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Weyl Spectrum of a Composition operators on $l^2(N)$

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Abstract

In this paper we obtain the Weyl spectrum of a composition operator on $l^2(N)$. A criterion for a composition operator to be a Weyl operator is established.

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1. Introduction and Preliminaries

Let N be the set of all positive integers and let $l^2(N)$ denote the Hilbert space of square summable sequences of complex numbers. Suppose $T : N \to N$ is a mapping from N into itself. Then we can define a composition transformation from $l^2(N)$ into the space of all complex valued sequences by $C_T f = f \circ T$ for every $f \in l^2(N)$. If C_T is bounded and range of C_T is contained in $l^2(N)$, then we shall call C_T a composition operator induced by T. It is proved in Theorem 2.1. of Singh and Komal [4] that C_T is bounded if and only if the sequence $\{f_0(n) : n \in N\}$ is a bounded sequence, where $f_0(n) = \#\big(T^{-1}(\{n\})\big)$, where $\#(E)$ denotes the cardinality of the set E. Two integers m and n in N are said to be in the same orbit of T if there exist two integers p and q such that $T^p(m) = T^q(n)$. For $n \in N$, the orbit of n with respect to T is defined as $O_T(n) = \{m \in N : T^p(m) = T^q(n) \text{ for some } p, q \in N\}$. The length of orbit $O_T(n)$ is $\#O_T(n)$.

A mapping $T : N \to N$ is said to be a periodic mapping at $n \in N$ if there exists a positive integer $m \in N$ such that $T^m(n) = n$. The smallest integer m is called the period of T at n. The set of all periods of T is denoted by $P(T)$. The set $\{e_n : n \in N\}$ is orthonormal basis of $l^2(N)$, where $e_n(m) = \{\delta_{nm}\}_{m=1}^{\infty}$, the Kronecker delta. For $E \subset N$, $l^2(E) = \{f \in l^2(N) : f(n) = 0 \ \forall n \notin E\}$. The restriction of operators $A \in B(l^2(N))$ to $l^2(E)$ is denoted by $A|l^2(E)$.

The set of all bounded linear operators on Hilbert space H into itself is denoted by $B(H)$. A bounded linear operator on H is called Fredhlom if $ker A$ and $ker A^*$ are finite dimensional and range of A is closed, where $ker A = \{x \in H : Ax = 0\}$. Let $A \in B(H)$. Then A is said to be Weyl operator if A is Ferdhlom operator and $dim\ ker A = dim\ ker A^*$. The Weyl spectrum of A is defined as $W(A) = \{\lambda \in \mathbb{C} : A - \lambda I$ is a Weyl operator$\}$ The point spectrum of A is the set of all eigenvalues of A and is denoted by $\Pi_0(A)$.

The spectrum and Weyl spectrum of a composition operator are not completely determined as yet. Some partial results on spectrum are obtained in Macluer[1],Panayappan[3] Takagi [7], Ridge [4],Singh[6]. In this paper we study Weyl spectrum of a composition operator on $l^2(N)$.

2. Weyl Composition operators

A necessary and sufficient condition for a composition operator to be a Weyl operator is investigated in this section.

Theorem 2.1. Let $C_T \in B(l^2(N))$. Then C_T is a Weyl operator if and only if

$$\#(N|T(N)) = \sum_{n \in T(N)} (f_0(n) - 1) < \infty.$$

Proof. Suppose C_T is a Weyl operator. Since $C_T e_n = \chi_{T^{-1}(\{n\})}$, where χ_E denotes the characteristic function of the set E. Hence $C_T e_n = 0$ if and only if $T^{-1}(\{n\}) = \emptyset$ or that $C_T e_n = 0$ if and only if $n \in N|T(N)$. This shows that $ker C_T = span\{e_n : n \in N|T(N)\} = l^2(N|T(N))$.

Next we prove that $dim\ ker C_T^* = \sum_{n \in T(N)} (f_0(n) - 1)$. Let $E = \{n \in N : f_0(n) \geq 2\}$. If E is an empty set, then T is injection. Therefore C_T is surjective in view of proof Theorem 2.2 of Singh and Komal [4]. Now $dim\ ker C_T^* = dim\ (ran C_T)^{\perp} = \{0\} = \sum_{n \in T(N)} (f_0(n) - 1)$. Let $g \in ran C_T$. Then we can find $f \in l^2(N)$ such that $g = C_T f$. Clearly, $g = C_T\big(\sum_{n=1}^{\infty} f_n e_n\big) = \sum_{n=1}^{\infty} f_n C_T e_n = \sum_{n=1}^{\infty} f_n \chi_{T^{-1}(\{n\})}$. This proves that $ran C_T = span\{\chi_{T^{-1}(\{n\})} : n \in N\}$. For $n_0 \in E$, $f_0(n_0) = k \geq 2$ or $T^{-1}(\{n_0\}) = \{n_1, n_2, \ldots, n_k\}$. The space $F = span\{e_{n_1}, e_{n_2}, \ldots, e_{n_k}\}$ is k dimensional and $\chi_{T^{-1}(\{n_0\})} = e_{n_1} + e_{n_2} + \cdots + e_{n_k}$ belongs to F. By Gram Schmidt process, we can find $k-1$ vectors in F say $g_1, g_2, \ldots, g_{k-1}$ which are pairwise orthogonal and also orthogonal to $\chi_{T^{-1}(\{n\})}$. Since supp$g_m \subset T^{-1}(\{n_0\})$ for $m = 1, \ldots, k-1$, so $\langle g_m, \chi_{T^{-1}(\{n\})}\rangle = 0$ for every $p \in N$. This proves that g_m is orthogonal to $ranC_T$. Hence $g_m \in (ranC_T)^{\perp} = kerC_T^*$ for $m = 1, \ldots, k-1$. This yields that $dim\ (kerC_T^*) = \sum_{n\in E}(f_0(n)-1) = \sum_{n\in T(N)}(f_0(n)-1)$. Since C_T is Fredholm, both E and $N|T(N)$ are finite sets. By hypothesis $dim\ kerC_T = dim\ kerC_T^* < \infty$ we can conclude that

$$\#(N|T(N)) = \sum_{n\in E}(f_0(n)-1) = \sum_{n\in T(N)}(f_0(n)-1) < \infty. \tag{2.1}$$

Conversely, suppose that the conditions of the theorem are satisfied. Then from the equation (2.1), $dim\ kerC_T = dim\ kerC_T^* < \infty$. Also $ranC_T$ is closed. Thus C_T is Fredhlom. Hence C_T is a Weyl operator. ■

Example 2.2. Suppose $k \in N$ and $k \geq 2$. Let $T : N \to N$ be defined by

$$T(n) = \begin{cases} k, & \text{for } n \leq k \\ n, & \text{for } n > k \end{cases}$$

Then C_T is Fredhlom. Also $dim\ kerC_T = dim(l^2(N|T(N))) = k-1$ and $dim\ kerC_T^* = \sum_{n\in T(N)}(f_0(n)-1) = k-1$. Hence C_T is a Weyl Operator.

Example 2.3. Let $T : N \to N$ be defined by $T(n) = n+2$. Then $C_T \in B(l^2(N))$ is Fredholm and $dim\ kerC_T = 2$. But $dim\ kerC_T^* = 0$. Therefore C_T is not a Weyl Operator.

3. Weyl spectrum of a Composition operator

In this section we shall compute Weyl spectrum of an invertible composition operator.

Theorem 3.1. Let $T : N \to N$ be invertible and T has at most finite number of orbits of finite length. Then

$$W(C_T) = \begin{cases} \bigcup_{n\in P(T)} \{\lambda \in \mathbb{C} : \lambda^n = 1\}, & \text{if } P(T) \neq \emptyset, \\ \{0\}, & \text{if } P(T) = \emptyset. \end{cases}$$

Proof. Let $n_0 \in N$. Then either T is periodic at n_0 or T is antiperiodic at n_0. We first suppose that T is periodic at n_0. Then we can find a smallest integer $m \in N$ such that $T^m(n_0) = n_0$. Clearly $T^m|O_T(n_0) = I|O_T(n_0)$ and $C_T^m|l^2(O_T(n_0)) = C_{T^m}|l^2_{O_T(n_0)} = I|l^2_{O_T(n_0)}$, where I is the identity operator.
Hence by spectral mapping theorem $\Pi_0(C_T|l^2_{O_T(n_0)}) = \{\lambda \in \mathbb{C} : \lambda^m = 1\}$. Let ψ be the

inverse of ϕ. Then $C_T^* = C_{T^{-1}} = C_\psi$ by a Theorem 2.1 of Gupta and Komal [2]. As proved earlier, we have $\Pi_0(C_T^*|l^2_{O_T(n_0)}) = \{\lambda \in \mathbb{C} : \lambda^m = 1\}$

Suppose $\lambda^m = 1$. Let $f_\lambda = \sum_{i=0}^{m-1} \lambda^i e_{n_i}$. Then it can be seen that $C_T f_\lambda = \lambda f_\lambda$. We show that $f \in ker(C_T - \lambda I)|l^2(O_T(n_0))$ if and only if $f = \alpha f_\lambda$ for some $\alpha \in C$. If $f = \alpha f_\lambda$, then $(C_T - \lambda I)f = 0$.
Next suppose that $0 \neq f \in ker\big((C_T - \lambda)|l^2(O_T(n_0))\big)$. Then $C_T f = \lambda f$. This implies that $f(T(n_0)) = \lambda f(n_0)$ or $f(n_1) = \lambda f(n_0)$. Again $f(T(n_1)) = \lambda f(n_1)$ implies that $f(n_2) = \lambda^2 f(n_0)$. Finally, $f(n_m) = \lambda^m f(n_0)$ or $f(n_0) = \lambda^m f(n_0)$. Hence $\lambda^m = 1$ as $f \neq 0$. This proves that $f = f_{n_0} \sum_{i=0}^{m-1} \lambda^i e_{n_i}$. Thus we have proved that for $\lambda^m = 1$ $ker(C_T - \lambda I)|l^2(O_T(n_0)) = span\ \{f_\lambda\}$.
In a similar manner we can prove that $ker(C_T^* - \lambda I)|l^2(O_T(n_0)) = span\ \{f_\lambda\}$ for $\lambda^m = 1$. This proves that $dim\ (ker(C_T - \lambda I)|l^2(O_T(n_0)) = 1 = dim\ (ker(C_T^* - \lambda I)|l^2(O_T(n_0))$. By the hypothesis T has only finite number of orbits of length $l < \infty$. Let $O_T(p_1), ..., O_T(p_k)$ be finite number of orbits of length $l < \infty$. By using above arguments we can conclude that $dim\big(\ (ker(C_T - \lambda I)|l^2(\bigcup_{i=1}^{k} O_T(p_i))\big) = k = dim\big(\ (ker(C_T^* - \lambda I)|l^2(\bigcup_{i=1}^{k} O_T(p_i))\big)$, where $\lambda^l = 1$.
Next if T is antiperiodic at $n_0 \in N$. Write $T^k(n_0) = n_k$ and $T^{k^{-1}}(n_0) = n_{-k}$. If $0 \neq f \in ker(C_T - \lambda I)$, then

$$f(n_k) = f(T^k(n_0)) = \lambda^k f_{n_0}.$$

Also $f(T^k(n_{-k})) = \lambda^k f_{n_{-k}}$ or $f(n_0) = \lambda^k f(n_{-k})$ so that $f(n_{-k}) = \frac{1}{\lambda^k} f(n_0)$. Now $f \in l^2(N)$. Therefore, $\infty > \sum_{n=1}^{\infty} |f(n)|^2 \geq \sum_{k=-\infty}^{\infty} |f(n_k)|^2$. This is possible if $f(n_0) = 0$, which is a contradiction. Hence $ker(C_T - \lambda I)|l^2(O_T(n_0)) = \{0\}$. Similarly, we can prove that $ker(C_T^* - \lambda I)|l^2(O_T(n_0)) = \{0\}$. Thus

$$dim\ (ker(C_T - \lambda I)|l^2(O_T(n_0)) = 0 = dim\ (ker(C_T^* - \lambda I)|l^2(O_T(n_0)).$$

Thus we have proved that for any $m \in N$, $\lambda^m = 1$, if there are k_m orbits of same length m, then $dim\ (ker(C_T - \lambda I) = k_m = dim\ (ker(C_T^* - \lambda I)$. It is easy to show that

$C_T - \lambda I$ has closed range. This proves that $C_T - \lambda I$ is a Weyl operator and hence

$$\begin{aligned} W(C_T) &= \sum_{m \in P(T)} \oplus\, W(C_T | l^2(E_m)) \cup \{0\} \\ &= \bigcup_{m \in P(T)} \{\lambda \in \mathbb{C} : \lambda^m = 1\} \cup \{0\}, \end{aligned}$$

where $E_m = \bigcup_{n \in N} \{O_T(n) : \# O_T(n) = m\}$ This completes the proof of the theorem. ■

Example 3.2. Let $T : N \to N$ be defined by

$$T(n) = \begin{cases} 2, & \text{if } n = 1 \\ 1, & \text{if } n = 2 \\ 3, & \text{if } n = 4 \\ n+2, & \text{if } n \text{ is odd } \geq 3 \\ n-2, & \text{if } n \text{ is even } \geq 6 \end{cases}$$

Then T has only one orbit of length 2 namely $O_T(1) = \{1, 2\}$. T is antiperiodic elsewhere. Hence $W(C_T) = \{1, -1, 0\}$.

Example 3.3. Let $T : N \to N$ be defined by

$$T(n) = \begin{cases} n+1, & \text{if } n \text{ is odd integer} \\ n-1, & \text{for } n \text{ is an even integer} \end{cases}$$

Then T is invertible and has infinitely many orbits of length two.
Let $f^{2n-1} = e_{2n-1} + \lambda e_{2n}$, where $\lambda^2 = 1$. Then $C_T f^{2n-1} - \lambda f^{2n-1} = \lambda f^{2n-1} - \lambda f^{2n-1} = 0$ for $n = 1, 2, \ldots$. But $\{f^{2n-1} : n \in N\}$ is an orthogonal family. Hence $dim\ (ker(C_T - \lambda I) = \infty$ Therefore $C_T - \lambda I$ is not Weyl Operator.

Competing interests
The authors declare that they have no conflict of interests related to the publication of this manuscript.

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