On generalized $\ast$-$n$-derivations in $\ast$-rings

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Abstract
Let $R$ be a $\ast$-ring. In this paper we introduce the notion of generalized $\ast$-$n$-derivation in $R$. An additive mapping $x \rightarrow x^\ast$ of $R$ into itself is called an involution on $R$ if it satisfies the conditions: (i) $(x^\ast)^\ast = x$, (ii) $(xy)^\ast = y^\ast x^\ast$ for all $x, y \in R$. A ring $R$ equipped with an involution $\ast$ is called a $\ast$-ring. It is shown that if a prime $\ast$-ring $R$ admits a nonzero generalized $\ast$-$n$-derivation $F$ (resp. a reverse generalized $\ast$-$n$-derivation) equipped with a $\ast$-$n$-derivation (resp. a reverse $\ast$-$n$-derivation) $D$, then $R$ is commutative. Further, some related properties of generalized $\ast$-$n$-derivation in a semiprime $\ast$-ring have also been investigated.

AMS subject classification:
Keywords: Associative ring, involution, derivation, reverse derivation, $\ast$-derivation, reverse $\ast$-derivation, $\ast$-$n$-derivation, reverse $\ast$-$n$-derivation, generalized $\ast$-$n$-derivation, reverse generalized $\ast$-$n$-derivation, prime $\ast$-ring, semi prime $\ast$-ring.

1. Introduction
Throughout the paper, $R$ will represent an associative ring with centre $Z(R)$. The ring $R$ is called a prime ring if $xRy = 0$ implies $x = 0$ or $y = 0$. It is called semi prime if $xRx = 0$ implies $x = 0$. Given an integer $n > 1$, the ring $R$ is said to be $n$-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. An additive mapping $D : R \rightarrow R$ is said to be a derivation (resp. a reverse derivation) on $R$ if $D(xy) = D(x)y + xD(y)$ (resp. $D(xy) = D(y)x + yD(x)$) holds for all $x, y \in R$. An additive mapping $x \rightarrow x^\ast$ of $R$ into itself is called an involution on $R$ if it satisfies the conditions:
A ring $R$ equipped with an involution $'*$ is called a $*$-ring. An additive mapping $D : R \to R$ is called a $*$-derivation (resp. a $*$-reverse derivation) on $R$ if $D(xy) = D(x)y^* + xD(y)$ (resp. $D(xy) = D(y)x^* + yD(x)$) holds for all $x, y \in R$. Let $R$ be a commutative $*$-ring. Then $D : R \to R$ defined by $D(x) = a(x - x^*)$, where $a \in R$, is a $*$-derivation on $R$ (see [7]). An additive map $T : R \to R$ is called a left (resp. right) $*$-multiplier if $T(xy) = T(x)y^*$ (resp. $T(xy) = x^*T(y)$) holds for all $x, y \in R$.

There are many works dealing with the commutativity of prime and semi prime rings admitting certain types of derivations (see [3-6,8,12,17]). Ali [2] defined symmetric $*$-bi derivation, a symmetric left (resp. right) $*$-bi multiplier and studied some properties of prime $*$-rings and semi prime $*$-rings, possessing symmetric $*$-bi derivation and a symmetric left (resp. right) $*$-bi multiplier. Motivated by these concepts and the notion of $n$-derivation given by Park (see [11]). Very recently Ashraf [18] defined the concept of $*$-$n$-derivation in prime $*$-rings and semi prime $*$-rings and study some of their properties. Thus the notion of $*$-$n$-derivation gives a generalization of $*$-derivation earlier known to us in $*$-rings.

Let $n$ be a fixed positive integer. An $n$-additive mapping $D : R \times R \times \cdots \times R \to R$ is called a $*$-$n$-derivation of $R$ if the relations

$$D(x_1, x_2, \ldots, x_n) = D(x_1, x_2, \ldots, x_n)(x'_1)^* + x_1D(x_1, x_2, \ldots, x_n)$$

$$D(x_1, x_2x_2', \ldots, x_n) = D(x_1, x_2, \ldots, x_n)(x'_2)^* + x_2D(x_1, x_2', \ldots, x_n)$$

$$\vdots$$

$$D(x_1, x_2, \ldots, x_n, x'_n) = D(x_1, x_2, \ldots, x_n)(x'_n)^* + x_nD(x_1, x_2, \ldots, x'_n)$$

holds for all $x_1, x'_1, x_2, x_2', \ldots, x_n, x'_n \in R$. Similarly, an $n$-additive mapping $D : R \times R \times \cdots \times R \to R$ is called the reverse $*$-$n$-derivation of $R$ if the relations

$$D(x_1x'_1, x_2, \ldots, x_n) = D(x_1', x_2, \ldots, x_n)x'_1^* + x'_1D(x_1, x_2, \ldots, x_n)$$

$$D(x_1, x_2x'_2, \ldots, x_n) = D(x_1, x'_2, \ldots, x_n)x'_2^* + x'_2D(x_1, x_2', \ldots, x_n)$$

$$\vdots$$

$$D(x_1, x_2, \ldots, x_n, x'_n) = D(x_1, x_2, \ldots, x'_n)x'_n^* + x'_nD(x_1, x_2, \ldots, x'_n)$$
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\[ D(x_1, x_2, \ldots, x_n x'_n) = D(x_1, x_2, \ldots, x'_n)x^*_n + x'_n D(x_1, x_2, \ldots, x_n) \]

holds for all \( x_1, x'_1, x_2, x'_2, \ldots, x_n, x'_n \in R \). As an example of *-n-derivation, consider \( C \) be the ring of complex numbers with involution * defined by \( z^* = \overline{z} \), where \( \overline{z} \) denotes the conjugate of the complex number \( z \). Now define \( D : C \times C \times \cdots \times C \to C \) such that
\[ D(z_1, z_2, \ldots, z_n) = \lambda (z_1 - \overline{z_1})(z_2 - \overline{z_2}) \cdots (z_n - \overline{z_n}) \]
where \( \lambda \) is any fixed complex number. We can easily verify that \( D \) is a *-n-derivation of \( C \).

An n-additive mapping \( T_1 : R \times R \times \cdots \times R \to R \) is called the left *-n-multiplier of \( R \) if
\[ T_1(x_1 x'_1, x_2, \ldots, x_n) = T_1(x_1, x_2, \ldots, x_n)(x'_1)^* \]
\[ T_1(x_1, x_2 x'_2, \ldots, x_n) = T_1(x_1, x_2, \ldots, x_n)(x'_2)^* \]
\[ \ldots \]
\[ T_1(x_1, x_2, \ldots, x_n x'_n) = T_1(x_1, x_2, \ldots, x_n)(x'_n)^* \]
holds for all \( x_1, x'_1, x_2, x'_2, \ldots, x_n, x'_n \in R \). An additive mapping \( T_2 : R \times R \times \cdots \times R \to R \) is called the right *-n-multiplier of \( R \) if
\[ T_2(x_1 x'_1, x_2, \ldots, x_n) = (x'_1)^* T_2(x_1, x_2, \ldots, x_n) \]
\[ T_2(x_1, x_2 x'_2, \ldots, x_n) = (x'_2)^* T_2(x_1, x_2, \ldots, x_n) \]
\[ \ldots \]
\[ T_2(x_1, x_2, \ldots, x_n x'_n) = (x'_n)^* T_2(x_1, x_2, \ldots, x_n) \]
holds for all \( x_1, x'_1, x_2, x'_2, \ldots, x_n, x'_n \in R \). As examples of a left *-n-multiplier and a right *-n-multiplier.

**Example 1.1.** Consider \( S \) to be a commutative ring which is not a zero ring and
\[ R = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid 0, x, y, z \in S \right\}. \]
\[
T_1\left( \begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

\[
T_2\left( \begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

and
\[
\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & z & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}.
\]

One can easily verify that \( * \) is an involution on \( R \). Also it is straightforward to check that \( T_1 \) is a nonzero left \( * \)-n-multiplier but not a right \( * \)-n-multiplier of the \( * \)-ring and \( T_2 \) is a nonzero right \( * \)-n-multiplier but not a left \( * \)-n-multiplier of the \( * \)-ring. Finally, an \( n \)-additive map \( T : R \times R \times \cdots \times R \to R \) is called a \( * \)-n-multiplier of \( R \) if it is both the left \( * \)-n-multiplier and right \( * \)-n-multiplier of \( R \). As an example of a \( * \)-n-multiplier, consider \( C \) to be the ring of complex numbers with involution \( * \) defined by \( z^* = \overline{z} \), where \( \overline{z} \) denotes the conjugate of the complex numbers \( z \). Now define \( T : C \times C \times \cdots \times C \to C \) such that \( T(z_1, z_2, \ldots, z_n) = \mu \overline{z}_1 \overline{z}_2 \cdots \overline{z}_n \), where \( \mu \) is any fixed complex number. One can easily verify that \( T \) is a \( * \)-n-multiplier of \( C \).

Now we define generalized \( * \)-derivation (resp. reverse generalized \( * \)-derivation). An additive map \( F : R \to R \) is said to be a generalized derivation if there exist derivation \( D \) on \( R \) such that
\[
F(xy) = F(x)y + xD(y) \quad \text{for all} \quad x, \ y, \ z \in R.
\]

An additive map \( F : R \to R \) is said to be a generalized \( * \)-derivation if there exist a \( * \)-derivation \( D \) on \( R \) such that
\[
F(xy) = F(x)y^* + xD(y) \quad \text{for all} \quad x, \ y, \ z \in R.
\]

or
\[
F(xy) = F(y)x^* + yD(x) \quad \text{for all} \quad x, \ y, \ z \in R.
\]

An additive map \( F : R \times R \times \cdots \times R \to R \) is called generalized \( * \)-n-derivation if there exist a \( * \)-n-derivation \( D : R \times R \times \cdots \times R \to R \) such that
\[
F(x_1x_1', x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)(x_1')^* + x_1D(x_1', x_2, \ldots, x_n)
\]

\[
F(x_1, x_2x_2', \ldots, x_n) = F(x_1, x_2, \ldots, x_n)(x_2')^* + x_2D(x_1, x_2', \ldots, x_n)
\]
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$F(x_1,x_2,\ldots,x_n) = F(x_1,x_2,\ldots,x_n)(x_n)^* + x_nD(x_1,x_2,\ldots,x_n)$

and reverse generalized $\ast$-n-derivation defined as:

$F(x_1x'_1,x_2,\ldots,x_n) = F(x_1,x_2,\ldots,x_n)(x_1)^* + x_1D(x_1,x_2,\ldots,x_n)$

$F(x_1x'_2,x_2,\ldots,x_n) = F(x_1,x_2,\ldots,x_n)(x_2)^* + x_2D(x_1,x_2,\ldots,x_n)$

$F(x_1,x_2,\ldots,x_n) = F(x_1,x_2,\ldots,x_n')D(x_1,x_2,\ldots,x_n)$

holds for all $x_i,x'_i \in R; i = 1, 2, \ldots, n$. As example of generalized $\ast$-n-derivation is:

**Example 1.2.** Let $S$ be a commutative ring which is not a zero ring and

$R = \left\{ \begin{pmatrix} 0 & x & y & \cdots & 0 \\ 0 & 0 & z & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \mid 0, x, y, z \in S \right\}.$

$F : R \times R \times \cdots \times R \rightarrow R$ and $r \rightarrow r^*$ such that

$F\left( \begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x_1x_2\cdots x_n \end{pmatrix}.$

there exist a $\ast$-n-derivation $D$ such that

$D\left( \begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & z_1z_2\cdots z_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

and

$\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & z & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}.$

In 1989, Bresar and Vukman (see [7, Proposition 1]) proved that if a prime $\ast$-ring $R$ admits a nonzero $\ast$-derivation (resp. $\ast$-derivation) $D$, then $R$ is commutative. In this paper, we prove its analogue in the setting of the $\ast$-n-generalized derivation for prime $\ast$-rings. We obtain some results related with $\ast$-n-multipliers in prime $\ast$-rings and semiprime $\ast$-rings. In fact our results generalize, extend and complement several results obtained earlier on the $\ast$-n-derivation, symmetric $\ast$-biderivation for prime $\ast$-rings and semiprime $\ast$-rings.
2. Preliminary Results

We begin with Lemma, most of which have been proved elsewhere.

**Lemma 2.1.** Let $R$ be a $*$-prime ring having generalized $*n$-derivations $F_1$ and $F_2$ and $D_1$ and $D_2$ corresponding $*n$-derivations. Further assume that $I_1, I_2, \ldots, I_n$ are non-zero ideals of $R$ such that $F_1(i_1, i_2, \ldots, i_n) = F_2(i_1, i_2, \ldots, i_n)$ for all $i_r \in I_r$, $1 \leq r \leq n$, then $D_1 = D_2$.

**Proof.** We have

$$F_1(i_1, i_2, \ldots, i_n) = F_2(i_1, i_2, \ldots, i_n) \quad \text{for all } i_r \in I_r; \quad 1 \leq r \leq n \quad (2.1)$$

Now putting $i_1 r_1$, where $r_1 \in R$, $i_1 \in I_1$ in relation (2.1), We obtain

$$F_1(i_1 r_1, i_2, \ldots, i_n) = F_2(i_1 r_1, i_2, \ldots, i_n)$$

using relation (2.1), we get

$$i_1 D_1(r_1, i_2, \ldots, i_n) = i_1 D_2(r_1, i_2, \ldots, i_n)$$

i.e.

$$i_1 D_1(r_1, i_2, \ldots, i_n) - i_1 D_2(r_1, i_2, \ldots, i_n) = 0.$$  

This shows that

$$i_1 R\{D_1(r_1, i_2, \ldots, i_n) - D_2(r_1, i_2, \ldots, i_n)\} = \{0\}.$$  

Since $I_1 \neq \{0\}$, the primeness of $R$ implies that

$$D_1(r_1, i_2, \ldots, i_n) = D_2(r_1, i_2, \ldots, i_n) \quad \text{for all } i_r \in I_r; \quad 1 \leq r \leq n. \quad (2.2)$$

Now putting $i_2 r_2$, where $r_2 \in R$, $i_2 \in I_2$ in (2.2), we get

$$D_1(r_1, i_2 r_2, \ldots, i_n) = D_2(r_1, i_2 r_2, \ldots, i_n)$$

i.e.

$$D_1(r_1, i_2, \ldots, i_n) r_2^* + i_2 D_1(r_1, r_2, \ldots, i_n) = D_2(r_1, i_2, \ldots, i_n) r_2^* + i_2 D_2(r_1, r_2, \ldots, i_n).$$

By (2.2), we get

$$i_2 D_1(r_1, r_2, i_3, \ldots, i_n) = i_2 D_2(r_1, r_2, i_3, \ldots, i_n)$$

$$i_2 D_1(r_1, r_2, i_3, \ldots, i_n) - i_2 D_2(r_1, r_2, i_3, \ldots, i_n) = 0.$$
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This shows that

$$i_2 R[D_1(r_1, r_2, i_3, \ldots, i_n) - D_2(r_1, r_2, i_3, \ldots, i_n)] = \{0\}.$$ 

Since $I_2 \neq \{0\}$, the primeness of $R$ implies that

$$D_1(r_1, r_2, i_3, \ldots, i_n) - D_2(r_1, r_2, i_3, \ldots, i_n) = 0.$$ 

Now, continuing this process, we conclude that

$$D_1 = D_2.$$ 

3. Main Result

In 1989, Bresar and Vukman (see [7, Proposition 1]) proved that if a prime $*$-ring $R$ admits a nonzero $*$-derivation (resp reverse $*$-derivation) $D$, then $R$ is commutative. We obtained the following.

**Theorem 3.1.** Let $R$ be a prime $*$-ring and if it admits a nonzero generalized $*$-$n$-derivation (resp. reverse generalized $*$-$n$-derivation) $F$ associated with a $*$-$n$-derivation $D$. Then $R$ is commutative ring.

**Proof.** By the hypothesis, we have for all $x_1, y, z, x_2, \ldots, x_n \in R$

$$F((x_1 y)z, x_2, \ldots, x_n) = F(x_1 y, x_2, \ldots, x_n)z^* + x_1 y D(z, x_2, \ldots, x_n)$$

$$F((x_1 y)z, x_2, \ldots, x_n) = \{F(x_1, x_2, \ldots, x_n)y^* + x_1 D(y, x_2, \ldots, x_n)\}z^* + x_1 y D(z, x_2, \ldots, x_n)$$

$$F((x_1 y)z, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)y^*z^* + x_1 D(y, x_2, \ldots, x_n)z^* + x_1 y D(z, x_2, \ldots, x_n)$$

(3.1)

Also,

$$F(x_1(yz), x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)(yz)^* + x_1 D(yz, x_2, \ldots, x_n)$$

$$F(x_1(yz), x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)z^*y^* + x_1\{D(y, x_2, \ldots, x_n)z^* + yD(z, x_2, \ldots, x_n)\}$$

$$F(x_1(yz), x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)z^*y^* + x_1\{D(y, x_2, \ldots, x_n)z^* + x_1 y D(z, x_2, \ldots, x_n)\}.$$ 

(3.2)
Combining (3.1) and (3.2), we get
\[ F(x_1, x_2, \ldots, x_n)z^*y^* = F(x_1, x_2, \ldots, x_n)y^*z^*. \]
Putting \(y\) and \(z\) instead of \(y^*\) and \(z^*\), respectively. We find that,
\[ F(x_1, x_2, \ldots, x_n)z y = F(x_1, x_2, \ldots, x_n) y z. \quad (3.3) \]
Now replacing \(y\) by \(y r\), where \(r \in R\) in (3.3), and using it again
\[ F(x_1, x_2, \ldots, x_n) y r z = F(x_1, x_2, \ldots, x_n) y z r \]
thus
\[ F(x_1, x_2, \ldots, x_n) R[r, z] = 0 \]
Since \(F \neq \{0\}\), \(*\)-primeness of \(R\) implies that \(rz = zr\) for all \(z, r \in R\). Therefore \(R\) is commutative. \(\qed\)

**Theorem 3.2.** Let \(R\) be a 2-torsion free prime \(*\)-ring possessing generalized \(*\)-\(n\)-derivations \(F_1\) and \(F_2\) associated with \(*\)-\(n\)-derivations \(D_1\) and \(D_2\). If \(F_1(x_1, x_2, \ldots, x_n) D_2(y_1, y_2, \ldots, y_n) + F_2(x_1, x_2, \ldots, x_n) D_1(y_1, y_2, \ldots, y_n) = 0\) for all \(x_1, x_2, \ldots, x_n;\)
\(y_1, y_2, \ldots, y_n \in R\) then either \(F_1 = 0\) or \(F_2\) is a left multiplier and either \(F_2 = 0\) or \(F_1\) is a left multiplier.

**Proof.** Suppose that
\[ F_1(x_1, x_2, \ldots, x_n) D_2(y_1, y_2, \ldots, y_n) + F_2(x_1, x_2, \ldots, x_n) D_1(y_1, y_2, \ldots, y_n) = 0. \]
Replacing \(y_1 z\) instead of \(y_1\), we get
\[ F_1(x_1, x_2, \ldots, x_n) D_2(y_1 z, y_2, \ldots, y_n) + F_2(x_1, x_2, \ldots, x_n) D_1(y_1 z, y_2, \ldots, y_n) = 0 \]
\[ F_1(x_1, x_2, \ldots, x_n)(D_2(y_1, y_2, \ldots, y_n)z^* + y_1 D_2(z, y_2, \ldots, y_n)) \]
\[ + F_2(x_1, x_2, \ldots, x_n)(D_1(y_1, y_2, \ldots, y_n)z^* \]
\[ + y_1 D_1(z, y_2, \ldots, y_n)) = 0 \]
\[ F_1(x_1, x_2, \ldots, x_n) D_2(y_1, y_2, \ldots, y_n)z^* + F_1(x_1, x_2, \ldots, x_n) y_1 D_2(z, y_2, \ldots, y_n) \]
\[ + F_2(x_1, x_2, \ldots, x_n) D_1(y_1, y_2, \ldots, y_n)z^* \]
\[ + y_1 D_1(z, y_2, y_3, \ldots, y_n) = 0 \]
\[ (F_1(x_1, x_2, \ldots, x_n) D_2(y_1, y_2, \ldots, y_n) + F_2(x_1, x_2, \ldots, x_n) D_1(y_1, y_1, \ldots, y_n))z^* \]
\[ + F_1(x_1, x_2, \ldots, x_n) y_1 D_2(z, y_2, \ldots, y_n) \]
\[ + F_2(x_1, x_2, \ldots, x_n) y_1 D_1(z, y_2, y_3, \ldots, y_n) = 0. \]
By the given hypothesis,

\[ F_1(x_1, x_2, \ldots, x_n)y_1D_2(z, y_2, \ldots, y_n) + F_2(x_1, x_2, \ldots, x_n)y_1D_1(z, y_2, \ldots, y_n) = 0. \tag{3.4} \]

Now we multiplying (3.4) from the right by \( pD_1(r_1, r_2, \ldots, r_n) \); and using (3.4), where \( r_i, p \in R \) we get

\[
F_1(x_1, x_2, \ldots, x_n)y_1D_2(z, y_2, \ldots, y_n)pD_1(r_1, r_2, \ldots, r_n) + F_2(x_1, x_2, \ldots, x_n)y_1D_1(z, y_2, \ldots, y_n)pD_1(r_1, r_2, \ldots, r_n) = 0.
\]

Since \( R \) is 2-torsion free, we get

\[
F_2(x_1, x_2, \ldots, x_n)y_1D_1(z, y_2, \ldots, y_n)pD_1(r_1, r_2, \ldots, r_n) = 0
\]

By \(*\)-primeness of \( R \), either

\[
F_2(x_1, x_2, \ldots, x_n)y_1D_1(z, y_2, \ldots, y_n) = 0
\]

or

\[
D_1(r_1, r_2, \ldots, r_n) = 0.
\]

But in first,

\[
F_2(x_1, x_2, \ldots, x_n)y_1D_1(z, y_2, \ldots, y_n) = 0
\]

\[
F_2(x_1, x_2, \ldots, x_n)\ast RD_1(z, y_2, \ldots, y_n) = \{0\}.
\]

By \(*\)-primeness of \( R \), either \( F_2(x_1, x_2, \ldots, x_n) = 0 \) or \( D_1(z, y_2, \ldots, y_n) = 0 \) this means that \( F_2 = 0 \) or \( F_1 \) is a left multiplier.

Now, multiplying (3.4) from left by \( D_2(r_1, r_2, \ldots, r_n)p \), where \( r_i, p \in R \), we get

\[
D_2(r_1, r_2, \ldots, r_n)pF_1(x_1, x_2, \ldots, x_n)y_1D_2(z, y_2, \ldots, y_n) + D_2(r_1, r_2, \ldots, r_n)pF_2(x_1, x_2, \ldots, x_n)y_1D_1(z, y_2, y_3, \ldots, y_n) = 0.
\]

Using (3.4) and the fact that \( R \) is 2-torsion free, we find

\[
D_2(r_1, r_2, \ldots, r_n)pF_1(x_1, x_2, \ldots, x_n)y_1D_2(z, y_2, \ldots, y_n) = 0
\]

i.e.

\[
D_2(r_1, r_2, \ldots, r_n)RF_1(x_1, x_2, \ldots, x_n)y_1D_2(z, y_2, \ldots, y_n) = \{0\}
\]

\[
D_2(r_1, r_2, \ldots, r_n)\ast RF_1(x_1, x_2, \ldots, x_n)y_1D_2(z, y_2, \ldots, y_n) = \{0\}.
\]
Since $R$ is $\ast$-prime, we get either $D_2(r_1, r_2, \ldots, r_n) = 0$ or $F_1(x_1, x_2, \ldots, x_n)y_1 D_2(z, y_2, \ldots, y_n) = 0$ from second relation,

$$F_1(x_1, x_2, \ldots, x_n)y_1 D_2(z, y_2, \ldots, y_n) = 0.$$ 

This gives

$$F_1(x_1, x_2, \ldots, x_n)^* RD_2(z, y_2, \ldots, y_n) = \{0\}$$

either $F_1(x_1, x_2, \ldots, x_n) = 0$ or $D_2(z, y_2, \ldots, y_n) = 0$ i.e. $F_1 = 0$ or $D_2 = 0$. □

**Theorem 3.3.** Let $R$ be a semiprime $\ast$-ring admitting a generalized $\ast$-$n$-derivation $F$. then $F(R, R, \ldots, R) \subseteq Z$.

**Proof.** Since $R$ is a $\ast$-ring and having a generalized $\ast$-$n$-derivation $F$, we have relation (3.3) in theorem (3.1).

Now putting $yF(x_1, x_2, \ldots, x_n)$ instead of $y$, we get

$$F(x_1, x_2, \ldots, x_n)y[F(x_1, x_2, \ldots, x_n), z] = 0 \text{ for all } x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z \in R.$$ 

This relation gives

$$ZF(x_1, x_2, \ldots, x_n)y[F(x_1, x_2, \ldots, x_n), z] = 0 \quad (3.5)$$

Now replacing $y$ by $zy$ in the relation $F(x_1, x_2, \ldots, x_n)y[F(x_1, x_2, \ldots, x_n), z] = 0$, we obtain

$$F(x_1, x_2, \ldots, x_n)zy[F(x_1, x_2, \ldots, x_n), z] = 0 \quad (3.6)$$

comparing (3.5) and (3.6), we get

$$F(x_1, x_2, \ldots, x_n)zy[F(x_1, x_2, \ldots, x_n), z] = zF(x_1, x_2, \ldots, x_n)y[F(x_1, x_2, \ldots, x_n), z].$$

This means that

$$[F(x_1, x_2, \ldots, x_n), z]y[F(x_1, x_2, \ldots, x_n), z] = 0$$

i.e.

$$[F(x_1, x_2, \ldots, x_n), z]R[F(x_1, x_2, \ldots, x_n), z] = \{0\}.$$ 

This gives

$$[F(x_1, x_2, \ldots, x_n), z]^* R[F(x_1, x_2, \ldots, x_n), z]^* = \{0\}.$$

Now by the $\ast$-semiprimeness of $R$, yields that

$$[F(x_1, x_2, \ldots, x_n), z] = 0$$
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i.e.

$$F(R, R, \ldots, R) \subseteq Z.$$  

Theorem 3.4. Let $R$ be a semiprime ring with involution $*$. If $F$ is a generalized $*$-$n$-derivation of $R$ associated with a $*$-$n$-derivation $D$ such that

$$F(x_1, x_2, \ldots, x_n)y_1 = x_1F(y_1, y_2, \ldots, y_n)$$

for all $x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n \in R,$

then $F$ is a left multiplier.

Proof. By the given hypothesis, we have

$$F(x_1, x_2, \ldots, x_n)y_1 = x_1F(y_1, y_2, \ldots, y_n)$$

for all $x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n \in R$.

Using $y_1z$ instead of $y_1$ where $z \in R$, we obtain

$$F(x_1, x_2, \ldots, x_n)y_1z = x_1F(y_1z, y_2, \ldots, y_n)$$

$$F(x_1, x_2, \ldots, x_n)y_1z = x_1(F(y_1, y_2, \ldots, y_n)z^* + y_1D(z, y_2, y_3, \ldots, y_n))$$

Using hypothesis

$$x_1F(x_1, x_2, \ldots, x_n)z = x_1F(y_1, y_2, \ldots, y_n)z^* + x_1y_1D(z, y_2, y_3, \ldots, y_n).$$

We replace $z^*$ in place of $z$, we get

$$x_1F(x_1, x_2, \ldots, x_n)z = x_1F(y_1, y_2, \ldots, y_n)z + x_1y_1D(z, y_2, y_3, \ldots, y_n)$$

$$x_1y_1D(z, y_2, y_3, \ldots, y_n) = 0.$$ 

Replacing $D(z, y_2, y_3, \ldots, y_n)$ instead of $x_1$, we obtain

$$D(z, y_2, y_3, \ldots, y_n)y_1D(z, y_2, y_3, \ldots, y_n) = 0.$$ 

This implies that

$$D(z, y_2, y_3, \ldots, y_n)RD(z, y_2, y_3, \ldots, y_n) = 0.$$ 

This gives

$$D(z, y_2, y_3, \ldots, y_n)^*RD(z, y_2, y_3, \ldots, y_n)^* = 0.$$ 

Using $*$-semiprimeness $F$ is a left multiplier.  

References