On the Forcing Vertex Steiner Number of A Graph

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Abstract
Let \( x \) be a vertex of \( G \). For a minimum \( x \)-Steiner set \( W \) of \( G \), a subset \( T \subseteq W \) is called a forcing subset for \( W \), if \( W \) is the unique minimum \( x \)-Steiner set containing \( T \). A forcing subset for \( W \) of minimum cardinality is a minimum forcing subset of \( W \). The forcing \( x \)-Steiner number of \( W \), denoted by \( f_{sx}(W) \), is the cardinality of a minimum forcing subset of \( W \). The forcing \( x \)-Steiner number of \( G \), denoted by \( f_{sx}(G) \), is \( f_{sx}(G) = \min f_{sx}(W) \), where the minimum is taken over all minimum \( x \)-Steiner sets \( W \) in \( G \). Some general properties satisfied by these concepts are studied. The forcing vertex Steiner number of some standard graphs are obtained. For every pair \( a, b \) of integers with \( 0 \leq a \leq b \), there exists a connected graph \( G \) such that \( f_s(G) = a \) and \( f_{sx}(G) = b \) for some vertex \( x \) in \( G \).

Keywords: Steiner distance, Steiner number, forcing Steiner number, vertex Steiner number, forcing vertex Steiner number.

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1. INTRODUCTION
By a graph \( G = (V, E) \), we mean a finite undirected connected graph without loops or multiple edges. The order and size of \( G \) are denoted by \( p \) and \( q \) respectively. The distance \( d(u, v) \) between two vertices \( u \) and \( v \) in a connected graph \( G \) is the
length of a shortest $u - v$ path in $G$. An $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic. It is known that the distance is a metric on the vertex set of $G$. For a vertex $v$ of $G$, the eccentricity $e(v)$ is the distance between $v$ and a vertex farthest from $v$. The minimum eccentricity among the vertices of $G$ is the radius, and denoted by $radG$ and the maximum eccentricity is its diameter, and denoted by $diamG$ of $G$. For basic graph theoretic terminology, we refer to Harary [2]. For a nonempty set $W$ of vertices in a connected graph $G$, the Steiner distance $d(W)$ of $W$ is the minimum size of a connected subgraph of $G$ containing $W$. Necessarily, each such subgraph is a tree and is called a Steiner tree with respect to $W$ or a Steiner-W-tree. It is to be noted that $d(W) = d(u, v)$ when $W = \{u, v\}$. If $v$ is an end vertex of a Steiner-W-tree, then $v \in W$. Also, if $< W >$ is connected, then any Steiner-W-tree contains the elements of $W$ only. The Steiner distance of a graph is introduced in [6]. The set of all vertices of $G$ that lie on some Steiner-W-tree is denoted by $S(W)$. If $S(W) = V$, then $W$ is called a Steiner set of $G$. A Steiner set of minimum cardinality is a minimum Steiner set or simply a s-set of $G$ and this cardinality is the Steiner numbers of $G$. If $W$ is a Steiner set of $G$ and $v$ a cut vertex of $G$, then $v$ lies in every Steiner-W-tree of $G$ and so $W \cup \{v\}$ is also a Steiner set of $G$. The Steiner number of a graph was introduced in [7] and further studied in [3,4,8,9,10,11,12,13]. Let $G$ be a connected graph and $W$ a minimum Steiner set of $G$. A subset $T \subseteq W$ is called a forcing subset for $W$, if $W$ is the unique minimum Steiner set containing $T$. A forcing subset for $W$ of minimum cardinality is a minimum forcing subset of $W$. The forcing Steiner number of $W$, denoted by $f_s(W)$, is the cardinality of a minimum forcing subset of $W$. The forcing Steiner number of $G$, denoted by $f_s(G)$, is $f_s(G) = \min f_s(W)$, where the minimum is taken over all minimum Steiner sets $W$ in $G$. A vertex $v$ is called a simplicial vertex of a graph $G$ if the subgraph induced by its neighbors is complete. Let $x$ be a vertex of a connected graph $G$ and $W \subset V(G)$ such that $x \notin W$. Then $W$ is called an $x$-Steiner set of $G$ if every vertex of $G$ lies on some Steiner-W-tree of $G$. The minimum cardinality of an $x$-Steiner set of $G$ is defined as the $x$-Steiner number of $G$ and denoted by $s_x(G)$. Any $x$-Steiner set of cardinality $s_x(G)$ is called an $s_x$-set of $G$.

Throughout the following $G$ denotes a connected graph.

The following theorems are used in the sequel.

**Theorem 1.1.** [7] Each simplicial vertex of a graph $G$ belongs to every Steiner set of $G$. In particular, each end-vertex of $G$ belongs to every Steiner set of $G$.

**Theorem 1.2.** [14] Let $G$ be a connected graph and $W$ be the set of all Steiner vertices of $G$. Then $f_s(G) \leq s(G) - |W|$.
Theorem 1.3.[14] For a complete graph $G = K_p (p \geq 2)$ or a non-trivial tree $G = T$, $f_s(G) = 0$.

Theorem 1.4.[11] Every simplicial vertex of $G$ other than the vertex $x$ (whether $x$ is extreme or not) belongs to every $x$-Steiner set for any vertex in $G$.

Theorem 1.5.[11] For any vertex $x$ in $G$, $s(G) \leq s_x(G) + 1$.

Theorem 1.6.[12] Let $G$ be a connected graph, $x$ a vertex of $G$ and $W$ the set of all $x$-Steiner vertices of $G$. Then $f_{sx}(G) \leq s_x(G) - |W|$.

Theorem 1.7.[12] For a complete graph $G = K_p (p \geq 2)$ or a non-trivial tree $G$, $f_{sx}(G) = 0$ for any vertex $x$ in $G$.

Theorem 1.8.[12] Let $G$ be a connected graph and $x$ a cut vertex of $G$. Then $f_s(G) = f_{sx}(G)$.

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Definition 2.1. Let $x$ be a vertex of $G$. For a minimum $x$-Steiner set $W$ of $G$, a subset $T \subseteq W$ is called a forcing subset for $W$, if $W$ is the unique minimum $x$-Steiner set containing $T$. A forcing subset for $W$ of minimum cardinality is a minimum forcing subset of $W$. The forcing $x$-Steiner number of $W$, denoted by $f_{sx}(W)$, is the cardinality of a minimum forcing subset of $W$. The forcing $x$-Steiner number of $G$, denoted by $f_{sx}(G)$, is $f_{sx}(G) = \min \{ f_{sx}(W) \}$, where the minimum is taken over all minimum $x$-Steiner sets $W$ in $G$.

Remark 2.2. From Theorem 1.5, there is a relationship between Steiner number and the vertex Steiner number. The following example shows there is no relationship between the forcing Steiner number and the forcing vertex Steiner number of a graph.

Example 2.3. For the graph $G$ given in Figure 2.1, $W_1 = \{v_1, v_2, v_5\}, W_2 = \{v_1, v_3, v_5\}$ and $W_3 = \{v_2, v_5, v_6\}$ are the $s$-sets of $G$ such that $f_s(W_1) = 2, f_s(W_2) = f_s(W_3) = 1$ so that $f_s(G) = 1$. For the vertex $x = v_3, W = \{v_1, v_5\}$ is the unique $s_x$-set of $G$ so that $f_{sx}(G) = 0$. Therefore $f_{sx}(G) < f_s(G)$.
Also for the graph $G$ given in Figure 2.2, $S_1 = \{z, z, u_1\}, S_2 = \{z, z, v_1\}$ are the $s$- sets of $G$ such that $f_s(S_1) = 1, f_s(S_2) = 1$, so that $f_s(G) = 1$. For the vertex $x = s, W_1 = \{z, z, u_1, w_1, w_2\}, W_2 = \{z, z, u_1, y_1, y_2\}, W_3 = \{z, z, v_1, w_1, w_2\}, W_4 = \{z, z, v_1, y_1, y_2\}, W_5 = \{z, z, u_1, w_1, y_2\}, W_6 = \{z, z, u_1, w_2, y_1\}, W_7 = \{z, z, v_1, y_1, w_2\}, W_8 = \{z, z, v_1, w_1, y_2\}$ are the $s_x$- sets of $G$ such that $f_{sx}(W_1) = f_{sx}(W_2) = f_{sx}(W_3) = f_{sx}(W_4) = f_{sx}(W_5) = f_{sx}(W_6) = f_{sx}(W_7) = f_{sx}(W_8) = 5$ so that $f_{sx}(G) = 5$. Therefore $f_s(G) < f_{sx}(G)$.

So we have the following realization result.

**Theorem 2.4.** For every pair $a, b$ of integers with $0 \leq a \leq b$, there exists a connected graph $G$ such that $f_s(G) = a$ and $f_{sx}(G) = b$ for some vertex $x$ in $G$.

**Proof. Case 1.** $a = b$. If $a = 0, b = 0$, let $G = K_a$. Then by Theorems 1.3 and 1.7, $f_s(G) = 0 = a$ and $f_{sx}(G) = 0 = b$. Now, assume that $a \geq 1$. Let $P: t, y, z$ be a path on three vertices and $P_i: u_i, v_i (1 \leq i \leq a)$ be a copy of path on two
vertices. Let $H$ be the graph obtained from $P$ and $P_i (1 \leq i \leq a)$ by joining the vertices $t$ and $u_i$ and the vertices $z$ and $v_i (1 \leq i \leq a)$. Now, let $G$ be the graph in Figure 2.3 obtained from $H$ by adding 2 edges $tz_1$ and $zz_2$. Let $Z = \{z_1, z_2\}$ and $H_i = \{u_i, v_i\} (1 \leq i \leq a)$. Let $x = t$.

First, we show that $s(G) = a + 2$. Since the vertices $u_i, v_i$ do not lie on any Steiner-$Z$ tree of $G$, it is clear that $Z$ is not a Steiner set of $G$. We observe that every $s$-set of $G$ must contain exactly one vertex from each $H_i = \{u_i, v_i\} (1 \leq i \leq a)$. Thus, $s(G) \geq a + 2$. On the other hand, since the set $W = Z \cup \{v_1, v_2, \ldots, v_a\}$ is a Steiner set of $G$, it follows that $s(G) \leq |W| = a + 2$. Hence $s(G) = a + 2$.

Next, we show $f_s(G) = a$. By Theorem 1.1, every Steiner set of $G$ contains $Z$ and so it follows from Theorem 1.2 that $f_s(G) \leq s(G) - |Z| = a$. Now, since $s(G) = a + 2$ and every $s$-set of $G$ contains $Z$, it is easily seen that every $s$-set $S$ is of the form $Z \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in H_i (1 \leq i \leq a)$. Let $T$ be any proper subset of $S$ with $|T| < a$. Then there is a vertex $c_j (1 \leq j \leq a)$ such that $c_j \not\in T$. Let $d_j$ be a vertex of $H_j$ distinct from $c_j$. Then $S_1 = (S - \{c_j\}) \cup \{d_j\}$ is a $s$-set properly containing $T$. Thus $S$ is not the unique $s$-set containing $T$ and so $T$ is not a forcing subset of $S$. This is true for all $s$-sets of $G$ and so $f_s(G) = a$.

Since $t$ is a cut vertex of $G$, by Theorem 1.8, $f_{sx}(G) = b$.

Case 2. $1 < a < b$.

Let $P: u, v, w$ be a path on three vertices and let $P_i: u_i, v_i (1 \leq i \leq a)$ be a copy of path on two vertices. Let $H$ be the graph obtained from $P$ and
First, show that $s(G) = c$. Let $S$ be any Steiner set of $G$. Then by Theorem 1.1, $Z \subseteq S$. It is clear that $Z$ is not a Steiner set of $G$. We observe that every Steiner set of $G$ must contain exactly one vertex from each $H_i (1 \leq i \leq a)$ and so $s(G) \geq c - a + a = c$. On the other hand, since the set $S_1 = Z \cup \{u_1, u_2, \ldots, u_a\}$ is a Steiner set of $G$, it follows that $s(G) \leq |S_1| = c$. Thus $s(G) = c$.

Next, we show that $f_s(G) = a$. Since every $s$-set of $G$ contains $Z$, it follows from Theorem 1.2 that $f_s(G) \leq s(G) - |Z| = c - (c - a) = a$. Now, since $s(G) = c$ and every $s$-set of $G$ contains $Z$, it is easily seen that every $s$-set $S$ of $G$ is of the form $Z \cup \{c_1, c_2, \ldots, c_a\}$ where $c_i \in H_i (1 \leq i \leq a)$. Let $T$ be any proper subset of $S$ with $|T| < a$. Then there is a vertex $x \in S$ such that $x \notin T$. Suppose that $x = c_j$, let $e_j$ be a vertex of $H_j$ distinct from $c_j$. Then $S_2 = (S - \{c_j\}) \cup \{e_j\}$ is a $s$-set properly containing $T$. Thus $S$ is not the unique $s$-set containing $T$ so that $T$ is not a forcing subset of $S$. This is true for all $s$-sets of $G$ and so it follows that $f_s(G) = a$.

Now, we show that $s_x(G) = c + b - a$. Let $S$ be any $x$-Steiner set of $G$. Then by Theorem 1.4, $Z \subseteq S$. It is clear that $Z$ is not an $x$-Steiner set of $G$. We observe that every $s_x$-set of $G$ must contain exactly one vertex from each $H_i (1 \leq i \leq a)$ and exactly one vertex from each $Q_j (1 \leq j \leq b - a)$. Thus $s_x(G) \geq c + b - a$. On the other hand, $S_3 = Z \cup \{u_1, u_2, \ldots, u_a\} \cup \{y_1, y_2, \ldots, y_{b-a}\}$ is an $x$-Steiner set of $G$ and so $s_x(G) = c + b - a$.

Next, we show that $f_{sx}(G) = b$. Since every $s_x$-set of $G$ contains $Z$, it follows from Theorem 1.6 that $f_{sx}(G) \leq s_x(G) - |Z| = (c + b - a) - (c - a) = b$. Now, since $s_x(G) = c + b - a$ and every $s_x$-set of $G$ contains $Z$, it is easily seen that every $s_x$-set $S$ of $G$ is of the form $Z \cup \{c_1, c_2, \ldots, c_a\} \cup \{d_1, d_2, \ldots, d_{b-a}\}$, where $c_i \in H_i (1 \leq i \leq a)$ and $d_j \in Q_j (1 \leq j \leq b - a)$. Let $T$ be any proper subset of $S$ with $|T| < b$. Then there is a vertex $y \in S$ such that $y \notin T$. If $y = c_i$, then let $e_i$ be a vertex of $H_i$ distinct from $c_i$. Then $S' = (S - \{c_i\}) \cup \{e_i\}$ is a smaller $x$-forcing set than $S$.

Finally, we show that $s_{sx}(G) = c + b - a$. Let $S$ be any $sx$-Steiner set of $G$. Then by Theorem 1.5, $Z \subseteq S$. It is clear that $Z$ is not an $sx$-Steiner set of $G$. We observe that every $s_{sx}$-set of $G$ must contain exactly one vertex from each $H_i (1 \leq i \leq a)$ and exactly one vertex from each $Q_j (1 \leq j \leq b - a)$. Thus $s_{sx}(G) \geq c + b - a$. On the other hand, $S_4 = Z \cup \{u_1, u_2, \ldots, u_a\} \cup \{y_1, y_2, \ldots, y_{b-a}\}$ is an $sx$-Steiner set of $G$ and so $s_{sx}(G) = c + b - a$.
\{c_i\} \cup \{e_i\} is an s_x-set properly containing T. If y = d_j then let f_j be a vertex of Q_j distinct from d_j. Then S'' = (S - \{d_j\}) \cup \{f_j\} is an s_x-set properly containing T. Thus S is not the unique s_x-set containing T and so T is not a forcing subset of S. This is true for all s_x-sets of G and so f_{sx}(G) = b.

We leave the following problem as an open problem.

**Problem 2.5.** For every pair a, b of integers with 0 ≤ a ≤ b, does there exist a connected graph G such that f_{sx}(G) = a and f_s(G) = b for some vertex x in G?

**REFERENCES**


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