Thermoelastic Response of a Thick Circular Plate due to Heat Generation and its Thermal Stresses

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Abstract

In this paper, an attempt has been made to study thermoelastic response of a thick circular plate occupying the space \( D: 0 \leq r \leq a, -h \leq z \leq h \), due to heat generation with radiation type boundary conditions. Here we apply integral transform techniques to find the thermoelastic solution.

Keywords: Thermo elastic problem, thick Circular Plate, Thermal Stresses, integral transform.

1. INTRODUCTION


This paper is concerned with steady-state thermoelastic problem and transient thermoelastic problem of a thick circular plate occupying the space $D:0 \leq r \leq a, -h \leq z \leq h$, due to heat generation with radiation type boundary conditions.

2. STATEMENT OF THE PROBLEM-I

Consider thick circular plate of thickness $2h$ occupying the space $D:0 \leq r \leq a, -h \leq z \leq h$, the material is homogenous and isotropic. The differential equation governing the displacement potential function $\phi(r,z,t)$ as Nowacki [14] is

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \left(\frac{1 + \nu}{1 - \nu}\right) \alpha_t T$$

(2.1)

where $\nu$ and $\alpha_t$ are Poisson’s ratio and linear coefficient of thermal expansion of the material of the plate and $T$ is the temperature of the plate satisfying the differential equation as Noda et al. [11] is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \chi(r,z) = 0$$

(2.2)

Subject to the boundary conditions

$$M_\phi(T,0,1,a) = g(z) \quad -h \leq z \leq h,$$

$$M_\phi(T,1,k_1,h) = f_1(r) \quad 0 \leq r \leq a$$

$$M_\phi(T,1,k_2,-h) = -\frac{Q_0}{\lambda} f_2(r) \quad 0 \leq r \leq a$$

(2.3)

where $k$ is thermal diffusivity of material of the plate.

The displacement function in the cylindrical coordinate system are represented by Love’s function as Khobragade [8] are
The Love’s function must satisfy
\[ \nabla^2 \nabla^2 L = 0 \] (2.7)
where
\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \]

The component of stresses are represented by the thermoelastic displacement potential \( \phi \) and Love’s function \( L \) as Noda et al. [11] are

\[ \sigma_{rr} = 2G \left\{ \frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi \right\} + \frac{\partial}{\partial z} \left( \nu \nabla^2 L - \frac{\partial^2 L}{\partial r^2} \right) \] (2.8)

\[ \sigma_{\theta\theta} = 2G \left\{ \frac{1}{r} \frac{\partial \phi}{\partial r} - \nabla^2 \phi \right\} + \frac{\partial}{\partial z} \left( \nu \nabla^2 L - \frac{1}{r} \frac{\partial^2 L}{\partial r^2} \right) \] (2.9)

\[ \sigma_z = 2G \left\{ \frac{\partial^2 \phi}{\partial z^2} - \nabla^2 \phi \right\} + \frac{\partial}{\partial z} \left( (z-\nu) \nabla^2 L - \frac{\partial^2 L}{\partial z^2} \right) \] (2.10)

Fig. 1: Shows the geometry of the problem
\[ \sigma_{rr} = 2G \left\{ \frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r} \left\{ (1-\nu)\nabla^2 L - \frac{\partial^2 L}{\partial z^2} \right\} \right\} \]  \hspace{1cm} (2.11)

For traction free surface stress function \( \sigma_z = \sigma_{r\theta} = 0 \) at \( z = \pm h \) for thick plate.

Equations (2.1) to (2.11) constitute the mathematical formulation of the problem under consideration.

3. SOLUTION OF THE PROBLEM

Applying Marchi-Fasulo transform to the equation (2.2), we get

\[ \frac{\partial^2 \overline{T}}{\partial r^2} + \frac{1}{r} \frac{\partial \overline{T}}{\partial r} - \lambda_n^2 \overline{T} = \psi \hspace{1cm} (3.1) \]

where, \( \psi = -\frac{P_n(h)}{k_1} f_1(r) - \frac{P_n(-h) O_0}{k_2} \frac{\lambda}{\lambda} f_2(r) \hspace{1cm} (3.2) \)

Equation (3.1) is a Bessel’s equation whose solution gives

\[ \overline{T} = A I_0(\lambda_n r) + B K_0(\lambda_n r) + \overline{F}(r) \hspace{1cm} (3.3) \]

Where \( \overline{F}(r) \) is the P.I.

As \( r \to 0, K_0 \to \infty \), But \( T \) is finite

\[ \therefore B = 0 \hspace{1cm} (3.4) \]

\[ A = \frac{g - \overline{F}'(a)}{I_0(\lambda_n a)} \hspace{1cm} (3.5) \]

\[ \therefore \overline{T} = \frac{g - \overline{F}'(a)}{I_0(\lambda_n a)} I_0(\lambda_n r) + \overline{F}(r) \hspace{1cm} (3.6) \]

Applying inverse Marchi-Fasulo transform to the equation (3.6) we get

\[ T = \sum_{n=1}^{\infty} \left[ \frac{P_n(z)}{\lambda_n} \frac{g - \overline{F}'(a)}{I_0(\lambda_n a)} I_0(\lambda_n r) + \overline{F}(r) \right] \hspace{1cm} (3.7) \]

and

\[ \phi = \frac{r^2}{4} \left( 1 + \nu \right) \left[ \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n} \frac{g - \overline{F}'(a)}{I_0(\lambda_n a)} I_0(\lambda_n r) + \overline{F}(r) \right] \hspace{1cm} (3.8) \]

We assume L as
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\[ L = \frac{r^2}{4} \left( \frac{1+v}{1-v} \right) a_1 \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n} \frac{g - F'(a)}{I'_0(\lambda_n a)} I_0(\lambda_n r) \]  

(3.9)

4. DETERMINATION OF DISPLACEMENT FUNCTION

Substituting equations (3.8) and (3.9) in equations (3.1) - (3.2) we get

\[ u_r = \frac{a_1}{4} \left( \frac{1+v}{1-v} \right) \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left[ \frac{g - F'(a)}{I'_0(\lambda_n a)} \left( r^2 I_0(\lambda_n r) + 2r I_0(\lambda_n r) \right) \right] \]

\[ \left( P_n(z) - P'_n(z) \right) + P_n(z) \overline{F'}(r) \}

(4.1)

\[ u_z = \frac{r^2}{4} \left( \frac{1+v}{1-v} \right) a_1 \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left[ \frac{g - F'(a)}{I'_0(\lambda_n a)} I_0(\lambda_n r) \left[ (1-2v)P''_n(z) + P'_n(z) \right] + \frac{P'_n(z) \overline{F}(r)}{\lambda_n} \right. \]

\[ + \frac{1+v}{2} a_1 \sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n} \left( \frac{g - F'(a)}{I'_0(\lambda_n a)} \right) \left[ r^2 I''_0(\lambda_n r) + 5rI'_0 + 4I_0(\lambda_n r) \right] \]

(4.2)

5. DETERMINATION OF STRESS FUNCTIONS

Substituting equations (3.8) and (3.9) in equations (2.8) – (2.11) we get

\[ \sigma_{rr} = \frac{Ga_1}{2} \left( \frac{1+v}{1-v} \right) \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left[ \frac{g - F'(a)}{I'_0(\lambda_n a)} \right. \]

\[ \left. \left[ P_n(z) r^2 I_0(\lambda_n r) - P'_n(z) \left( r^2 I_0(\lambda_n r) + \frac{r^2 I'_0(\lambda_n r) \overline{F}(r)}{\overline{g - F'}(a)} \right) \right] \]

\[ + P'_n(z) \left[ (\nu - 1) r^2 I''_0(\lambda_n r) + (5\nu r - 4r) I'_0(\lambda_n r) + 2(2\nu - 1)I_0(\lambda_n r) \right] \]

\[ - P_n(z) \left[ rI'_0(\lambda_n r) + 2I_0(\lambda_n r) \right] \]

(5.1)

\[ \sigma_{00} = \frac{Ga_1}{2} \left( \frac{1+v}{1-v} \right) \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left( \frac{g - F'(a)}{I'_0(\lambda_n a)} \right) \]

\[ \left[ P_n(z) \left( \nu r^2 I_0(\lambda_n r) \right) - P'_n(z) \left( r^2 I_0(\lambda_n r) + \frac{r^2 I'_0(\lambda_n r) \overline{F}(r)}{\overline{g - F'}(a)} \right) \right] \]

\[ + P'_n(z) \left( (\nu - 1) r^2 I''_0(\lambda_n r) + (5\nu r - 4r) I'_0(\lambda_n r) + \left( 4\nu - \frac{2}{r} \right) I_0(\lambda_n r) \right) \]

\[ - P_n(z) \left[ r^2 I''_0(\lambda_n r) + 4r I'_0(\lambda_n r) + 2I_0(\lambda_n r) \right] \]
\[ + \frac{I_0^\prime (\lambda_n a)}{\bar{g} - F(a)} \left( r^2 F''(r) + 4r F'(r) + 2\bar{F}(r) \right) \]  
(5.2)

\[ \sigma_{zz} = \frac{G a}{2} \left( 1 + \nu \right) \sum_{n=1}^{\infty} \frac{\bar{g} - F'(a)}{\lambda_n I_0' (\lambda_n a)} \left\{ P_n^\prime (z) (1 - \nu) r^2 I_0 (\lambda_n r) + P_n^\prime (z) (2 - \nu) \right\} \]
\[ + P_n^\prime (z) \left[ (4 - \nu) I_0' (\lambda_n r) + (5\nu - 4) I_0' (\lambda_n r) + \left( 4\nu - \frac{2}{r} \right) I_0 (\lambda_n r) \right] \]
\[ - P_n (z) \left[ r^2 I_0' (\lambda_n r) + 5 r I_0' (\lambda_n r) + 4 I_0 (\lambda_n r) \right] \]
\[ + \frac{I_0^\prime (\lambda_n a)}{\bar{g} - F'(a)} \left( r^2 F''(r) + 5r F'(r) + 4\bar{F}(r) \right) \]  
(5.3)

\[ \sigma_{rz} = \frac{G a}{2} \left( 1 + \nu \right) \sum_{n=1}^{\infty} \frac{\bar{g} - F'(a)}{\lambda_n I_0' (\lambda_n a)} \left\{ P_n^\prime (z) \nu r \left[ r^2 I_0' (\lambda_n r) + 2 I_0' (\lambda_n r) \right] \right\} \]
\[ + P_n (z) \left[ r I_0' (\lambda_n r) + 2 I_0 (\lambda_n r) + \frac{I_0' (\lambda_n a)}{\bar{g} - F'(a)} \left( r^2 F''(r) + 2\bar{F}'(r) \right) \right] \]
\[ - P_n^\prime (z) (1 - \nu) \left[ r^2 I_0^\prime (\lambda_n r) + 7 r I_0^\prime (\lambda_n r) + 10 I_0^\prime (\lambda_n r) + \frac{2}{r} I_0 (\lambda_n r) \right] \]  
(5.4)

6. SPECIAL CASE

Set \( F(r, z) = z^2 (1 - r^2) \)

Applying Marchi-Fasulo transform, we obtain

\[ \bar{F}(r, n) = (1 - r^2) \int_{-h}^{h} z^2 P_n (z) dz \]

\[ \bar{F}(r, n) = (1 - r^2) \Phi_n \left[ \frac{2h^2 \sin(a_n h)}{a_n} + \frac{4h \cos(a_n h)}{a_n^2} - \frac{4 \sin(a_n h)}{a_n^3} \right] \]  
(6.2)

Where

\( P_n (z) = Q_n \cos(a_n z) - W_n \sin(a_n z), \)

\[ Q_n = a_n \left( \alpha_1 + \alpha_2 \right) \cos(a_n h) + (\beta_1 - \beta_2) \sin(a_n h) \]
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\[ W_n = (\beta_1 - \beta_2) \cos(a_n h) + a_n (\alpha_1 - \alpha_2) \sin(a_n h) \]

Again on applying Hankel transform, we obtain

\[ \overline{F}^+(m,n) = \prod_n \left[ \frac{a}{\xi_m} J_1(a \xi_m) - \frac{a(a^2 \xi_m^2 - 4)}{\xi_m^3} J_1(a \xi_m) - \frac{2a^2}{\xi_m^2} J_0(a \xi_m) \right] \]

(6.3)

Where

\[ \Pi_n = \Phi_n \left[ \frac{2h^2 \sin(a_n h)}{a_n} + \frac{4h \cos(a_n h)}{a_n^2} - \frac{4 \sin(a_n h)}{a_n^3} \right] \]

And

\[ \Phi_n = a_n (\alpha_1 + \alpha_2) \cos(a_n h) + (\beta_1 - \beta_2) \sin(a_n h). \]

Using equation (6.3) in equation (3.7), one obtains

\[ T = \sum_{n=1}^{\infty} \left[ \frac{P_n(z)}{\lambda_n} \frac{\bar{g} - \bar{F}'(a)}{I_0(\lambda_n r) + (1 - r^2)} \Phi_n \left[ \frac{2h^2 \sin(a_n h)}{a_n} + \frac{4h \cos(a_n h)}{a_n^2} - \frac{4 \sin(a_n h)}{a_n^3} \right] \right] \]

(6.4)

7. NUMERICAL RESULTS

Set \( a = 2, k = 15.9 \times 10^6, t = 1 \) second in equation (6.4) we get

\[ T = \sum_{n=1}^{\infty} \left[ \frac{P_n(z)}{\lambda_n} \frac{\bar{g} - \bar{F}'(a)}{I_0(\lambda_n r) + (1 - r^2)} \Phi_n \left[ \frac{2h^2 \sin(a_n h)}{a_n} + \frac{4h \cos(a_n h)}{a_n^2} - \frac{4 \sin(a_n h)}{a_n^3} \right] \right] \]

(7.1)

8. STATEMENT OF THE PROBLEM-II

Consider thick circular plate of thickness \( 2h \) occupying the space \( D: 0 \leq r \leq a, -h \leq z \leq h \), the material is homogenous and isotropic. The differential equation governing the displacement potential function \( \phi(r,z,t) \) as Nowacki [14] is

\[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \left( \frac{1+v}{1-v} \right) \alpha_t T \]

(8.1)

where \( v \) and \( \alpha_t \) are Poisson’s ratio and linear coefficient of thermal expansion of the material of the plate and \( T \) is the temperature of the plate satisfying the differential equation as is
\[
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \chi(r, z, t) = \frac{1}{k} \frac{\partial T}{\partial t}
\] (8.2)

Subject to initial condition
\[
M_z(T, 1, 0, 0) = F(r, z) \quad 0 \leq r \leq a, -h \leq z \leq h.
\] (8.3)

The boundary conditions are
\[
M_z(T, 0, 1, a) = g(z, t) \quad -h \leq z \leq h, \quad t > 0
\] (8.4)
\[
M_z(T, 1, k_1, h) = f_1(r, t) \quad 0 \leq r \leq a, \quad t > 0
\] (8.5)
\[
M_z(T, 1, k_2, -h) = -\frac{Q}{k} f_2(r, t) \quad 0 \leq r \leq a, \quad t > 0
\] (8.6)

where \( k \) is thermal diffusivity of material of the plate.

The displacement function in the cylindrical coordinate system are represented by Love’s function as Khobragade [8] are

\[
u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 L}{\partial r \partial z}
\] (8.6)
\[
u_z = \frac{\partial \phi}{\partial z} + 2(1 - \nu) \nabla^2 L - \frac{\partial^2 L}{\partial z^2}
\] (8.7)

Love’s function \( L \) [10] must satisfy
\[
\nabla^2 \nabla^2 L = 0
\] (8.8)

where
\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}
\]

The component of stresses are represented by the thermoelastic displacement potential \( \phi \) and Love’s function \( L \) as Noda et al. [11] are

\[
\sigma_{rr} = 2G \left[ \frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi \right] + \frac{\partial}{\partial z} \left[ \nu \nabla^2 L - \frac{\partial^2 L}{\partial r^2} \right]
\] (8.9)
\[
\sigma_{\theta\theta} = 2G \left[ \frac{1}{r} \frac{\partial \phi}{\partial \theta} - \nabla^2 \phi \right] + \frac{\partial}{\partial z} \left[ \nu \nabla^2 L - \frac{1}{r} \frac{\partial^2 L}{\partial r^2} \right]
\] (8.10)
\[
\sigma_{zz} = 2G \left[ \frac{\partial^2 \phi}{\partial z^2} - \nabla^2 \phi \right] + \frac{\partial}{\partial z} \left[ (2 - \nu) \nabla^2 L - \frac{\partial^2 L}{\partial z^2} \right]
\] (8.11)
\[ \sigma_{zz} = 2G \left[ \frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r} \left\{ \left(1 - \nu \right) \nabla^2 L - \frac{\partial^2 L}{\partial z^2} \right\} \right] \]

(8.12)

For traction free surface stress function
\[ \sigma_z = \sigma_{r0} = 0 \text{ at } z = \pm h \text{ for thick plate.} \]

Equations (8.1) to (8.12) constitute the mathematical formulation of the problem under consideration.

9. SOLUTION OF THE PROBLEM

Applying Hankel transform to the equation (8.2), we get

\[ -\xi_m^2 \overline{T}(\xi_m, z, t) + \frac{d^2 \overline{T}}{dz^2}(\xi_m, z, t) + \frac{1}{\kappa} \overline{T} \frac{1}{k} \frac{d \overline{T}}{dt} = \frac{1}{k} \frac{d \overline{T}}{dt} \]

(9.1)

Again applying Marchi-Fasulo transform to above equation, we obtain

\[ \frac{dT^*}{dt} + kp^2 T^* = \Pi \]

(9.2)

where
\[ p^2 = \xi_m^2 + a_n^2 \]

Equation (9.2) is a linear equation whose solution is given by
\[
\overline{T}(\xi_m, n, t) = e^{-kp^2t} \int_0^t \prod e^{kp^2t} dt' + Ce^{-kp^2t}
\] (9.3)

Using (8.3), we get
\[
C = F^*(m,n)
\]

Thus we have
\[
\overline{T}(\xi_m, n, t) = e^{-kp^2t} \left[ \int_0^t \prod e^{kp^2t} dt' + F^*(m,n) \right]
\] (9.4)

Applying inversion of Marchi-Fasulo transform and Hankel transform to the differential equation (9.4), we get
\[
T(r, z, t) = \frac{2}{a^2} \sum_m \sum_n \frac{J_0(r \xi_m)}{[J_1(a \xi_m)]^2} \frac{P_n(z)}{\lambda_n} e^{-kp^2t} \times \left[ \int_0^t \prod e^{kp^2t} dt' + F^*(m,n) \right]
\] (9.5)

This is the desired solution of the given problem.

**Temperature Distribution (Cooling Process)**

\(T'\) is the temperature (cooling) of the plate satisfying the differential equation as \(Noda et al. [11]\) is

\[
\frac{\partial^2 T'}{\partial r^2} + \frac{1}{r} \frac{\partial T'}{\partial r} + \frac{\partial^2 T'}{\partial z^2} = \frac{1}{k} \frac{\partial T'}{\partial t}
\] (9.6)

Subject to initial condition
\[
T'(r, z, t)_{t=t_0} = T(r, z, t)_{t=t_0} = T(r, z, t_0) = G(r, z)
\] (9.7)

The boundary conditions are
\[
M_r(T',0,1,a) = g_1(z, t), \quad -h \leq z \leq h, \quad t > 0
\] (9.8)
\[
M_z(T',1,k_1,h) = f_3(r, t) \quad 0 \leq r \leq a, \quad t > 0
\] (9.9)
\[
M_z(T',1,k_2,-h) = f_3(r, t) \quad 0 \leq r \leq a, \quad t > 0
\]
10. SOLUTION OF THE PROBLEM

Applying Hankel transform and Marchi-Fasulo transform one obtains

\[
\frac{d\bar{T}^{**}}{dt} + kp^2 \bar{T}^* = \Pi \tag{10.1}
\]

where

\[
p^2 = \xi_m^2 + \lambda_n^2
\]

and

\[
\Pi = k \left( \frac{p_n(h) f_3}{k_1} - \frac{p_n(-h) f_4}{k_2} \right)
\]

Equation (10.1) is a linear equation whose solution is given by

\[
\bar{T}^* (\xi_m, n, t) = \Psi + \int_0^t \Pi e^{-kp^2(t-t')} dt' \tag{10.2}
\]

where

\[
\Psi = \bar{G}^* - \int_0^{t_0} \Pi e^{-kp^2(t_0-t')} dt
\]

Applying inversion of Marchi-Fasulo transform and Hankel transform to the differential equation (10.2), we get

\[
T'(r, z, t) = \frac{2}{a^2} \sum_m \sum_n \frac{J_0(r \xi_m)}{[J_1(a \xi_m)]^2} \frac{P_n(z)}{\lambda_n} e^{-kp^2 t} \times \left( \Psi + \int_0^t \Pi e^{-kp^2(t-t')} dt' \right) \tag{10.3}
\]

This is the desired solution of the given problem.

Let us assume Love’s function \( L \) which satisfy condition (8.10) as

\[
L(r, z) = \frac{2}{a^2} \sum_m \sum_n \frac{J_0(r \xi_m)}{[J_1(a \xi_m)]^2} \frac{P_n(z)}{\lambda_n} \Omega \tag{10.4}
\]

where

\[
\Omega = e^{-kp^2} \left[ \int_0^t \Pi e^{kp^2 t'} dt' + \bar{F}^* (m, n) \right]
\]

Using (8.1) and (9.5), we get displacement potential \( \phi \) as
\[
\phi = A \sum_m \sum_n \frac{J_0(r_{\xi_m}^n)}{J_1(a_{\xi_m}^n)^2} \left[ \frac{P_n'(z)}{\lambda_n} \Omega + B(t) \right] 
\]  

(10.5)

where

\[
A = \left( \frac{1 + \nu}{1 - \nu} \right) \frac{2\alpha_t}{a^2}
\]

\[
B(t) = \int e^{-i\rho t} \int_{-\infty}^{\infty} e^{-i\rho t'} dt' + \tilde{F}_{m,n}(m,n) dt
\]

11. DETERMINATION OF DISPLACEMENT FUNCTION

Substituting equations (10.4) and (10.5) in equation (8.9), (8.7) we get

\[
u = A \sum_m \sum_n \frac{\xi_m^2 J_1(r_{\xi_m})}{J_1(a_{\xi_m})^2} \left[ \frac{P_n'(z)}{\lambda_n} \Omega + B(t) \right] 
\]

(11.1)

\[
u = A \sum_m \sum_n \frac{J_0(r_{\xi_m})}{J_1(a_{\xi_m})^2} \left[ \frac{P_n''(z)}{\lambda_n} \Omega + B(t) \right] 
\]

(11.2)

Substituting equations (10.4) and (10.5) in equations (8.9) to (8.12), we obtain

\[
\sigma_{rr} = 2G \left\{ \frac{2(\nu - 1)}{a^2} \sum_m \sum_n \frac{\xi_m^2 J_1'(r_{\xi_m}) P_n'(z)}{J_1(a_{\xi_m})^2} \Omega \right. 
\]

\[+ \frac{2}{a^2r} \sum_m \sum_n \frac{\xi_m^2 J_1(r_{\xi_m}) P_n'(z)}{J_1(a_{\xi_m})^2} \Omega 
\]

\[+ \frac{2}{a^2} \sum_m \sum_n \frac{J_0(r_{\xi_m}) P_n'(z)}{J_1(a_{\xi_m})^2} \Omega \]
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\[
- \frac{A}{r} \sum_{m} \sum_{n} \frac{\xi_{m} J_{1}(r \xi_{m})}{J_{1}(a \xi_{m})^{2}} \left[ \frac{P'_{n}(z)}{\lambda_{n}} \Omega + B(t) \right] \\
- A \sum_{m} \sum_{n} \frac{J_{0}(r \xi_{m})}{J_{1}(a \xi_{m})^{2}} \left[ \frac{P'_{n}(z)}{\lambda_{n}} \Omega + B(t) \right]
\]

\[
\sigma_{\theta \theta} = 2G \left\{ \frac{2\nu}{a^{2}} \sum_{m} \sum_{n} \frac{\xi_{m} J_{1}(r \xi_{m})}{J_{1}(a \xi_{m})^{2}} \frac{P_{n}^{1}(z)}{\lambda_{n}} \Omega \\
+ \frac{2(\nu - 1)}{a^{2}} \sum_{m} \sum_{n} \frac{\xi_{m} J_{1}(r \xi_{m})}{J_{1}(a \xi_{m})^{2}} \frac{P_{n}^{1}(z)}{\lambda_{n}} \Omega \right\}
\]

\[
\sigma_{zz} = 2G \left\{ \frac{(2 - \nu)}{a^{2}} \sum_{m} \sum_{n} \frac{\xi_{m} J_{1}(r \xi_{m})}{J_{1}(a \xi_{m})^{2}} \frac{P_{n}^{1}(z)}{\lambda_{n}} \Omega \\
+ \frac{2(2 - \nu)}{a^{2} r} \sum_{m} \sum_{n} \frac{\xi_{m} J_{1}(r \xi_{m})}{J_{1}(a \xi_{m})^{2}} \frac{P_{n}^{1}(z)}{\lambda_{n}} \Omega \right\}
\]

\[
- A \sum_{m} \sum_{n} \frac{\xi_{m} J_{1}(r \xi_{m})}{J_{1}(a \xi_{m})^{2}} \left[ \frac{P_{n}^{1}(z)}{\lambda_{n}} \Omega + B(t) \right] \\
- A \sum_{m} \sum_{n} \frac{\xi_{m} J_{1}(r \xi_{m})}{J_{1}(a \xi_{m})^{2}} \left[ \frac{P_{n}^{1}(z)}{\lambda_{n}} \Omega + B(t) \right]
\]
\[
\sigma_{rz} = 2G \left\{ \frac{2(1 - \nu)}{a^2} \sum \frac{\xi_m^3 J_1^1 (r \xi_m)}{\left[ J_1 (a \xi_m) \right]^2} P_1^1 (z) \frac{1}{\lambda_n} \right\} \\
+ 2\left(\frac{-\nu}{a^2}\right) \sum \frac{\xi_m J_1 (r \xi_m)}{\left[ J_1 (a \xi_m) \right]^2} P_1^1 (z) \frac{1}{\lambda_n} \\
+ 2\left(\frac{1 - \nu}{a^2}\right) \sum \frac{\xi_m J_1 (r \xi_m)}{\left[ J_1 (a \xi_m) \right]^2} \left\{ \frac{\xi_m r J'_1 (r \xi_m)}{r_2} \right\} \frac{P_1 (z)}{\lambda_n} \\
+ A \sum \frac{\xi_m J_1 (r \xi_m)}{\left[ J_1 (a \xi_m) \right]^2} \left[ \frac{P_1 (z)}{\lambda_n} \Omega + B(t) \right]\}
\]

where

\[
A = \left( \frac{1 + \nu}{1 - \nu} \right) \frac{2a}{a^2},
\]

\[
\Omega = e^{-k \rho t} \left[ \int_0^t \prod e^{-k \rho t'} dt' + \Phi^* (m, n) \right],
\]

\[
B(t) = \int \Omega \ dt
\]

12. SPECIAL CASE

Set \( F(r, z) = z^2 (1 - r^2) \)

Applying Marchi-Fasulo transform, to (12.1) we obtain

\[
\overline{F}(r, n) = (1 - r^2) \int_{-h}^{h} z^2 P_n (z) dz
\]

\[
\overline{F}(r, n) = (1 - r^2) \Phi \left[ \frac{2h^2 \sin(a_n h)}{a_n} + \frac{4h \cos(a_n h)}{a_n^2} - \frac{4\sin(a_n h)}{a_n^3} \right]
\]

Where

\[
P_n (z) = Q_n \cos(a_n z) - W_n \sin(a_n z),
\]

\[
Q_n = a_n (\alpha_1 + \alpha_2) \cos(a_n h) + (\beta_1 - \beta_2) \sin(a_n h)
\]

\[
W_n = (\beta_1 - \beta_2) \cos(a_n h) + a_n (\alpha_1 - \alpha_2) \sin(a_n h)
\]
Again on applying Hankel transform, we obtain

\[ F_r(m,n) = \Pi_n \left[ \frac{a}{\xi_m} J_1(a\xi_m) - \frac{a(\xi_m^2 - 4)}{\xi_m^2} J_1(a\xi_m) - \frac{2a^2}{\xi_m^2} J_0(a\xi_m) \right] \]  

(12.2)

Where

\[ \Pi_n = \Phi_n \left[ \frac{2h^2 \sin(a_n h)}{a_n} + \frac{4h \cos(a_n h)}{a_n^2} - \frac{4\sin(a_n h)}{a_n^3} \right] \]

(12.3)

And

\[ \Phi_n = a_n(\alpha_1 + \alpha_2)\cos(a_n h) + (\beta_1 - \beta_2)\sin(a_n h) \]

Using equation (7.6.2) in equation (7.3.5) – (7.5.6) one obtains

\[ T(r,z,t) = \frac{2}{a^2} \sum_m \sum_n \frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2} \frac{P_n(z)}{\lambda_n} e^{-k r^2} \times \left[ \int_0^t \Pi e^{k r^2 t} dt \right] + \Pi_n \left[ \frac{a}{\xi_m} J_1(a\xi_m) - \frac{a(\xi_m^2 - 4)}{\xi_m^2} J_1(a\xi_m) - \frac{2a^2}{\xi_m^2} J_0(a\xi_m) \right] \]

(13.1)

13. NUMERICAL RESULTS

Set \( a = 2, k = 15.9 \times 10^6, t = 1 \) second in equation (12.3) we get

\[ T(r,z,t) = 2 \sum_m \sum_n \frac{J_0(r\xi_m)}{[J_1(2\xi_m)]^2} \frac{P_n(z)}{\lambda_n} e^{-(15.9 \times 10^6) r^2} \times \left[ \int_0^t \Psi e^{(15.9 \times 10^6) r^2 t} dt \right] + \Pi_n \left[ \frac{2}{\xi_m} J_1(2\xi_m) - \frac{8(\xi_m^2 - 1)}{\xi_m^3} J_1(2\xi_m) - \frac{8}{\xi_m^3} J_0(2\xi_m) \right] \]

(13.1)

14. CONCLUSION

In both the problems, the temperature distribution, displacement and thermal stresses of a thick circular plate are investigated with known boundary conditions. Finite integral transform techniques are used to obtain numerical results. The results are obtained in terms of Bessel’s function in the form of infinite series. Numerical estimated are calculated and depicted graphically.

Any particular cases of special interest can be assigned to the parameters and functions in expressions. The results that are obtained can be useful to the design of structure or machines in engineering applications.
REFERENCES


Graph 1: Temperature distribution versus r

Graph 2: The displacement potential function versus r

Graph 3: The displacement function versus r
Graph 4: The displacement function versus r

Graph 5: The component of stresses versus r

Graph 6: The component of stresses versus r
Graph 7: The component of stresses versus \( r \)

Graph 8: The component of stresses versus \( r \)

Graph 9: Temperature distribution versus \( r \)
**Graph 10:** The displacement potential function versus $r$

**Graph 11:** The displacement function versus $r$

**Graph 12:** The displacement function versus $r$
Graph 13: The component of stresses versus r

Graph 14: The component of stresses versus r

Graph 15: The component of stresses versus r
Graph 16: The component of stresses versus $r$