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Contractivity of entire functions of semigroup in the L^P– spaces

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Abstract

In this study, we show and prove that a maximizers of several variables belong to the space of all entire functions on the unitary space has a general characterization.

Key words: Entire functions**,** Semigroup, measurable functions.

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1. INTRODUCTION AND PRELIMINARIES

Classical Sobolev inequalities state that if a function f on $\mathbb{R}^n$ together with its first derivatives are in $L_p(\mathbb{R}^n)$ then f is in $L_q(\mathbb{R}^n)$ for $q \in [p, \left(\frac{1}{p} - \frac{1}{n}\right)^{-1}]$ if $0 < (\frac{1}{p} - \frac{1}{n})$, $q \in [p, \infty)$ if $(\frac{1}{p} - \frac{1}{n}) < 0$, and $q \in [p, \infty]$ if $(\frac{1}{p} - \frac{1}{n}) = 0$. The fundamental role that Sobolev inequalities have played in the study of differential operators, especially elliptic differential operators, is well known.The number operator N in the free boson field may be viewed as a second-order elliptic differential operator in infinitely many variables. It is important for the constructive quantum field theory to study the semigroup$\{e^{-tN}\}$ generated by the number operator N. Because of the difficulty due to the infiniteness of the dimension, one needs to obtain some estimate independent of the dimension.

When the real wave representation is used, i.e., the free boson field is given by (K, Γ, W, v) over a complex Hilbert space H, where$K = L_2(H_r, dg')$, and $H_r$is the real part of H, dg' is the weak Gauss distribution over H, with variance 1, let N_n be the restriction of the operator N to $L_2(\mathbb{R}^n, dg'_n)$, where $\mathbb{R}^n$ is identified with any n-dimensional subspace of Hr ; much of the progress made in the study of nonlinear fields depends on the contractivity of the operation e^{-tN_n} from $L_2(\mathbb{R}^n, dg'_n)$ to $L_{2e^{\varepsilon t}}(\mathbb{R}^n, dg'_n)$, where $\varepsilon > 0$ is a constant independent of t and the dimension n, $L_p(\mathbb{R}^n, dg'_n)$ denotes all functions on $\mathbb{R}^n$ such that

$$\|f\|_p^p = \int_{\mathbb{R}^n} |f(x)|^p dg'_n(x) < \infty,$$

and g'_n is the Gauss measure on $\mathbb{R}^n$. Specifically

$$dg'_n(x) = (2\pi)^{-n/2} \exp[-|x|^2/2]\, dx.$$

In the case of n = 1, N_1 is given by

$$(N_1 f)(x) = -f''(x) + xf'(x),$$

Which is a second-order differential operator. Let $H_j(x)$ denote the jth normalized Hermite polynomial corresponding to the unit Gauss measure dg'_1on $\mathbb{R}$; then H_j is a eigenfunction of N_1 corresponding to the eigenvalue j. In general $(e^{-tN_1}f)(x)$ is given by a integral equation.

$$\int_{\mathbb{R}} M(t, x, y) f(y)\, dy,$$

where M(t, x, y) is called the Mehler kernel. The hypercontractive theorem states that $e^{-tN_n}, t > 0$, is a contraction from $L_p(\mathbb{R}^n, dg'_n)$ to $L_q(\mathbb{R}^n, dg'_n)$ if and only if

$$q - 1 \le (p-1)e^{2t}.$$

Nelson first proved this Theorem using Fock space techniques in [5]. Gross then found its equivalence to his logarithmic Sobolev inequalities in [8]. Afterwards Beckner [1] , Brascamp and Lieb [6] showed the hypercontractive theorem to be a consequence of their sharp Young's inequalities. Epperson [7] generalized the result to operators $\Gamma(\alpha): H_j(x) \to \alpha^j H_j(x), |\alpha| \le 1, \alpha \in \mathbb{C}$, and showed that $\Gamma(\alpha)$ is a contraction from $L_p(\mathbb{R}, dg')$ to $L_q(\mathbb{R}, dg')$ if α satisfies the condition

$$(\mathrm{Im}\, \alpha z)^2 + (p-1)(\mathrm{Re}\, \alpha z)^2 \le (\mathrm{Im}\, z)^2 + (q-1)(\mathrm{Re}\, z)^2 \text{for all } z \in \mathbb{C}.$$

Z. Zhou[10] study the contractivity of the number operator in the complex wave representation for the free boson field. This representation has remarkable mathematical properties, although physically it is less readily interpreted than the other two representations. One of these, the Fock-Cook, or particle representation, diagonalizes the occupation numbers; the other, the renormalized Schrodinger functions integration, or real wave representation, can serve to dianonalizes the field variables at a fixed time. The complex wave representation diagonalizes the creation operators. And the number operator can be easily represented in the complex wave representation. If H is an n-dimensional complex Hilbert space, then K is unitarily equivalent to $HL_2(\mathbb{C}^n, dg_n)$, where $HL_2(\mathbb{C}^n, dg_n)$ is the space of all entire functions on $\mathbb{C}^n$ such that

$$\|f\|_2^2 = \int_{\mathbb{C}^n} |f(z)|^2 dg_n(z),$$

and $dg_n(z) = \pi^n e^{|z|^2} |dz|$. The number operator is given by

$$N: z_1^{j_1} \cdots z_n^{j_n} \to (j_1 + \cdots + j_n) z_1^{j_1} \cdots z_n^{j_n},$$

Where $j_1 \cdots j_n$ are nonnegative integers. The notation T(t), $t \geq 0$, and $HL_p(\mathbb{C}^n, dg_n)$ will be used throughout this section as the semigroup of operators $HL_p(\mathbb{C}^n, dg_n)$ generated by the number operator N, i.e.,

$$(T(t)f)(z) = f(e^{-t}z)$$

for f in the space $HL_p(\mathbb{C}^n, dg_n)$ of all entire functions such that

$$\|f\|_p^p = \int_{\mathbb{C}^n} |f(z)|^p dg'_n(z) < \infty.$$

We are interested in the embedding property of operators T(t), t > 0, from $HL_r(\mathbb{C}^n, dg_n)$ to $HL_p(\mathbb{C}^n, dg_n)$ and determine all maximizers. So, it is easy to see that $\sup\|T(t)f\|_p/\|f\|_r \geq 1$ for any r, $p \in [1, \infty)$ since $\|T(t)f\|_p/\|f\|_r = 1$ if $f \equiv 1$.

We assume $p \in [2, 2e^{2t}]$ and use $f_j(z), j = 0, 1, 2, \ldots,$ to denote entire functions in $\mathbb{C}$ given by

$$f_j(z) = \frac{z^j}{\sqrt{j!}}.$$

Thus, it can be easily checked that f_j is a unit vector in $HL_2(\mathbb{C}, dg_1(z))$. Furthermore, $\{f_j\}_0^\infty$ form an orthonormal basis for $HL_2(\mathbb{C}, dg_1(z))$.

Janson [9] derived the sharp inequality $\|T(t)f\|_{pe^{2t}} \leq \|f\|_p$ for any entire function $f \in HL_p(\mathbb{C}^n, dg_n)$ in 1983 from the contractivity of real wave representation in three lines. But Janson's method fails to determine the cases of equality. E. Carlen also proved Theorem 1.1 by using an identity [2] and the logarithmic Soboleve inequality [3] for entire functions.

Lemma 1.1: If T(t): $HL_2(\mathbb{C}^n, dg_n) \to HL_p(\mathbb{C}^n, dg_n)$ is a contraction for n = 1 then so is T(t) for any n. Furthermore, $f \in HL_2(\mathbb{C}^n, dg_n)$ such that $\|T(t)f\|_p / \|f\|_2 = 1$; i.e., f is a maximize for T(t): $HL_2(\mathbb{C}^n, dg_n) \to HL_p(\mathbb{C}^n, dg_n)$, if and only if $f(z_1 \dots, z_n) = \prod_{j=1}^n h_j(z_j)$, where each h_j is a maximizer for T(t): $HL_2(\mathbb{C}^1, dg_1) \to HL_p(\mathbb{C}^1, dg_1)$.

Proof. Let us prove this lemma by induction on dimension. If n =1, then there is nothing to prove. Assume that this is true for n = k. Then for n = k + 1, write any entire function $f \in HL_2(\mathbb{C}^n, dg_n)$ as $f(w, z) = \sum_0^\infty g_j(w) f_j(z)$,

where $w \in \mathbb{C}^k$ and $z \in \mathbb{C}$. Therefore for $p \in [2, 2e^{2t}]$,

$$\|T(t)f\|_p^p = \int_{\mathbb{C}^k} \left[\int_{\mathbb{C}} \left|\sum_j g_j(e^{-t}w)(f_j(e^{-1}z)\right|^p dg_1(z) dg_k(w) \quad (1)$$

$$\leq \int_{\mathbb{C}^k} \left[\sum_j \left|g_j(e^{-t}w)\right|^2\right]^{p/2} dg_k(w) \quad (2)$$

$$\leq \left(\sum_j \left[\int_{\mathbb{C}^k} \left|g_j(e^{-t}w)\right|^p dg_k(w)\right]^{2/p}\right)^{p/2} \quad (3)$$

$$\leq \left(\sum_j \int_{\mathbb{C}^k} \left|g_j(w)\right|^2 dg_k(w)\right)^{p/2} = \|f\|_2^p. \quad (4)$$

The inequality from (1) to (2) is the consequence of the hypothesis of the theorem; the inequality from (2) to (3) is simply the Holder's inequality; the inequality from (3) to (4) follows from the induction hypothesis. Also it is easy to see that if f is a maximize, namely $\|T(t)f\|_p = \|f\|_2$, then all inequalities from (1) to (4) must be equalities. From (2) = (3) it follows that $g_j = c_j g$ for $j = 0, 1, \ldots$, where c_j are complex constants and g is an entire function in $\mathbb{C}^k$. (3) = (4) implies that g must be a maximize for k-dimension. Hence

$$f(w, z) = g(w) \sum_j c_j f_j(z),$$

where g is a maximize for k-dimension. Finally (1) = (2) implies that for each fixed $w, f(w, \cdot)$ should be a maximize for 1-dimension, which concludes that

$$h(z) = \sum_j c_j f_j(z)$$

is a maximize for 1-dimension. Therefore, f (w, z) is the product of maximizers for 1-dimension.

If T(t) is a bounded operator from $HL_2(\mathbb{C}^k, dg_k)$ to $HL_p(\mathbb{C}^k, dg_k)$, we define

$$B_{2,p}(k) = \sup\left\{\frac{\|T(t)h\|_p}{\|h\|_2}; h \in HL_2(\mathbb{C}^k, dg_k), \|h\|_2 \neq 0\right\}.$$

It should be remarked that from the proof of Lemma 1.1, especially by checking those inequalities carefully, one can draw the following conclusions:

Corollary 1.1: If T(t) is a bounded mapping from $HL_2(\mathbb{C}^1, dg_1)$ to $HL_p(\mathbb{C}^1, dg_1)$ with the norm

$$B_{2,p}(1) = \sup\left\{\frac{\|T(t)h\|_p}{\|h\|_2}; h \in HL_2(\mathbb{C}^1, dg_1), \|h\|_2 \neq 0\right\},$$

then T(t) is also a bounded mapping from $HL_2(\mathbb{C}^k, dg_k)$ to $HL_p(\mathbb{C}^k, dg_k)$ with the norm

$$B_{2,p}(k) = \sup\left\{\frac{\|T(t)h\|_p}{\|h\|_2}; h \in HL_2(\mathbb{C}^k, dg_k), \|h\|_2 \neq 0\right\} = B_{2,p}^k(1).$$

Corollary 1.2: If $h_j \in HL_2(\mathbb{C}^1, dg_1)$, $j = 1, 2, \ldots, k$, are maximizers for $B_{2,p}(1)$, i.e., $B_{2,p}(1) = \frac{\|T(t)h_j\|_p}{\|h_j\|_2}$,

then$h_1(z_1)h_2(z_2) \ldots h_k(z_k)$ is a maximizer for $B_{2,p}(k)$. Conversely, any maximize f $(z_1, \ldots, z_k)$ for $B_{2,p}(k)$ is of the form

$$f(z_1, \ldots, z_k) = g_1(z_1) \ldots g_k(z_k), \qquad (5)$$

where each g_j is a maximize for $B_{2,p}(1)$, which says that any maximize for higher dimension is just the product of maximizers for 1-dimension.

Now we can characterize maximizers for $B_{2,p}(1)$.

Lemma 1.2: If T(t) is a bounded mapping from $HL_2(\mathbb{C}^1, dg_1)$ to $HL_p(\mathbb{C}^1, dg_1)$, and f is a maximizer for $B_{2,p}(1)$, i.e.,

$$\frac{\|T(t)f\|_p}{\|f\|_2} = \sup\left\{\frac{\|T(t)h\|_p}{\|h\|_2};\ h \in HL_2(\mathbb{C}^1, dg_1), \|h\|_2 \neq 0\right\} = B_{2,p}(1),$$

then f has the form

$$f(z) = ce^{az^2\ bz},$$

Where a, b, and c are complex constants, and $c \neq 0$.

Proof. Here we use the doubling technique of Lieb [4]. If f (z) is the maximizer for $B_{1,p}(1)$, then $F(z_1, z_2) = f(z_1)f(z_2)$ is a maximizer for $B_{2,p}(2)$ by Lemma 1.1. Note that the function

$$G(z_1, z_2) = f\left(\frac{z_1+z_2}{\sqrt{2}}\right) f\left(\frac{z_1-z_2}{\sqrt{2}}\right)$$

is a maximize for $B_{2,p}(2)$by virtue of the invariance of dg2. By Corollary 1.2,

$$f\left(\frac{z_1+z_2}{\sqrt{2}}\right) f\left(\frac{z_1-z_2}{\sqrt{2}}\right) = \alpha(z_1)\beta(z_2) \qquad (6)$$

whereα and β are two maximizers for $B_{2,p}(1)$.

Equation (6) implies that f must have the indicated form. For details, see [5].

It follows from (9) that T(t): $HL_2(\mathbb{C}^1, dg_1) \to HL_p(\mathbb{C}^1, dg_1)$ is not bounded if $q > 2e^{2t}$. The first question one can ask is whether T(t): $HL_2(\mathbb{C}^1, dg_1) \to HL_{2e^{2t}}(\mathbb{C}^1, dg_1)$ is bounded. The answer to this is positive. As a first step towards the proof of it, let us try to find the smallest $t_0 > 0$ such that $T(t_0)$ is a bounded mapping from $HL_2(\mathbb{C}^1, dg_1)$ to $HL_4(\mathbb{C}^1, dg_1)$. A necessary condition for t0 is $e^{2t_0} \geq 2$ from (9). Surprisingly it turns out that $e^{2t_0} \geq 2$ is also a sufficient condition. Actually even more surprising $T(t_0)$ is a contraction from $HL_2(\mathbb{C}^1, dg_1)$ to $HL_4(\mathbb{C}^1, dg_1)$ if $e^{2t_0} \geq 2$. This is where we first realized that T(t) might have the property that T(t): $HL_2(\mathbb{C}^n, dg_n) \to HL_p(\mathbb{C}^n, dg_n)$ is bounded if and only if T(t): $HL_2(\mathbb{C}^n, dg_n) \to HL_p(\mathbb{C}^n, dg_n)$ is a contraction , and the biggest possible p is $2e^{2t}$.

Lemma 1.3: Let $t_0 > 0$ such that $e^{2t_0} \geq 2$. Then $T(t_0)$ is a contraction mapping from $HL_2(\mathbb{C}^1, dg_1)$ to $HL_4(\mathbb{C}^1, dg_1)$.

Proof. It is enough to prove this lemma for $2e^{2t} = 2$ since $\|T(t)f\|_2 \leq \|f\|_2$ for any $t > 0$. For any function $f \in HL_2(\mathbb{C}^1, dg_1)$, let us express it by using the orthonormal basis $\{f_j\}_0^\infty$. That is ,

$$f(z) = \sum_0^\infty a_j f_j(z),$$

where a_j's are complex constants. It is easy to check that

$$\left\|f_j\right\|_4 = \left(\frac{(2j)!}{(j!)^2}\right)^{1/4}.$$

Therefore if $\|f\|_2^2 = \sum_0^\infty \left|a_j\right|^2 = 1$, noting the orthogonality of z^j and z^k if $j \neq k$,

$$\|T(t_0)f\|_4^4 = \int \sum_{m,n,j,k} e^{-(m+n+j+k)t_0} a_m a_n \overline{a_j a_k} f_m f_n \overline{f_j f_k} dg_1$$

$$= \int \sum_{l=0}^\infty \sum_{m+n=l} \sum_{j+k=l} e^{-2/t_0} a_m a_n \overline{a_j a_k} f_m f_n \overline{f_j f_k} dg_1$$

$$= \sum_{l=0}^\infty \left|\sum_{m+n=l} \left(\frac{1}{2}\right)^l a_m a_n \left(\frac{l!}{m!n!}\right)^{1/2}\right|^2$$

$$\leq \sum_{l=0}^\infty \left(\frac{1}{2}\right)^l \left(\sum_{m+n=l} \frac{l!}{m!n!}\right) \left(\sum_{m+n=l} |a_m|^2 |a_n|^2\right)$$

$$= \sum_{l=0}^\infty \left(\sum_{m+n=l} |a_m|^2 |a_n|^2\right) = \|f\|_2^4,$$

which completes the proof of the lemma.

From the proof of Lemma 1.3, one can see easily that $\|T(t_0)f\|_4 = \|f\|_2$ for $e^{2t_0} =$

2 if and only if $a_j = cb^j$ for some constants c and b for all j = 0, 1, 2, …, where a_j's are coefficients in the expansion $f(z) = \sum_0^\infty a_j f_j(z)$. That means $f(z) = ce^{bz}$, which is called the "coherent" state in physics. So, f is a maximize for $T(t_0)$: $HL_2(\mathbb{C}^n, dg_n) \to HL_4(\mathbb{C}^n, dg_n)$ and $e^{2t_0} = 2$ if and only if f is an exponential function. Furthermore, by induction it is easy to prove that for any positive integer j

$$\|T(jt_0)f\|_{2^{j+1}} = \|T(jt_0)f\|_{2e^{jt_0}} \leq \|f\|_2 \qquad (7)$$

Since $f^2(z)$ is entire if f is entire and $T(t)(f^2) = (T(t)f)^2$. The estimate in (7) is a part of Theorem 1.1. Actually theorem 1.1 is motivated by Lemma 1.3 and the estimate in (7).

It should be pointed out that the ordinary interpolation theorem does not apply here since $|f|$ if not entire if f is entire on $\mathbb{C}^n$. Even if there is a corresponding interpolation theorem, one could only expect that T(t) is a contraction from $HL_2(\mathbb{C}^n, dg_n)$ to $HL_{p(t)}(\mathbb{C}^n, dg_n)$ where $p(t) = 4t_0/(2t_0 - t)$ for $0 \leq t \leq t_0$. But it is easy to see that $4t_0/(2t_0 - t) < 2e^{2t}$ for $t \in (0, t_0)$. In fact $S(t) = (2t_0 - t)^{e^{2t}} - 2t_0$ satisfies $S(0) = S(t_0) = 0$ and $S''(t) < 0$ on $[0, t_0]$. Therefore, $S(t) > 0$ in $(0, t_0)$. For this operator T(t) we have

Lemma 1.4: $T(t): HL_2(\mathbb{C}^1, dg_1) \to HL_{2e^{2t}}(\mathbb{C}^1, dg_1)$ is bounded for all $t \in [0, \infty)$.

Proof. It is enough to show this lemma for $t \in [0, t_0]$, where t_0 is given in Lemma 1.3. In fact if the Lemma holds for $t \in [0, t_0]$, then for any $t \in [t_0, \infty)$, there exists an integer j such that $t = jt_0 + t_1$ and $t_1 \in [0, t_0]$. Therefore, if $\|f\|_2 \leq 1, T(t)f = T(t_1)T(jt_0)f$ and

$$\left\|[T(t)f]^{2j}\right\|_{2e^{2t_1}} = \left\|T(t_1)[T(jt_0)f]^{2j}\right\|_{2e^{2t_1}} \leq c\left\|[T(jt_0)f]^{2j}\right\|_2 \leq c$$

by (7), which implies that $\|T(t)f\|_{2^{j+1}e^{2t_1}} \leq c$, where c is a constant independent of f. Finally $2^{j+1}e^{2t_1} = 2e^t$ is number required.

Now let us consider the case in which $t \in [0, t_0]$. Recall that

$$\|T(t)f\|_{2e^{2t}}^{2e^{2t}}$$

$$= \frac{1}{\pi}\int |f(e^{-1}z)|^{2e^{2t}} e^{-|z|^2} dx\ dy$$

$$= \frac{e^{2t}}{\pi}\int |f(z)|^{2e^{2t}} e^{-|z|^2} dx\ dy$$

$$\leq c\left[\int |f(z)|^2 e^{-|z|^2} dx\, dy\right]^{\theta e^{2t}} \left[\int |f(z)|^4 e^{-2|z|^2} dx\, dy\right]^{(1-\theta)e^{2t}/2} \qquad (8)$$

by Holder's inequality, where the nonnegative number θ satisfies $1/2e^{2t} = \theta/2 + (1-\theta)/4$. Inequality (8) and Lemma 1.3 imply that $\|T(t)f\|_{2e^{2t}} \le c\|f\|_2$.

Lemma 1.5: $T(t): HL_2(\mathbb{C}^1, dg_1) \to HL_p(\mathbb{C}^1, dg_1)$ is compact and $B_{2,p}(1)$ is attained by a maximizer if $p \in [2, e^{2t})$.

The existence of a maximizer follows from the compactness of $T(t): HL_2(\mathbb{C}^1, dg_1) \to HL_p(\mathbb{C}^1, dg_1)$.

Theorem 1.1: For any $t \ge 0$ and $r \in [1, \infty)$, the operator $T(t)$: $HL_r(\mathbb{C}^n, dg_n) \to HL_p(\mathbb{C}^n, dg_n)$ is bounded if $p > re^{2t}$, and is a contraction if $p \in [1, re^{2t}]$, i.e.,

$$\sup\left\{\frac{\|T(t)f\|_p}{\|f\|_r} \,|\, \|f\|_r \ne 0\right\} = 1$$

for any $p \in [1, re^{2t}]$. Furthermore, in the case of $p \in [1, re^{2t}]$,

$$\frac{\|T(t)f\|_p}{\|f\|_2} = 1$$

If and only if f is a constant function. In the case of $p = re^{2t}$ and $t > 0$

$$\frac{\|T(t)f\|_p}{\|f\|_r} = 1$$

if and only if f is an exponential function, i.e., $f(z_1, \ldots, z_n) = \alpha e^{\beta_1 z_1 \ldots + \beta_n z_n}$, where $\alpha, \beta_1, \ldots, \beta_n$ are complex constants.

The unboundedness of $T(t)$: $HL_r(\mathbb{C}^n, dg_n) \to HL_p(\mathbb{C}^n, dg_n)$ for $p > re^{2t}$ can be proved easily. Consider a sequence of functions $\{h_j\}$ in $HL_r(\mathbb{C}^n, dg_n)$ given by

$$h_j(z_1, \ldots, z_n) = e^{jz_i}.$$

Then it is easy to see that

$$\frac{\|T(t)h_j\|_p}{\|h_j\|_r} = e^{j^2(pe^{-2t}-r)/4} \to \infty \qquad (9)$$

as $j \to \infty$ if $p > re^{2t}$.

Proof. For r = 2, $p \in [2, e^{2t}]$. As the first step towards the complete proof of Theorem 1.1, we first consider the case $T(t): HL_2(\mathbb{C}^1, dg_1) \to HL_p(\mathbb{C}^1, dg_1)$. For $p \in [2, e^{2t}]$. Let f be any maximize for $B_{2,p}(1)$ whose existence is guaranteed by

Lemma 1.5. From Lemma 1.2, $f(z) = ce^{az^2+bz}$. An easy evaluation of a Gaussian integral shows that

$$\|f\|_q = \frac{|c|}{(1-|aq|^2)^{1/(2q)}} \exp\left[\frac{qb_r^2}{4(1-|aq|)} + \frac{qb_i^2}{4(1-|aq|)}\right], \quad (10)$$

Where $b_r = \mathrm{Re}\left(be^{-\frac{i\theta}{2}}\right), b_i = \mathrm{Im}(be^{-i\theta/2}),\ \theta = \arg(a)$. Therefore

$$B_{2,p}(1) = \frac{\|T(t)f\|_p}{\|f\|_2} = \frac{(1-4|a|)^{1/4}}{(1-p^2e^{-4t}|a|^2)^{1/2p}} \exp(\alpha b_r^2 + \beta b_i^2), \quad (11)$$

where

$$\alpha = \frac{pe^{-2t}}{4(1-pe^{-2t}|a|^t)} - \frac{2}{4(1-2|a|)},\ \beta = \frac{pe^{-2t}}{4(1+pe^{-2t}|a|)} - \frac{2}{4(1+2|a|)}$$

it is easy to see that $\alpha < 0,\ \beta < 0$, and $\frac{(1-4|\alpha|)^{\frac{1}{4}}}{(1-p^2e^{-4t}|a|^2)^{\frac{1}{p}}} \leq 1$if $p \in [2, e^{2t})$. Hence (11) implies that $B_{2,p}(1) \leq 1$. On the other hand, $B_{2,p}(1) \geq 1$ since $\|T(t)f\|_p/\|f\|_2 = 1$ if $f \equiv 1$, which implies that $B_{2,p}(1)$ =1 and f is a maximize if and only if f (z) = ce^{az^2+bz} with $(1-4|a|)^{1/4}/(1-p^2e^{\ 4t}|a|^2)^{1/2p} = 1$ and $b_r = b_i = 0$, i.e., $b = 0$ and $a = 0$. This completes the proof of Theorem 1.1 for $p \in [2, e^{2t})$ and r = 2.

Finally, let us come back to the case p = $2e^{2t}$, r = 2. For any polynomial f (z) of z with $\|f\|_2 = 1$, consider the function on [2, ∞),

$$g(q) = \frac{1}{\pi}\int_{\mathbb{C}} |T(t)f(z)|^q e^{-|z|^2}\, dx\ dy,$$

which is continuous by the dominated convergence theorem. By what has been proved for $p < 2e^{2t}, g(q) \leq 1$for $q < 2e^{2t}$. Therefore,

$$g(2e^{2t}) = \lim_{q\to 2e^{2t}} g(q) \leq 1,$$

which implies that $B_{2,2e^{2t}}(1) = 1$ by the density of polynomials in $HL_2(\mathbb{C}^1, dg_1)$. The function $f \equiv 1$ is clearly a maximize for $B_{2,2e^{2t}}(1)$. By Lemma 1.2 and Eq. (10), f (z) is a maximize for $B_{2,2e^{2t}}(1)$ if and only if $f(z) = ce^{az^2+bz}$ with

$$\frac{\|T(t)f\|_{2e^{2t}}}{\|f\|_2} = (1-4|a|)^{(1-e^{-2t})/4} = 1$$

If and only if a = 0, i.e., $f(z) = ce^{bz}$.

2. MAIN RESULT:

For general $r \in [1, \infty)$, the first natural question is whether $T(t): HL_r(\mathbb{C}^n, dg_n) \to HL_r(\mathbb{C}^n, dg_n)$ is a contraction. The following two lemmas [10] give a positive answer.

Lemma 2.1: If f is an entire function on $\mathbb{C}^n$ $r \in [1, \infty)$, then

$$F(x_1 y_1, \dots, x_n, y_n) = |f(z_1, \dots, z_n)|^r$$

is a sub harmonic function on $\mathbb{R}^{2n}$. Furthermore,

$$g(u) = \int_{S^{2n-1}} |f(uz)|^r \, d\sigma \qquad (12)$$

is an increasing function of u, where S2n - 1 is the set $\{z \in \mathbb{C}^n \,||z| = 1\}$, and $d\sigma$ is the standard measure on S^{2n-1}.

Proof. It is easy to see that

$$F_{x_j} = \frac{r}{2}|f(z_1, \dots, z_n)|^{r-2}\left[f_{z_j}\bar{f} + f\overline{f_{z_j}}\right];$$

$$F_{y_j} = \frac{r}{2}|f(z_1, \dots, z_n)|^{r-2}\left[if_{z_j}\bar{f} - if\overline{f_{z_j}}\right];$$

$$F_{x_j x_j} = \frac{r}{2}\left(\frac{r}{2} - 1\right)|f(z_1, \dots, z_n)|^{r-4}\left[f_{z_j}\bar{f} + f\overline{f_{z_j}}\right]^2$$

$$+ \frac{r}{2}|f(z_1, \dots, z_n)|^{r-2}\left[f_{z_j z_j}\bar{f} + f\overline{f_{z_j z_j}} + 2\left|f_{z_j}\right|^2\right];$$

$$F_{y_j y_j} = \frac{r}{2}\left(\frac{r}{2} - 1\right)|f(z_1, \dots, z_n)|^{r-4}\left[if_{z_j}\bar{f} + if\overline{f_{z_j}}\right]^2$$

$$+ \frac{r}{2}|f(z_1, \dots, z_n)|^{r-2}\left[-f_{z_j z_j}\bar{f} + f\overline{f_{z_j z_j}} - 2\left|f_{z_j}\right|^2\right].$$

Therefore

$$\Delta F = \sum_1^n \left[\partial_{x_j x_j} + \partial_{y_j y_j}\right] F = r^2 |f|^{r-2} \sum_1^n \left|f_{z_j}\right|^2 \geq 0.$$

To prove that g(u) is increasing we consider the fundamental solution of Laplace's equation in $\mathbb{R}^{2n}$,

$$\psi(u) = \begin{cases} u^{2-2n}/(2-2n) & \text{if } n \geq 1 \\ \log u, & \text{if } n = 1, \end{cases}$$

Where u = $\left[\sum_1^n (x_j^2 + y_j^2)\right]^{1/2}$. Using Green's identity on the region $u_1 \leq u \leq u_2$

$$\int_{u_1 \leq u \leq u_2} (\psi \Delta F - F \Delta \psi)\, dv$$

$$= \int_{u=u_2} \frac{dF}{du}\, \psi\, dS - \int_{u=u_1} \frac{dF}{du}\, \psi\, dS - \int_{u=u_2} \frac{d\psi}{du}\, F\, dS + \int_{u=u_1} \frac{d\psi}{du}\, F\, dS,$$

i.e.,

$$g(u_2) - g(u_1) = \psi(u_2) \int_{u=u_2} \frac{dF}{du}\, dS - \psi(u_1) \int_{u=u_1} \frac{dF}{du}\, dS - \int_{u_1 \leq u \leq u_2} \psi(u) \Delta F dv$$

Let the right side of the equation be $I(u_2, u_1)$. In order to show $g(u_2) \geq g(u_1)$ it suffices to show that $(\partial / \partial s)\, I(s, u_1) \geq 0$ for $s \geq u_1$. Note that

$$\frac{\partial}{\partial s} I(s, u_1) = \psi'(s) \int_{u \leq s} \Delta F\, dv \geq 0.$$

Lemma 2.2:T(t) is a contraction mapping from $HL_r(\mathbb{C}^n, dg_n)$to $HL_r(\mathbb{C}^n, dg_n)$ for any $t \geq 0$ and $r \geq 1$.

Proof. It follows immediately from the previous lemma. In fact

$$\int |T(t)f|^r\, dg\,(z)$$

$$= \int_0^\infty \left[\int_{S^{2n-1}} |f(e^{-t}uz)\,|^r\, d\sigma\right] u^{2n-1} e^{-u^2} du$$

$$= \int_0^\infty g(e^{-1}u) u^{2n-1} e^{-u^2} du \leq \int_0^\infty g(u) u^{2n-1} e^{-u^2} du = \int |f|^r\, dg(z).$$

Lemma 2.2 has the following corollary which is a part of Theorem 1.1.

Corollary 2.1: For any $t \geq 0$, $T(t): HL_r(\mathbb{C}^n, dg_n) \rightarrow HL_p(\mathbb{C}^n, dg_n)$ is contractive if $p \in [1, r)$, and its maximize must be a constant function.

Proof. Since dg_ndgn is a probability measure on $\mathbb{C}^n$ for any $1 \leq p < r$ and $f \in HL_r(\mathbb{C}^n, dg_n)$,

$$\|T(t)f\|_p \leq \|T(t)f\|_r \leq \|f\|_r \qquad (13)$$

by Holder's inequality and Lemma 2.2.

$f \in HL_r(\mathbb{C}^n, dg_n)$is a maximizer if and only if all inequalities in (13) become equalities. It is well known that $\|T(t)f\|_p = \|T(t)f\|_r$ if and only if $|f|^r \equiv 1$, i.e., $f \equiv$ constant by the analyticity of f.

It should be remarked that since T(t) is a contraction from $HL_2(\mathbb{C}^n, dg_n)$to $HL_{2e^{2t}}(\mathbb{C}^n, dg_n)$ and from $HL_q(\mathbb{C}^n, dg_n)$ to $HL_q(\mathbb{C}^n, dg_n)$ for any $q \in [2, \infty)$, one would expect that T(t) is a contraction mapping from Lp to $L_{pe^{2t}}$ by using the Riesz convexity theorem. If T is a linear operator defined on all nice functions defined on, e.g., $\mathbb{R}^n$, such that $\|Tf\|_q \leq \|f\|_p$ and $\|Tf\|_s \leq \|f\|_r$, then for any simple function f such that $\|f\|_{p_1} \leq 1$, where $p_1^{-1} = \theta/p + (1-\theta)/r, \theta \in [0, 1]$, the estimate $\|Tf\|_{q_1} \leq 1$ holds if $q_1^{-1} = \theta/q + (1-\theta)/s$. The corollary is proved by considering the analytic function of w defined by

$$h(w) = \int T\left(|f(x)|^{p_1(w/p+(1-w)/r)} e^{i\alpha(x)}\right) |g(x)|^{q_1(1-w/q-(1-w)/s} e^{i\beta(x)},$$

where α and β are chosen so that $f(x) = |f(x)|e^{i\alpha(x)}$ and $g(x) = |g(x)|e^{i\beta(x)}$, and g(x) is a simple function with $\|g\|_{q_1} = 1$. It is easy to check that $|h(w)| \leq 1$ when Re(w) = 0 and Re(w) = 1. By the three line theorem $|h(w)| \leq 1$ for all w in the infinite strip $0 \leq Re(w) \leq 1$, which implies that $\|Tf\|_{q_1} \leq 1$. But the operation $f(z) \rightarrow |f(z)|^{p_1(w/p+(1-w)/r)} e^{i\alpha(z)}$ takes an entire function out of the space of entire functions.

Even though the classic convexity theorem does not apply here, it is still true that $T(t) : HL_p(\mathbb{C}^n, dg_n) \rightarrow HL_{pe^{2t}}(\mathbb{C}^n, dg_n)$ is a contraction if p is an even integer by using the fact $T(t)(f^k) = (T(t)f)^k$ and Theorem 1.1 for r = 2.

Lemma 2.3: If $f \in HL_r(\mathbb{C}^n, dg_n)$ then

$$|f(z)| \leq c\, \|f\|_r e^{(|z|+2)^2/r},$$

where c is a constant independent of f.

Proof. We just give a proof for n = 1; the proof for general n is similar. For any $z \in \mathbb{C}$, using Cauchy's integral theorem,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z + ue^{i\theta}\right) d\theta$$

for any u > 0. Therefore

$$|f(z)| = \left|\frac{1}{2\pi}\int_1^2 \int_0^{2\pi} f(z + ue^{i\theta})\, d\theta\, du\right|$$

$$\leq \int_{1\leq|w-z|\leq 2} |f(w)||dw|$$

$$= \int_{1\leq|w-z|\leq 2} |f(w)|e^{-|w|^2/r}e^{|w|^2/r}|dw|$$

$$\leq \left[\int_{1\leq|w-z|\leq 2} |f(w)|^r e^{-|w|^2}|dw|\right]^{1/r} \times \left[\int_{1\leq|w-z|\leq 2} e^{\acute{r}|w|^2/r}|dw|\right]^{1/\acute{r}}$$

$$\leq c\|f\|_r e^{(|z|+2)^2/r}.$$

Lemma 2.4: (we have Minkowski's inequality). Let $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{C}$ be a function in $L_p \cap L_1$ and $p > 1$. Then

$$\left(\int_{\mathbb{R}^m}\left[\int_{\mathbb{R}^n}|f(x, y|\, dx\right]^p dy\right)^{1/p} \leq \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^m}|f(x, y|^p\, dy\right]^{1/p} dx. \qquad (14)$$

Furthermore, (14) becomes an equality if and only if

$$|f(x,\ y)| = A(x)\, B(y),$$

where A and B are two nonnegative measurable functions.

Minkowski's inequality plays the same role as inequalities (2)-(5) did to characterize maximizers.

Theorem 2.1: Suppose that t > 0, $r \in [1, \infty)$, and $p \in [r, re^{2t}]$. Then

(i) f is a maximize for $T(t) : HL_r(\mathbb{C}^n, dg_n) \to HL_p(\mathbb{C}^n, dg_n)$ if and only if

$$f(z_1, \dots, z_n) = \prod_{j=1}^n h_j\left(z_j\right),$$

Where h_j's are maximizers for $T(t) : HL_r(\mathbb{C}^n, dg_n) \to HL_p(\mathbb{C}^n, dg_n)$;

(ii) $T(t) : HL_r(\mathbb{C}^n, dg_n) \to HL_p(\mathbb{C}^n, dg_n)$; is compact if p < re2t.

(To prove this theorem we need an estimate on functions in $HL_p(\mathbb{C}^n, dg_n)$ and Minkowski's inequality.)

Proof. If $T(t) : HL_r(\mathbb{C}^1, dg_1) \to HL_q(\mathbb{C}^1, dg_1)$ is bounded with

$$B_{r,p}(1) = \sup\left\{\frac{\|T(t)f\|_p}{\|f\|_r}; f \in HL_r(\mathbb{C}^1, dg_1), \|f\|_r \neq 0\right\},$$

then by induction on dimension n, $T(t) : HL_r\left(\mathbb{C}^k, dg_k\right) \to HL_q\left(\mathbb{C}^k, dg_k\right)$ is also bounded with

$$B_{r,p}(n) = \sup\left\{\frac{\|T(t)f\|_p}{\|f\|_r}; f \in HL_r(\mathbb{C}^n, dg_n), \|f\|_r \neq 0\right\} = B^n_{r,p} \quad (1).$$

In fact, suppose that this is the case for $n \leq k$, for $n = k + 1$, and for any $f \in HL_r\left(\mathbb{C}^{k+1}, dg_{k+1}\right)$ writing f as f (w, z), where $w \in \mathbb{C}^k$ and $z \in \mathbb{C}$. Therefore

$$\int_{\mathbb{C}^k}\int_{\mathbb{C}} |f(e^{-t}w, e^{-t}z|^p dg_k(w)\, dg(z)$$

$$\leq B^p_{r,p}(1)\int_{\mathbb{C}^k}\left[\int_{\mathbb{C}} |f(e^{-t}w, z)|^r dg_1(z)\right]^{p/r} dg_k(w) \qquad (15)$$

$$\leq B^p_{r,p}(1)\left[\int_{\mathbb{C}} \left[\int_{\mathbb{C}^k}|f(e^{-t}w, z)|^p dg_k(w)\right]^{r/p} dg_1(z)\right]^{p/r} \qquad (16)$$

$$\leq B^{(k+1)p}_{r,p}(1)\left[\int_{\mathbb{C}} \left[\int_{\mathbb{C}^k}|f(w, z)|^r dg_k(w)\right] \; dg_1(z)\right]^{p/r}$$

$$= B^{(k+1)p}_{r,p}(1)\|f\|^p_r. \qquad (17)$$

Equation (15) follows from the hypothesis, (16) is an application of Minkowski's inequality, and (17) is the consequence of the induction hypothesis. Inequalities (15) to (17) imply that $B_{r,p}(k+1) \leq B^{(k+1)}_{r,p}(1)$. On the other hand, it is trivial to see that $B_{r,p}(n) \geq B^n_{r,p}(1)$ by using the function of the form $f(z_1, \ldots, z_n) = \prod_1^n g(z_j)$, where $g(z) \in HL_r(\mathbb{C}^1, dg_1)$. Hence $B_{r,p}(n) = B^n_{r,p}(1)$.

Now, we are in position to prove the following corollary.

Corollary 2.2: If $f \in HL_r(\mathbb{C}^n, dg_n)$, let $f: R^n \times R^m \to \mathbb{C}$ is a maximizer if and only if $f(z_1, \ldots, z_n) = \prod_{j=1}^n A_j(z_j)$ where $A_1, \ldots, A_n \in HL_r$ are non-negative measurable functions, then

$$|f(z_1, \dots, z_n)| \le c \prod_{j=1}^{n} \|A_j\|_r e^{\frac{\sum_{j=1}^{n}(|z_j|+2)^2}{r}} .$$

Proof. For $A_1, \dots, A_n$ lemma 2.3 show that

$$|f(z_1, \dots, z_n)| = A_1(z_1) \dots A_n(z_n) \le c_1 \|A_1\|_r e^{\frac{(|z_1|+2)^2}{r}} \dots c_n \|A_n\|_r e^{\frac{(|z_1|+2)^2}{r}}$$
$$= c \prod_{j=1}^{n} \|A_j\|_r e^{\frac{\sum_{j=1}^{n}(|z_j|+2)^2}{r}}$$

For n=1 the proof of lemma 2.3 gives the result and for general n is similar.

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