2|^l T^2 x)\|]^{\frac{2(1-\frac{m}{l})}{(l+1)}} \text{ [By equation (M)].} \\
&= \||T^2|^l T^2 x\|^{\frac{1}{(l+1)}} \\
&= F(l).
\end{aligned}$$

Hence $F(l)$ is an increasing function for $l \geq k > 0$. ■

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