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Analytical Approximate Solutions for the Nonlinear Fractional Differential-Difference Equations Arising in Nanotechnology

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Abstract

The aim of this article is by using the fractional complex transform FCT and the optimal homotopy analysis transform method OHATM to find the analytical approximate solutions for nonlinear fractional differential-difference equations FNDDEs arising in physical phenomena such as wave phenomena in fluids, coupled nonlinear optical waveguides and nanotechnology fields. Fractional complex transformation is proposed to convert nonlinear fractional differential-difference equation to nonlinear differential-difference equation. This optimal approach has general meaning and can be used to get the fast convergent series solution of the different type of nonlinear differential-difference equations. This technique finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. HATM method provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameter h. So the valid region for h where the series converges is the horizontal segment of each h curve. The results obtained by the HATM show that the approach is easy to implement and computationally very attractive.

The numerical solutions show that the proposed method is very efficient and computationally attractive. The results reveal that this method is very effective and powerful to obtain the approximate solutions.

Keywords: nonlinear fractional differential-difference equation, method, Laplace transform, optimal homotopy analysis method.

1. INTRODUCTION

Fractional calculus has attracted much attention for its potential applications in various scientific fields such as fluid mechanics, biology, viscoelasticity, engineering, and other areas of science [1-2]. In the recent years, fractional differential equations have been demonstrated applications in numerous seemingly diverse fields of engineering sciences, finance, applied mathematics, bio-engineering and others. Of extreme important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science, probability and statistics, electrochemistry

Of corrosion, chemical physics, quantum chemistry, quantum mechanics, damping laws, diffusion processes and signal processing are well described by differential equations of fractional order [3–4].

So it becomes important to find some efficient methods for solving fractional differential equations. A great deal of effort has been spent on constructing of the numerical solutions and many effective methods have been developed such as fractional wavelet method [5--8], fractional differential transform method [9], fractional operational matrix method [10, 11], fractional improved homotopy perturbation method [12, 13], fractional variational iteration method [14, 15], and fractional Laplace Adomian decomposition method [16-19].

Many analytic approximate approaches for solving nonlinear differential equations have been proposed and the most outstanding one is the homotopy analysis method (HAM). In recent years, many authors have paid attention to studying the solutions of nonlinear partial differential equations and nonlinear differential difference equations by various methods [20]. The proposed method is coupling of the homotopy analysis method and Laplace transform method. The advantage of this proposed method is its capability of combining two powerful methods for obtaining the approximate solutions. Homotopy analysis method (HAM) was first proposed and applied by Liao [21, 22], based on homotopy, a fundamental concept in topology and differential geometry. The HAM has been successfully applied by many researchers for solving linear and nonlinear partial differential equations.

The objective of the present paper is to extend the application of the HATM to obtain an analytic approximate solution of the following nonlinear difference differential equations in mathematical physics: In this work, we analyze the nonlinear difference differential equations by using the fractional complex transform FCT and homotopy analysis transforms method HATM. The HATM is an innovative amalgamation of the Laplace transform scheme and the HAM. The supremacy of this technique is its potential of combining two robust computational techniques for solving fractional problems.

(i) The general fractional lattice equation [2-3]:

$$\frac{d^{\alpha}u_n}{dt^{\alpha}} = (\varepsilon + \beta u_n + \gamma u_n^2)(u_{n+1} - u_{n-1}), \quad 0 < \alpha \le 1. \tag{1.1}$$

Where ε, β, and γ are arbitrary constant. The HATM is a combination of the Laplace transform method, HAM and He's polynomials. The advantage of this technique is its capability of combining two powerful methods for obtaining exact and approximate analytical solutions for nonlinear equations. The fact that the HATM solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method. The HATM provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions. It is worth mentioning that the proposed approach is capable of reducing the volume of computational work compared to classical methods while still maintaining the high accuracy in the numerical result; the size reduction amounts to an improvement of the performance of the approach.

We use the optimal homotopy analysis method combined with the Laplace transform for solving nonlinear differential difference equations in this paper. The main advantage of this problem is that we can accelerate the convergence rate, minimize iterative times, accordingly save the computation time and evaluate the efficiency.

2. PRELIMINARIES AND NOTATIONS

In this section, we give some basic definitions of fractional calculus theory which are be used further in this work. Local fractional derivative of $f(x)$ order α in interval $[a,b]$ is defined by [29, 30][23]

$$D^{\alpha} f(x_0) = \frac{d^{\alpha} f}{dx^{\alpha}}\Big|_{x=x_0} = \lim_{x\to x_0} \frac{D^{\alpha}(f(x) - f(x_0))}{(x-x_0)^{\alpha}}, \tag{2.2}$$

where $D^{\alpha}(f(x) - f(x_0)) = \Gamma(\alpha+1) D(f(x) - f(x_0))$.

Also the inverse of local fractional derivative to of $f(x)$ order α in interval [a, b] is defined by [30, 31][25-26]

$$ {}_aI_b^{\alpha} f(x) = \frac{1}{\Gamma(\alpha+1)} \int_a^b f(t)(dt)^{\alpha} = \frac{1}{\Gamma(\alpha+1)} \lim_{Dt\to 0} \sum_{j=0}^{N-1} f(t_j)(Dt_j)^{\alpha}, \tag{2.3}$$

where $Dt_j = t_{j-1} - t_j$, $Dt = \max\{Dt_1, Dt_2, ...\}$, $j = 0,1,....,N-1, t_0 = a, t_N = b$ are the partition of the interval [a, b].

3. THE OPTIMAL HOMOTOPY ANALYSIS TRANSFORM METHOD (OHATM) FOR DIFFERENTIAL DIFFERENCE EQUATIONS

To illustrate the basic idea of the OHATM for the fractional partial differential equation as:

$$\mathbf{D}_t^{\mathbf{n\alpha}}\mathbf{u}(x,t)+\mathbf{R}[x]\mathbf{u}(x,t)+\mathbf{N}[x]\mathbf{u}(x,t)=\mathbf{g}(x,t), t\succ 0, x\in R, n-1\prec n\alpha\leq n, \tag{3.4}$$

where $\mathbf{D}_t^{\mathbf{n\alpha}}=\frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$, $R[x]$ is the linear operator in x , $N[x]$ is the general nonlinear operator in x , and $g(x,t)$ are continuous functions. For simplicity we ignore all initial and boundary conditions, which can be treated in similar way. Now the methodology consists of applying Laplace transform first on both sides of Eq. (3.4), we get

$$\mathbf{L}[\mathbf{D}_t^{\mathbf{n\alpha}}\mathbf{u}(x,t)]+\mathbf{L}[\mathbf{R}[x]\mathbf{u}(x,t)+\mathbf{N}[x]\mathbf{u}(x,t)]=\mathbf{L}[\mathbf{g}(x,t)]. \tag{3.5}$$

Now, using the differentiation property of the Laplace transform, we have

$$\mathbf{L}[\mathbf{u}(x,t)]-\frac{1}{s^{\mathbf{n\alpha}}}\sum_{k=0}^{n-1}\mathbf{s}^{(\mathbf{n\alpha-k-1})}\mathbf{u}^{k}(\mathbf{x},0)+\frac{1}{s^{\mathbf{n\alpha}}}\mathbf{L}[\,\mathbf{R}[x]\mathbf{u}(x,t)+\mathbf{N}[x]\mathbf{u}(x,t)-\mathbf{g}(x,t)]=0. \tag{3.6}$$

We define the nonlinear operator

$$N[\phi(x,t;q)]=\mathbf{L}[\phi(x,t;q)]-\frac{1}{s^{\mathbf{n\alpha}}}\sum_{k=0}^{n-1}\mathbf{s}^{(\mathbf{n\alpha-k-1})}\mathbf{u}^{k}(\mathbf{x},0)+\frac{1}{s^{\mathbf{n\alpha}}}\mathbf{L}[\,\mathbf{R}[x]\mathbf{u}(x,t)+ \mathbf{N}[x]\mathbf{u}(x,t)-\mathbf{g}(x,t)], \tag{3.7}$$

where $q\in[0,1]$ be an embedding parameter $\phi(x,t;q)$ and is the real function of x,t and q . By means of generalizing the traditional homotopy methods, Liao constructed the zero order deformation equation

$$(1-q)\mathbf{L}(\phi(x,t;q)-u_0(x,t))=\mathbf{q}\hbar\mathbf{H}(\mathbf{x},\mathbf{t})\mathbf{N}[\mathbf{D}_t^{\alpha}\phi(x,t;q)], \tag{3.8}$$

where $\hbar\neq\mathbf{0}$ is an auxiliary parameter, $\mathbf{H}(\mathbf{x},\mathbf{t})\neq\mathbf{0}$ is an auxiliary function, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $\phi(x,t;q)$ is an unknown function. It is important that one has great freedom to choose auxiliary thing in FHATM. Obviously, when $q=0$ and $q=1$, it holds

$$\boldsymbol{\phi}(\mathbf{x},\mathbf{t};\mathbf{0})=\mathbf{u}_0(\mathbf{x},\mathbf{t})\ \textbf{and}\ \boldsymbol{\phi}(\mathbf{x},\mathbf{t};\mathbf{1})=\mathbf{u}(\mathbf{x},\mathbf{t}), \tag{3.9}$$

respectively. Thus, as q increases from 0 to 1 , the solution varies from the initial guess $\mathbf{u}_0(\mathbf{x},\mathbf{t})$ to the solution $\mathbf{u}(\mathbf{x},\mathbf{t})$. Expanding $\boldsymbol{\phi}(\mathbf{x},\mathbf{t};\mathbf{q})$ in Taylor's series with respect to q , we have

$$\boldsymbol{\phi}(\mathbf{x},\mathbf{t};\mathbf{q}) = \mathbf{u}_0(\mathbf{x},\mathbf{t}) + \sum_{\mathbf{m=1}}^{\infty} \mathbf{u}_m(\mathbf{x},\mathbf{t})\mathbf{q}^m, \tag{3.10}$$

where

$$\mathbf{u}_m(\mathbf{x},\mathbf{t}) = \frac{1}{m!}\frac{\partial^m \boldsymbol{\phi}(\mathbf{x},\mathbf{t};\mathbf{q})}{\partial q^m}|_{q=0}\ . \tag{3.11}$$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter h and the auxiliary function $H(x,t)$ are selected such that the series (3.10) is convergent at $q=1$, then we have

$$\mathbf{u}(\mathbf{x},\mathbf{t}) = \mathbf{u}_0(\mathbf{x},\mathbf{t}) + \sum_{\mathbf{m=1}}^{\infty} \mathbf{u}_m(\mathbf{x},\mathbf{t}). \tag{3.12}$$

Let us define the vector

$$\overleftrightarrow{u_n}(t) = \{u_0(x,t), u_1(x,t), u_2(x,t), \ldots, u_n(x,t)\}. \tag{3.13}$$

Differentiating Eq. (3.8) *m*-times with respect to q , then setting $q=0$ and dividing then by $m!$, we have the m^{th} - order deformation equation

$$\mathbf{L}[\mathbf{u}_m(\mathbf{x},\mathbf{t}) - \boldsymbol{\chi}_m \mathbf{u}_{m-1}(\mathbf{x},t)] = \hbar \mathbf{H}(\mathbf{x},\mathbf{t})\mathbf{R}_m(\overleftrightarrow{u}_{m-1}). \tag{3.14}$$

Applying inverse Laplace transform

$$\mathbf{u}_m(\mathbf{x},\mathbf{t}) = \boldsymbol{\chi}_m \mathbf{u}_{m-1} + \hbar \mathbf{L}^{-1}[\mathbf{H}(\mathbf{x},\mathbf{t})\mathbf{R}_m(\vec{\mathbf{u}}_{m-1})] \tag{3.15}$$

where

$$\mathbf{R}_m(\overleftrightarrow{\mathbf{u}}_{m-1}) = \frac{1}{(\boldsymbol{m-1})!}\frac{\partial^{m-1}\mathbf{N}[\boldsymbol{\phi}(\mathbf{x},\mathbf{t};\mathbf{q})]}{\partial \mathbf{q}^{m-1}}|_{q=0}, \tag{3.16}$$

and

$$\boldsymbol{\chi}_m = \begin{cases} 0 & m \le 1, \\ 1 & m > 1. \end{cases}$$

In this way, it is easily to obtain $u_m(x,t)$ for $m \ge 1$, at $m^{th} -$ order, we have

$$\mathbf{u}(\mathbf{x},\mathbf{t}) = \sum_{\mathbf{m=0}}^{M} \mathbf{u}_m(\mathbf{x},\mathbf{t}).$$

The m^{th}-order deformation Eq. (3.15) is linear iteration problems and thus can be easily solved, especially by means of symbolic computation software such as Mathematica, or Maple.

S.J. Liao [24] and Mohamed S. Mohamed et al. [25-26] they suggested the so called optimization method to find out the optimal convergence control parameters by minimum of the square residual error integrated in the whole region having physical meaning. Their method depends on the square residual error. Let $\Delta(h)$ denote the square residual error of the governing equation (3.4) and express as

$$\Delta(h) = \int_{\Omega} (N[\tilde{u}_n(t)])^2 \, d\Omega, \tag{3.17}$$

Where

$$\tilde{u}_m(t) = u_0(t) + \sum_{k=1}^{m} u_k(t) \tag{3.18}$$

the optimal value of h is given by a nonlinear algebraic equation as:

$$\frac{d\Delta(h)}{dh} = 0. \tag{3.19}$$

4. THE FRACTIONAL COMPLEX TRANSFORM

The fractional complex transform was first proposed in [27, 28]. Consider the following general fractional differential equation

$$f(u, u_t^{(\alpha)}, u_x^{(\beta)}, u_y^{(\gamma)}, u_z^{(\lambda)}, u_t^{(2\alpha)}, u_x^{(2\beta)}, u_y^{(2\gamma)}, u_z^{(2\lambda)}, ...) = 0, \tag{4.20}$$

where $u_t^{(\alpha)} = \frac{\partial^{\alpha} u(x,y,z,t)}{\partial^{\alpha}}$ denotes the Local fractional derivative. $0 < \alpha \le 1$, $0 < \beta \le 1$, $0 < \gamma \le 1$, $0 < \lambda \le 1$. The fractional complex transform requires

$$\begin{aligned} T &= \frac{wt^{\alpha}}{\Gamma(1+\alpha)}, \\ X &= \frac{px^{\beta}}{\Gamma(1+\beta)}, \\ Y &= \frac{ky^{\gamma}}{\Gamma(1+\gamma)}, \\ Z &= \frac{lz^{\lambda}}{\Gamma(1+\lambda)}. \end{aligned} \tag{4.21}$$

were $p, q, k,$ and l are unknown constants. Using the basic properties of the fractional derivative and the above transforms, we can convert fractional derivatives into classical derivatives:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = w\frac{\partial u}{\partial T},$$
$$\frac{\partial^{\beta} u}{\partial x^{\beta}} = p\frac{\partial u}{\partial X},$$
$$\frac{\partial^{\gamma} u}{\partial y^{\gamma}} = k\frac{\partial u}{\partial Y}, \tag{4.22}$$
$$\frac{\partial^{\lambda} u}{\partial t^{\lambda}} = l\frac{\partial u}{\partial Z}.$$

Therefore, we can easily convert the fractional differential equations into partial differential equations, so that everyone familiar with advanced calculus can deal with fractional calculus without any difficulty which can be solved by optimal homotopy analysis.

5. Numerical Results

In this section, In this section, two examples on time-fractional wave equations are solved to demonstrate the performance and efficiency of the FCT and OHATM.

Example1: The general fractional- difference lattice equation:

$$\frac{d^{\alpha} u_n}{dt^{\alpha}} = (\varepsilon + \beta u_n + \gamma u_n^2)(u_{n+1} - u_{n-1}), \quad 0 < \alpha \le 1, \tag{5.23}$$

with the initial condition,

$$\mathbf{u_n(0)} = \frac{\sqrt{\boldsymbol{\beta}^2 - \mathbf{4}\varepsilon\boldsymbol{\gamma}}\ \mathbf{sn}(\boldsymbol{\lambda},\boldsymbol{\mu})\,\mathbf{cn}(\boldsymbol{\lambda}\mathbf{n},\boldsymbol{\mu})\,\mathbf{dn}(\boldsymbol{\lambda}\mathbf{n},\mu)}{\mathbf{2}\boldsymbol{\gamma}\,\mathbf{cn}(\boldsymbol{\lambda},\boldsymbol{\mu})\,\mathbf{dn}(\boldsymbol{\lambda},\boldsymbol{\mu})\,\mathbf{sn}(\boldsymbol{\lambda}\mathbf{n},\boldsymbol{\mu})} - \frac{\beta}{\mathbf{2}\boldsymbol{\gamma}} \tag{5.24}$$

with the exact solution at $\alpha = 1$

$$\mathbf{u_n(t)} = \frac{\sqrt{\boldsymbol{\beta}^2 - \mathbf{4}\varepsilon\boldsymbol{\gamma}}\ \mathbf{sn}(\boldsymbol{\lambda},\mathbf{m})\,\mathbf{cn}(\zeta,\mathbf{m})\,\mathbf{dn}(\zeta,\mathbf{m})}{\mathbf{2}\boldsymbol{\gamma}\,\mathbf{cn}(\boldsymbol{\lambda},\boldsymbol{\mu})\,\mathbf{dn}(\boldsymbol{\lambda},\boldsymbol{\mu})\,\mathbf{sn}(\zeta,\boldsymbol{\mu})} - \frac{\beta}{\mathbf{2}\boldsymbol{\gamma}} \tag{5.25}$$

where $\mathbf{sn}(\boldsymbol{\lambda}, \boldsymbol{\mu})$, $\mathbf{cn}(\boldsymbol{\lambda}\mathbf{n}, \boldsymbol{\mu})$, and $\mathbf{dn}(\boldsymbol{\lambda}\mathbf{n}, \mu)$ are Jacobain functions and ε $\beta,$ $\gamma,$ $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are arbitrary constant and

$$\zeta = \boldsymbol{\lambda}\,\mathbf{n} - \frac{t\,(\mathbf{4}\varepsilon\boldsymbol{\gamma} - \boldsymbol{\beta}^2)\,\mathbf{sn}(\boldsymbol{\lambda},\boldsymbol{\mu})}{\mathbf{2}\boldsymbol{\gamma}\,\mathbf{cn}(\boldsymbol{\lambda},\boldsymbol{\mu})\,\mathbf{dn}(\boldsymbol{\lambda},\boldsymbol{\mu})}.$$

To apply FCT to eq. (5.23), we use the above transformations, so we have the following partial differential equation:

$$w\frac{\partial u_n}{\partial T}=(\varepsilon+\beta u_n+\gamma u_n^2)(u_{n+1}-u_{n-1}),$$

So we have,

$$\frac{\partial u_n}{\partial T}=\frac{1}{w}((\varepsilon+\beta u_n+\gamma u_n^2)(u_{n+1}-u_{n-1})). \tag{5.26}$$

To solve equation (5.26) by means of the homotopy analysis transform method we consider the following $w=1,$ and linear

$$\Im[\Phi(\mathbf{n},\mathbf{T};\mathbf{q})]=\mathbf{L}\Phi(\mathbf{n},\mathbf{T};\mathbf{q})=\frac{\partial\Phi(\mathbf{n},\mathbf{T};\mathbf{q})}{\partial\mathbf{t}}$$

with the property that $\Im[c]=0$, where c is a constant. This implies that

$$\Im^{-1}(\bullet)=\int_0^t(\bullet)dt$$

Taking Laplace transform of equation (5.26) both of sides subject to the initial condition, we get

$$\mathbf{L}[\mathbf{u}_n(\mathbf{n},\mathbf{T})]-\frac{1}{s}\left[\frac{\sqrt{\boldsymbol{\beta}^2-\mathbf{4}\varepsilon\boldsymbol{\gamma}}\,\mathbf{sn}(\boldsymbol{\lambda},\boldsymbol{\mu})\mathbf{cn}(\boldsymbol{\lambda}\mathbf{n},\boldsymbol{\mu})\mathbf{dn}(\boldsymbol{\lambda}\mathbf{n},\mu)}{\mathbf{2}\boldsymbol{\gamma}\ \mathbf{cn}(\boldsymbol{\lambda},\boldsymbol{\mu})\ \mathbf{dn}(\boldsymbol{\lambda},\boldsymbol{\mu})\ \mathbf{sn}(\boldsymbol{\lambda}\mathbf{n},\boldsymbol{\mu})}\right]$$

$$-\frac{1}{s^{\alpha}}\mathbf{L}\left[(\varepsilon+\boldsymbol{\beta}\mathbf{u}_n+\boldsymbol{\gamma}\mathbf{u}_n^2)(\mathbf{u}_{n-1}-\mathbf{u}_{n+1})\right]=\mathbf{0}. \tag{5.27}$$

Where $T=\dfrac{t^{\alpha}}{\Gamma(1+\alpha)}$

We now define the nonlinear operator as:

$$\mathbf{N}[\boldsymbol{\varphi}_{\mathbf{n}}(\mathbf{T};\mathbf{q})]=\mathbf{L}[\boldsymbol{\varphi}_{\mathbf{n}}(n,\mathbf{T};\mathbf{q})]-\frac{1}{s}\left[\frac{\sqrt{\boldsymbol{\beta}^2-\mathbf{4}\varepsilon\boldsymbol{\gamma}}\,\mathbf{sn}(\boldsymbol{\lambda},\boldsymbol{\mu})\mathbf{cn}(\boldsymbol{\lambda}\mathbf{n},\boldsymbol{\mu})\mathbf{dn}(\boldsymbol{\lambda}\mathbf{n},\mu)}{\mathbf{2}\boldsymbol{\gamma}\mathbf{cn}(\boldsymbol{\lambda},\boldsymbol{\mu})\mathbf{dn}(\boldsymbol{\lambda},\boldsymbol{\mu})\mathbf{sn}(\boldsymbol{\lambda}\mathbf{n},\boldsymbol{\mu})}\right]$$

$$-\frac{1}{s^{\alpha}}L[(\varepsilon+\boldsymbol{\beta}\boldsymbol{\varphi}_n(n,\mathbf{T};\mathbf{q})+\boldsymbol{\gamma}\boldsymbol{\varphi}_n(n,\mathbf{T};\mathbf{q})^2)\ \ (\boldsymbol{\varphi}_{n-1}(n,\mathbf{T};\mathbf{q})-\boldsymbol{\varphi}_{n+1}(n,\mathbf{T};\mathbf{q})]=\mathbf{0} \tag{5.28}$$

and then the mth-order deformation equation is given by

$$\mathbf{L}[\mathbf{u}_m(\mathbf{n},\mathbf{T}) - \chi_m \mathbf{u}_{m-1}(\mathbf{n},\mathbf{t})] = \hbar \mathbf{H}(\mathbf{n},\mathbf{T}) \mathbf{R}_m(\overleftrightarrow{\mathbf{u}}_{m-1}). \tag{5.29}$$

Taking inverse Laplace transform of Eq. (5.29), we get

$$\mathbf{u}_m(\mathbf{n},\mathbf{T}) = \chi_m \mathbf{u}_{m-1} + \hbar \mathbf{L}^{-1}[\mathbf{H}(\mathbf{n},\mathbf{T}) \mathbf{R}_m(\overleftrightarrow{\mathbf{u}}_{m-1})] \tag{5.30}$$

Where

$$\mathbf{R}_m(\overleftrightarrow{\mathbf{u}}_{m-1}(\mathbf{n},\mathbf{T})) = \mathbf{L}[\mathbf{u}_m(\mathbf{n},\mathbf{T})] - \frac{1}{s}\left[\frac{\sqrt{\boldsymbol{\beta}^2 - \mathbf{4}\varepsilon\boldsymbol{\gamma}}\,\mathbf{sn}(\boldsymbol{\lambda},\boldsymbol{\mu})\mathbf{cn}(\boldsymbol{\lambda}\mathbf{n},\boldsymbol{\mu})\mathbf{dn}(\boldsymbol{\lambda}\mathbf{n},\mu)}{\mathbf{2}\boldsymbol{\gamma}\mathbf{cn}(\boldsymbol{\lambda},\boldsymbol{\mu})\mathbf{dn}(\boldsymbol{\lambda},\boldsymbol{\mu})\mathbf{sn}(\boldsymbol{\lambda}\mathbf{n},\boldsymbol{\mu})}\right](\mathbf{1} - \chi_m)$$

$$-\frac{1}{s^{\alpha}}\mathbf{L}[(\varepsilon + \boldsymbol{\beta}\mathbf{u}_{m-1}(\mathbf{n},\mathbf{T}) + \boldsymbol{\gamma}\mathbf{u}^2_{m-1}(\mathbf{n},\mathbf{T}))(\mathbf{u}_{m-1}(\mathbf{n}-\mathbf{1},\mathbf{T}) - \mathbf{u}_{m-1}(\mathbf{n}+\mathbf{1}, \tag{5.31}$$

Let us take the initial approximation as

$$\mathbf{u}_0(\mathbf{n},\mathbf{T}) = \frac{\sqrt{\boldsymbol{\beta}^2 - \mathbf{4}\varepsilon\boldsymbol{\gamma}}\,\mathbf{sn}(\boldsymbol{\lambda},\boldsymbol{\mu})\mathbf{cn}(\boldsymbol{\lambda}\mathbf{n},\boldsymbol{\mu})\mathbf{dn}(\boldsymbol{\lambda}\mathbf{n},\mu)}{\mathbf{2}\boldsymbol{\gamma}\mathbf{cn}(\boldsymbol{\lambda},\boldsymbol{\mu})\mathbf{dn}(\boldsymbol{\lambda},\boldsymbol{\mu})\mathbf{sn}(\boldsymbol{\lambda}\mathbf{n},\boldsymbol{\mu})},$$

the other components are given by

$$\mathbf{u}_1(n,T) = -\mathbf{hT}(\alpha + \frac{(\beta^2 - 4\varepsilon\gamma)\mathbf{cn}[n\lambda,\mu]^2\,\mathbf{dn}[n\lambda,\mu]^2\,\mathbf{sn}[\lambda,\mu]^2}{4\gamma\mathbf{cn}[\lambda,\mu]^2\,\mathbf{dn}[\lambda,\mu]^2\,\mathbf{sn}[n\lambda,\mu]^2} +$$

$$\frac{\beta\sqrt{\beta^2 - 4\varepsilon\gamma}\,\mathbf{cn}[n\lambda,\mu]\mathbf{dn}[n\lambda,\mu]\mathbf{sn}[\lambda,\mu]}{\mathbf{2}\boldsymbol{\gamma}\mathbf{cn}[\lambda,\mu]\mathbf{dn}[\lambda,\mu]\mathbf{sn}[\mathbf{n}\boldsymbol{\lambda},\boldsymbol{\mu}]})$$

$$(\frac{\sqrt{\beta^2 - 4\varepsilon\gamma}\,\mathbf{cn}[(-1+n)\lambda,\mu]\mathbf{dn}[(-1+n)\lambda,\mu]\mathbf{sn}[\lambda,\mu]}{\mathbf{2}\boldsymbol{\gamma}\mathbf{cn}[\lambda,\mu]\mathbf{dn}[\lambda,\mu]\mathbf{sn}[(-\mathbf{1}+\mathbf{n})\boldsymbol{\lambda},\boldsymbol{\mu}]} -$$

$$\frac{\sqrt{\beta^2 - 4\varepsilon\gamma}\,\mathbf{cn}[(1+n)\lambda,\mu]\mathbf{dn}[(1+n)\lambda,\mu]\mathbf{sn}[\lambda,\mu]}{2\gamma\mathbf{cn}[\lambda,\mu]\mathbf{dn}[\lambda,\mu]\mathbf{sn}[(1+n)\lambda,\mu]})$$

…

Therefore, the approximate solution is

$$\mathbf{u}(\mathbf{n},\mathbf{t}) = \mathbf{u}_0(\mathbf{n},\mathbf{t}) + \mathbf{u}_1(\mathbf{n},\mathbf{t}) + \mathbf{u}_2(\mathbf{n},\mathbf{t}) + \ldots \tag{5.32}$$

We calculate the numerical results of our proposed method HATM for different values of $n \in [-30,30]$, ($-1 \le h \le 1$), $\varepsilon = 0.2, \beta = 0.5, \gamma = 0.2, \lambda = 0.3$ and $t = 0.45$ are presented in Figs.1 and 2.

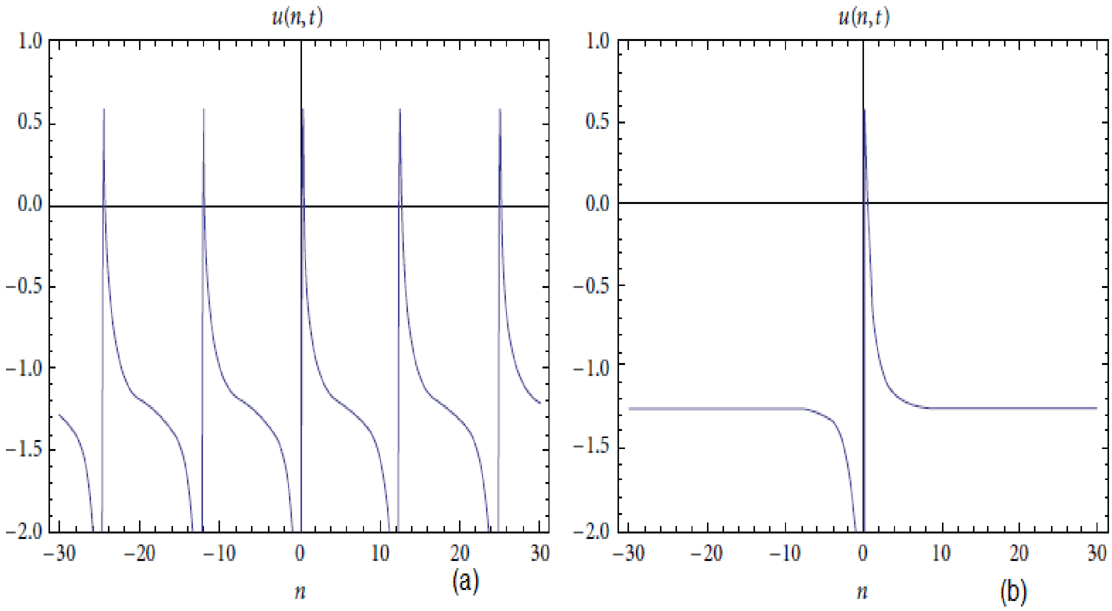

Figure 1: The 2nd-order approximate solution (5.32), respectively, when (a) μ = 0.5 and (b) μ =1 at $h|_{\text{optimal}}$ = −0.975.

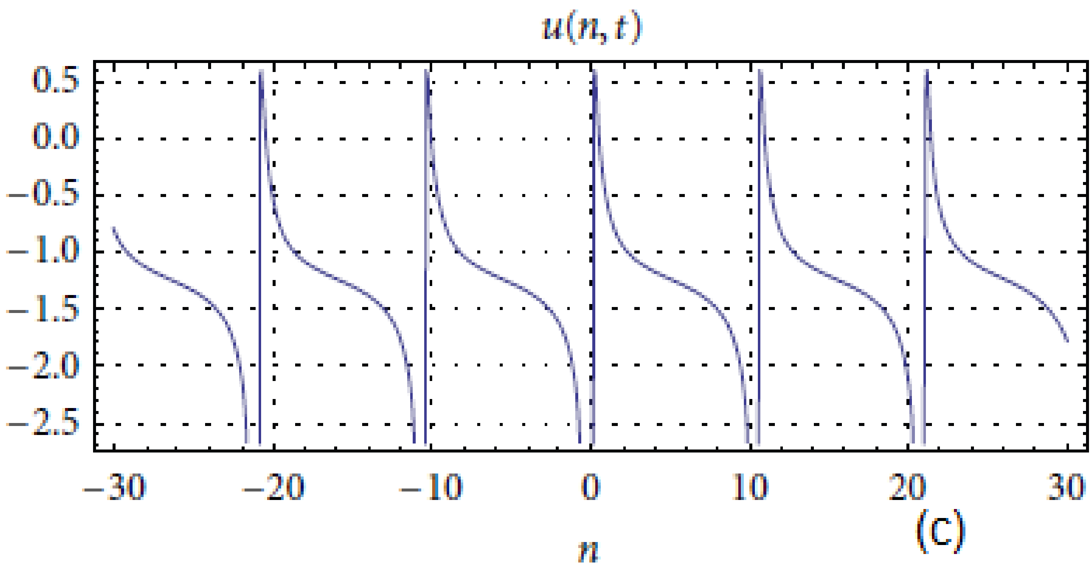

Figure 2: The 2nd-order approximate solution (5.32), when (c) μ = 0, at $h|_{\text{optimal}}$ = −0.975.

Example2:

In equation (5.23), if $\varepsilon = 1, \beta = 0, \gamma = 1.$ The general fractional-difference lattice equation is written as:

$$\frac{d^{\alpha} u_n}{dt^{\alpha}} = (1 - u_n^2)(u_{n+1} - u_{n-1}), \quad 0 < \alpha \leq 1, \tag{5.33}$$

with the initial condition,

$$\mathbf{u_n}(\mathbf{0}) = A\,\mathrm{Tanh}[\mathrm{k\,n}], \tag{5.34}$$

Where k is an arbitrary constant and $A = \mathrm{Tanh(k)},$ with the exact solution at $\alpha = 1$ [29, 30].

$$\mathbf{u_n}(\mathbf{t}) = A\,\mathrm{Tanh}[\mathrm{kn} + 2\mathrm{At}]. \tag{5.35}$$

In a related technique as above, we choose linear operator as

$$\Im[\Phi(\mathbf{n},\mathbf{t};\mathbf{q})] = \frac{\partial \Phi(\mathbf{n},\mathbf{t};\mathbf{q})}{\partial \mathbf{t}},$$

with the property that $\Im[c] = 0$, where c is a constant. This implies that

$$\Im^{-1}(\bullet) = \int_0^t (\bullet) dt.$$

Taking Laplace transform of equation (5.33) both of sides subject to the initial condition, we get

$$\mathbf{L}[\mathbf{u}_n(\mathbf{n},\mathbf{t})] - \frac{1}{s}\left[A\,\mathrm{Tanh(kn)}\right] - \frac{1}{s^{\alpha}}\mathbf{L}\left[(1 - \mathbf{u}_n^2)(\mathbf{u}_{n-1} - \mathbf{u}_{n+1})\right] = \mathbf{0}. \tag{5.36}$$

We now define the nonlinear operator as:

$$\mathbf{N}[\boldsymbol{\varphi}_{\mathbf{n}}(n,\mathbf{t};\mathbf{q})] = \mathbf{L}[\boldsymbol{\varphi}_{\mathbf{n}}(n,\mathbf{t};\mathbf{q})] - \frac{1}{s}[A\ \mathrm{Tanh(kn)}] - \frac{1}{s^{\alpha}} L[(1 - \boldsymbol{\varphi}_n(n,\mathbf{t};\mathbf{q})^2)\ \ (\boldsymbol{\varphi}_{n-1}(n,\mathbf{t};\mathbf{q}) - \boldsymbol{\varphi}_{n+1}(n,\mathbf{t};\mathbf{q})] = \mathbf{0} \tag{5.37}$$

and then the m$^{\text{th}}$-order deformation equation is given by

$$\mathbf{L}[\mathbf{u}_m(\mathbf{n},\mathbf{t}) - \chi_m \mathbf{u}_{m-1}(\mathbf{n},\mathbf{t})] = \hbar \mathbf{H}(\mathbf{n},\mathbf{t}) \mathbf{R}_m(\overleftrightarrow{\mathbf{u}}_{m-1}). \tag{5.38}$$

Taking inverse Laplace transform of Eq. (5.38), we get

$$\mathbf{u}_m(\mathbf{n},\mathbf{t}) = \chi_m \mathbf{u}_{m-1} + \hbar \mathbf{L}^{-1}[\mathbf{H}(\mathbf{n},\mathbf{t})\mathbf{R}_m(\overset{\leftrightarrow}{\mathbf{u}}_{m-1})] \tag{5.39}$$

where

$$\mathbf{R}_m(\overset{\leftrightarrow}{\mathbf{u}}_{m-1}(\mathbf{n},\mathbf{t})) = \mathbf{L}[\mathbf{u}_m(\mathbf{n},\mathbf{t})] - \frac{1}{s}(A\,\mathrm{Tanh}[\mathrm{k\,n}])(\mathbf{1}-\chi_m) - \frac{1}{s^{\alpha}}\mathbf{L}[(1-\mathbf{u}^2_{m-1}(\mathbf{n},\mathbf{t}))$$
$$(\mathbf{u}_{m-1}(\mathbf{n}-\mathbf{1},\mathbf{t}) - \mathbf{u}_{m-1}(\mathbf{n}+\mathbf{1},\mathbf{t}))]. \tag{5.40}$$

Let us take the initial approximation as

$$\mathbf{u}_0(\mathbf{n},\mathbf{t}) = A\,\mathrm{Tanh}[\mathrm{k\,n}],$$

the other components are given by

$$\mathbf{u}_1(n,t) = \frac{Aht^{\alpha}(-1+A^2\tanh^2[k\,n])(\tanh[k(-1+n)]-\tanh[k(1+n)])}{Gamma[1+\alpha]},$$

$$\begin{aligned}\mathbf{u}_2(n,t) = {} & \frac{Aht^{\alpha}}{Gamma[1+\alpha]^2 Gamma[1+2\alpha]Gamma[1+3\alpha]}(-(1+h)\cosh[k](-1+A^2+(-1+A^2)\\ & Cosh[2kn]Gamma[1+\alpha]\,\mathrm{Gamma}[1+2\alpha]\mathrm{Gamma}[1+3\alpha]\mathrm{Sech}[\mathrm{k}(-1+\mathrm{n})]\mathrm{Sech}[\mathrm{k\ n}]^2\\ & Sech[k(1+n)]Sinh[k] - 2A^2h^2t^{2\alpha}Gamma[1+2\alpha]^2 Tanh[k\,n](-1+A^2Tanh[k\,n]^2)(\\ & Tanh[k(-1+n)] - Tanh[k(1+n)](Tanh[k(-2+n)](1-A^2Tanh[k(-1+n)]^2)+Tanh[kn](\\ & -2+A^2Tanh[k(-1+n)]^2 + A^2Tanh[k(1+n)]^2)+(1-A^2Tanh[k(1+n)]^2)Tanh[k(2+n)]) -\\ & ht^{\alpha}Gamma[1+\alpha]^2Gamma[1+3\alpha](Tanh[k(2+n)](-1+A^2Tanh[k(-1+n)]^2)-Tanh[kn]\\ & (-2+A^2Tanh[k(-1+n)]^2 + A^2Tanh[k(1+n)]^2)+(-1+A^2Tanh[k(1+n)]^2)Tanh[k(n+2)]),\end{aligned}$$

$$\cdots$$

Therefore, the approximate solution is

$$\mathbf{u}(\mathbf{n},\mathbf{t}) = \mathbf{u}_0(\mathbf{n},\mathbf{t}) + \mathbf{u}_1(\mathbf{n},\mathbf{t}) + \mathbf{u}_2(\mathbf{n},\mathbf{t}) + \ldots \tag{5.41}$$

By means of the so-called *h*_curves it is easy to find out the so called valid regions of *h* to gain a convergent solution series. When the valid region of *h* is a horizontal line segment, then the solution is converged. Figure 3 shows the *h*_curve obtained from the 4th order FHATM approximation solution. In our study, it is obvious from Figure 3 that the acceptable range of auxiliary parameter *h* is $-50 \le h < 50$.

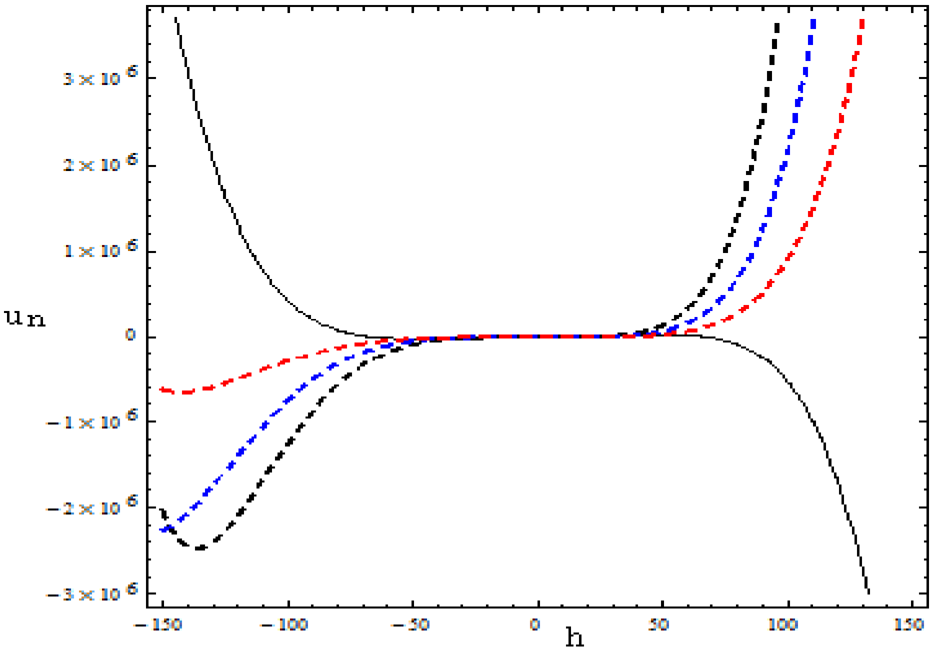

Figure 3: Plot of h_curve for different values of α, for $k = 0.2, t = 0.4,$ and $n = -0.5$

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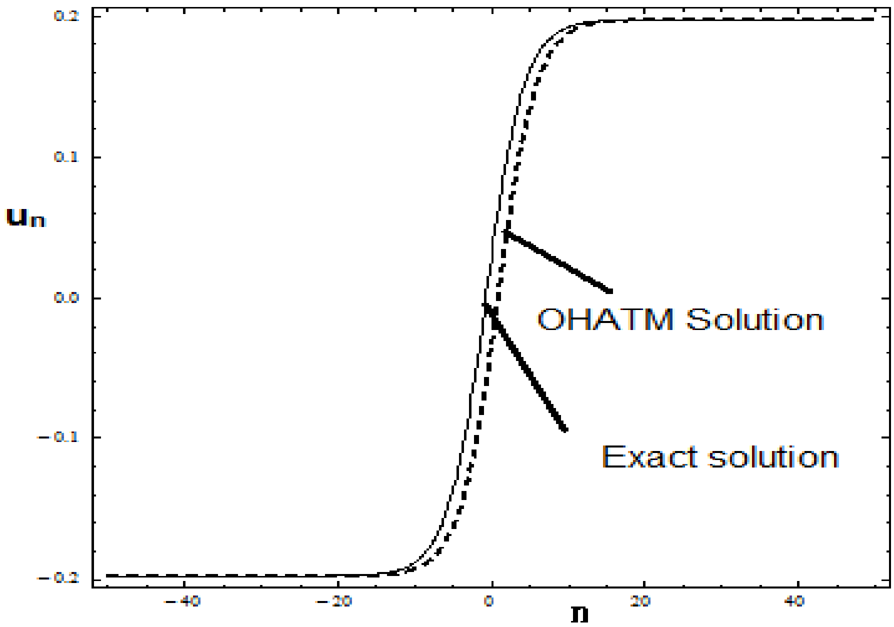

Figure 4: Comparison between exact solution (5.35) and OHATM solution (5.41) at $k = 0.2, t = 0.4, \alpha = 1$ and $h|_{\text{optimal}} = -0.575$.

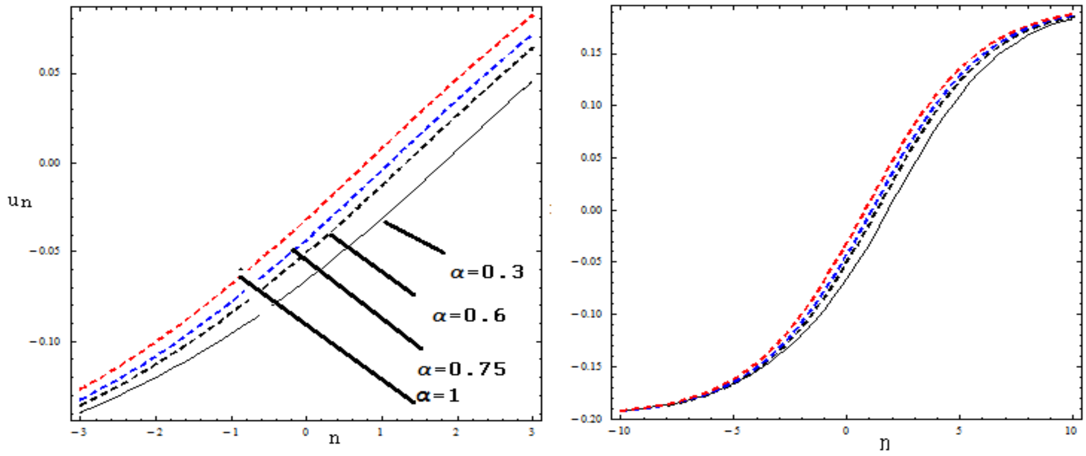

Figure 5: Plot of solutions at different values of α, for $k=0.2, t=0.4$, and $h|_{\text{optimal}} = -0.575$.

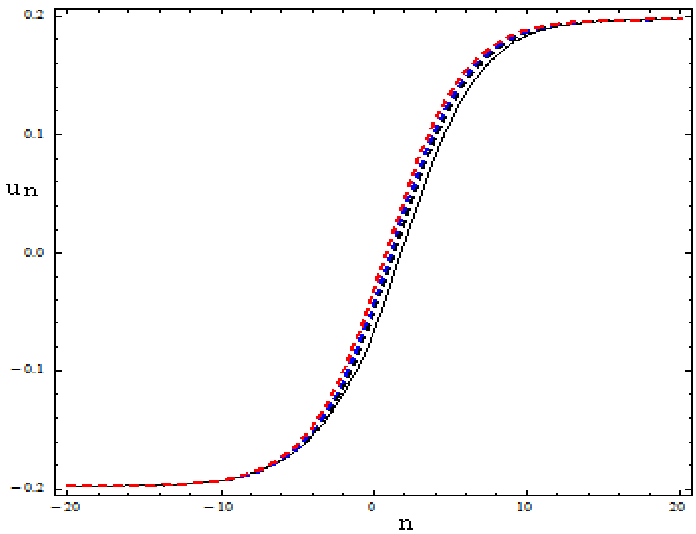

Figure 6: Plot of solutions at different values of α, for $k=0.2, t=0.4$, and $h|_{\text{optimal}} = -0.575$.

The method provides the solutions in terms of series of nonlinear FDDEs which are easily computable components in a direct way without using linearization, perturbation or restrictive assumptions. Moreover, it is observed that the summation of an infinite series got by OHATM usually converges rapidly to the exact solution. In addition, numerical results have confirmed the theoretical results and high accuracy of the proposed scheme. It may be fulfilled that the algorithm of OHATM is very powerful and successful in finding approximate solutions of fractional differential difference equations arising in the field of science, engineering and technology.

6. CONCLUSIONS

In this paper, the OHATM has been successfully applied for solving discontinued problems arising in nanotechnology. The result shows that the OHATM is a powerful and efficient technique in finding exact and approximate solutions for nonlinear differential equations. Also, it can be observed that there is good agreement between the results obtained using the present method and the exact solution. The proposed technique solves the problems using FCT and OHATM. The homotopy analysis transform method utilizes a simple and powerful method to adjust and control the convergence region of the infinite series solution using an auxiliary parameter. The numerical solutions obtained by this modified proposed method indicate that the approach is easy to implement, highly accurate, and computationally very attractive.

A good agreement between the obtained solutions and some well-known results has been obtained. The OHATM requires less computational work compared to other analytical methods. In conclusion, the OHATM may be considered a nice refinement in existing numerical techniques and may find wide applications. It may be fulfilled that the algorithm of OHATM is very powerful and successful in finding approximate solutions of fractional differential difference equations arising in the field of science, engineering and technology. The solution is very rapidly convergent by utilizing the modified HAM by modification of Laplace operator. It may be concluded that the modified HATM methodology is very powerful and efficient in finding approximate solutions as well as analytical solutions of many fractional physical models.

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