**Abstract**

A hypergraph $H = (V, E)$ is said to be a $T_0$ hypergraph if for any two distinct vertices $u$ and $v$ of $V$ there exists a hyperedge containing one of them but not the other. In this paper we give examples of $T_0$ hypergraphs. Sufficient conditions for dual of a hypergraph to be $T_0$ is derived. The proof of line graph of a $T_0$ hypergraph is $T_0$ is also given. Sufficient conditions for join of two hypergraphs, corona, incidence graph, middle graph, and total graph of hypergraph to be $T_0$ are derived. Sufficient conditions for various products of two hypergraphs to be $T_0$ are derived.

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**Keywords:** $T_0$ hypergraph, Dual of a hypergraph, Join of hypergraphs, Corona, Line graph, Primal graph, Incidence graph, Middle graph, Total graph, Cartesian Product, Minimal rank preserving direct product, Maximal rank preserving direct product, Non-rank preserving direct product, Wreath product.

1. **Introduction**

All the graphs considered here are finite and simple. In this paper we denote the set of vertices of the graph $G$ by $V(G)$, the set of edges of $G$ by $E(G)$, the maximum degree of $G$ by $\Delta(G)$ and the minimum degree of $G$ by $\delta(G)$.
The degree \[ \text{deg}(v) \] of a vertex \( v \) in graph \( G \) is the number of edges incident with \( v \). A simple graph is said to be complete [1] if every pair of distinct vertices of \( G \) are adjacent in \( G \). A graph \( H \) is called a subgraph[5] of \( G \) if \( V(H) \subseteq V(G) \), \( E(H) \subseteq E(G) \). A maximal complete subgraph of graph is a clique [4] of the graph. That is if \( Q \) is a clique in \( G \), then any supergraph of \( Q \) is not complete.

A graph \( G \) is said to be \( T_0[14] \) if for any two distinct vertices \( u \) and \( v \) of \( G \), one of the following three conditions hold: (1) At least one of \( u \) and \( v \) is isolated (2) There exist two nonadjacent edges \( e_1 \) and \( e_2 \) of \( G \) such that \( e_1 \) is incident with \( u \) and \( e_2 \) is incident with \( v \). From the definition of \( T_0 \) graph, if \( G \) is \( T_0 \) graph with no isolated vertices, then any supergraph of \( G \) is \( T_0 \).

Hypergraphs are generalization of graphs, hence many of the definitions of graphs carry verbatim to hypergraphs. The basic idea of the hypergraph concept is to consider such a generalization of a graph in which any subset of a given set may be an edge rather than two-element subsets [16]. A hypergraph [3] \( H \) is a pair \( (V, \mathcal{E}) \), where \( V \) is a set of elements called nodes or vertices, and \( \mathcal{E} \) is a set of nonempty subsets of \( V \) called hyperedges or edges. Therefore, \( \mathcal{E} \) is a subset of \( P(X)\setminus\{\emptyset\} \), where \( P(X) \) is the power set of \( X \). In drawing hypergraphs, each vertex is a point in the plane and each edge is a closed curve separating the respective subset from the remaining vertices. The cardinality of the finite set \( V \), is denoted by \( |V| \), is called the order [15] of the hypergraph. The number of edges is usually denoted by \( m \) or \( m(H) \) [15] and is called the size of the hypergraph.

A simple hypergraph [2] is a hypergraph with the property that if \( e_i \) and \( e_j \) are hyperedges of \( H \) with \( e_i \subseteq e_j \), then \( i = j \). Two vertices in a hypergraph are adjacent[16] if there is a hyperedge which contains both vertices. Two hyperedges in a hypergraph are incident[16] if their intersection is nonempty.

A \( k \)-uniform hypergraph [10] or a \( k \)-hypergraph is a hypergraph in which every edge consists of \( k \) vertices. So a 2-uniform hypergraph is a graph, a 3-uniform hypergraph is a collection of unordered triples, and so on. The rank [16] \( r(H) \) of a hypergraph is the maximum of the cardinalities of the edges in the hypergraph. The co-rank [16] \( cr(H) \) of a hypergraph is the minimum of the cardinalities of a hyperedge in the hypergraph. If \( r(H) = cr(H) = k \), then \( H \) is \( k \)-uniform.

The degree \[ d_H(v) \] of a vertex \( v \) in a hypergraph \( H \) is the number of edges of \( H \) that containing the vertex \( v \). A hypergraph \( H \) is \( k \)-regular if every vertex has degree \( k \). A vertex of a hypergraph which is incident to no edges is called an isolated vertex.[16] The degree of an isolated vertex is trivially zero. The degree \[ d(e) \] of a hyperedge, \( e \in \mathcal{E} \) is its cardinality.

A hyperedge \( e \) of \( H \) with \( |e| = 1 \) is called a loop; more specifically a hyperedge \( e = \{v\} \) is a loop at the vertex \( v \). A vertex of degree 1 is called a pendant vertex.

A simple hypergraph \( H \) with \( |e| = 2 \) for each \( e \in \mathcal{E} \) is a simple graph.

For a hypergraph \( H = (V, \mathcal{E}) \), any subhypergraph \( H' \subseteq H \) such that \( H' = (V, \mathcal{E}') \) is called a partial subhypergraph [15]. Any spanning subgraph of a graph is a partial subhypergraph.

**Theorem 1.1.** [12] Let \( G \) be a graph. If \( K_2 \) is not a component of \( G \) then it is \( T_0 \).
In this paper we consider only those hypergraphs with no isolated vertices.

2. \( T_0 \) Hypergraphs

In this section we introduce the concept of \( T_0 \) hypergraphs. We have investigated the \( T_0 \) property of various graphs derived from the given hypergraph and different products of hypergraphs.

**Definition 2.1.** A hypergraph \( H = (V, \mathcal{E}) \) is said to be a \( T_0 \) hypergraph or said to satisfy \( T_0 \) axiom if for any two distinct vertices \( u \) and \( v \) of \( V \) there exists a hyperedge containing one of them but not the other.

If \( H = (V, \mathcal{E}) \) is a \( T_0 \) hypergraph, then the topology generated by the elements of \( \mathcal{E} \) is a \( T_0 \) topology on \( V \).

![Hypergraphs](a) A \( T_0 \) hypergraph. (b) A non-\( T_0 \) hypergraph.

In the case of graphs, if \( \delta(G) \geq 2 \), then \( G \) is \( T_0 \). But this is not true in the case of hypergraphs. For example, graph shown in Figure 1(b) is a non-\( T_0 \) hypergraph with \( \text{deg}_H(v) \geq 2 \), for every \( v \in V \).

Next consider the \( T_0 \) property of the dual of a given hypergraph.

**Theorem 2.2.** Let \( H = (V, \mathcal{E}) \) be a hypergraph. If for any two distinct edges \( e_i \) and \( e_j \) of \( H \) there exists a vertex \( v_p \) which belongs to one of them but not the other. Then its dual \( H^* \) is \( T_0 \).

**Proof.** Consider two distinct vertices \( e_i \) and \( e_j \) of \( H^* \). By hypothesis, there exists a vertex \( v_p \) which belongs to one of them but not the other. Without loss of generality assume that \( v_p \in e_i \). Then, \( V_p \) is a hyperedge of \( H^* \) containing \( e_i \) but not \( e_j \). Hence the theorem.

Hereafter all the hypergraphs considered in this section are simple hypergraphs.
Now, we derive sufficient condition under which line graph of a given hypergraph to be $T_0$.

Figure 2 shows that line graph of a $T_0$ hypergraph need not be $T_0$.

![Graphs](image)

Figure 2: A $T_0$ hypergraph $H$ and its non-$T_0$ line graph $L(H)$.

**Theorem 2.3.** The line graph $L(H)$ of the hypergraph $H$ is $T_0$ provided $H$ has the property that for any two intersecting hyperedges $e_i$ and $e_j$ of $H$ with cardinality $> 1$, there exists a hyperedge $e_k$ distinct from $e_i$ and $e_j$ such that either $e_i \cap e_k \neq \emptyset$ or $e_j \cap e_k \neq \emptyset$.

*Proof.* Let $H$ be the given hypergraph and $L(H)$ be its line graph. Let $e_i$ and $e_j$ be two adjacent vertices of $L(H)$. By the hypothesis there exists an edge $e_k$ of $L(H)$ distinct from $e_i$ and $e_j$ which is incident with either $e_i$ or $e_j$. This implies $K_2$ is not a component of $L(H)$. Therefore, by Theorem ??, $L(H)$ is $T_0$. \[\square\]

Now, we investigate the $T_0$ property of the primal graph of the given hypergraph. Figure 3 shows that the primal graph of a non-$T_0$ hypergraph may be $T_0$.

![Graphs](image)

Figure 3: A non-$T_0$ hypergraph and its $T_0$ primal graph.
Theorem 2.4. The primal graph $H$ of a $T_0$ hypergraph is $T_0$.

Proof. Let $u$ and $v$ be two distinct vertices of the primal graph $H$ of the hypergraph $H$. As $H$ is $T_0$ there exists a hyperedge containing one of them but not the other. Without loss of generality assume that $e_1$ is a hyperedge of $H$ containing $u$ but not $v$. If $e_1$ is a loop then there is nothing to prove. So assume that $e_1$ is not a loop. Choose a vertex $w$ of $e_1$ distinct from $u$. Then $uw$ is an edge of $H$ incident with $u$ but not with $v$. Since $u$ and $v$ are arbitrary, theorem follows.

As in the case of $T_0$ hypergraph, here also we discuss the sufficient condition under which the join and the corona of two hypergraphs to be $T_0$.

Theorem 2.5. The join $H_1 \lor H_2$ of any two hypergraphs $H_1$ and $H_2$ is $T_0$.

Proof. Let $u$ and $v$ be two distinct vertices of $H_1 \lor H_2$. If $u \in V(H_1)$ and $v \in V(H_2)$ then any hyperedge of $H_1$ containing $u$ serve the purpose. On the other hand if $u, v \in V(H_1)$ (or $V(H_2)$) then for any vertex $w$ of $H_2$ (or $H_1$), $\{u, w\}$ is a hyperedge of $H_1 \lor H_2$ containing $u$ but not $v$. Therefore, $H_1 \lor H_2$ is $T_0$.

Theorem 2.6. The corona $H_1 \circ H_2$ of any two hypergraphs $H_1$ and $H_2$ is $T_0$.

Proof. Let $u$ and $v$ be two distinct vertices of $H_1 \circ H_2$. We need only consider the cases where both $u$ and $v$ belong to the copy of $H_1$ or both lie in the same copy of $H_2$. In the first case fix a vertex $w$ of some copy of $H_2$. Then $\{u, w\}$ is a hyperedge of $H_1 \circ H_2$ containing $u$ but not $v$. In the second case we fix a vertex $x$ of copy of $H_1$. Then $\{x, u\}$ is a hyperedge of $H_1 \circ H_2$ containing $u$ but not $v$. Therefore, $H_1 \circ H_2$ is $T_0$.

Next, we discuss $T_0$ property of the incidence graph, the middle graph and the total graph of the given hypergraph.

Theorem 2.7. If $H$ is a hypergraph with no loops, then its incidence graph $I(H)$ is $T_0$.

Proof. As $H$ is a hypergraph with no loops each hyperedge of $H$ contain at least two vertices. Therefore, each vertex $e$ of $E$ in the bipartite graph $I(H)$ with bipartition $(V, E)$ is adjacent to at least two vertices of $I(H)$. Therefore, $K_2$ is not a component of $I(H)$. This implies $I(H)$ is $T_0$.

Remark 2.8. If $e = \{v\}$ is a loop of the hypergraph $H$, then $ev$ is an edge of $I(H)$ which is not incident with any other edges of $I(H)$. This implies $K_2$ is a component of $I(H)$. Hence it is not $T_0$.

Remark 2.9. As the incidence graph $I(H)$ is the spanning subgraph of its middle graph $M(H)$ and its total graph $T(H)$, if $H$ is a hypergraph with no loops, then $M(H)$ and $T(H)$ are $T_0$.

Now, we investigate $T_0$ property of various products of hypergraphs.
3. \( T_0 \) Property of Product of Hypergraphs

3.1. Cartesian Product

The Cartesian product of two hypergraphs was introduced by J. Cooper and A. Dutle [7]. It is a natural generalization of the notion of Cartesian products simple graphs. In this section we prove that the Cartesian product of any two hypergraphs is \( T_0 \).

**Definition 3.1.** [7] The Cartesian product \( H_1 \square H_2 \) of two hypergraphs \( H_1 = (V_1, E_1) \) and \( H_2 = (V_2, E_2) \) is the hypergraph \( H = (V, E) \) with vertex set \( V = V_1 \times V_2 \) and edge set \( E = \{ \{u\} \times f : u \in V_1, f \in E_2\} \cup \{ e \times \{v\} : e \in E_1, v \in V_2\} \).

**Theorem 3.2.** Let \( H_1 \) and \( H_2 \) be two hypergraphs. Then the Cartesian product \( H = H_1 \square H_2 \) of \( H_1 \) and \( H_2 \) is a \( T_0 \) hypergraph.

**Proof.** Let \( \{u_1, u_2, \ldots, u_n\} \) and \( \{v_1, v_2, \ldots, v_m\} \) be the set of all vertices of \( H_1 \) and \( H_2 \) respectively. Let the hyperedges of \( H_1 \) be \( e_1, e_2, \ldots, e_p \) and that of \( H_2 \) be \( f_1, f_2, \ldots, f_q \). Let us denote the hyperedges \( \{u_i\} \times f_j \) and \( e_r \times \{v_s\} \) by \( f_{i,j} \) and \( e_{r,s} \) respectively.

Consider two distinct vertices \( (u_i, v_j) \) and \( (u_r, v_s) \) of \( H \).

**Case 1.** \( v_j = v_s \)

Let \( f_j \) be a hyperedge of \( H_2 \) containing \( v_j \). Then \( f_{i,j} \) is a hyperedge of \( H_1 \square H_2 \) containing \( (u_i, v_j) \) but not \( (u_r, v_s) \).

**Case 2.** \( v_j \neq v_s \)

Suppose \( u_i = u_r \). This case can be dealt as in Case 1.

Now, suppose \( u_i \neq u_r \).

Let \( e_i \) be a hyperedge of \( H_1 \) containing \( u_i \). Then \( e_{i,j} \) is a hyperedge of \( H_1 \square H_2 \) containing \( (u_i, v_j) \) but not \( (u_r, v_s) \).

Hence the theorem. \( \blacksquare \)

There are many ways to generalize the direct product of graphs to a product of hypergraphs. As we want such products to coincide with the usual direct product when the factors have rank 2 (and are therefore graphs) it is necessary to impose some rank restricting conditions on the edges. This can be accomplished in different ways and leads to different variants of the direct and strong products of graphs. First of all we consider the minimal rank preserving direct product.

3.2. Minimal Rank Preserving Direct Product

This section begins with the definition of minimal rank preserving direct product of two hypergraphs. The restriction of this product to simple graphs coincides with the direct product of graphs.

**Definition 3.3.** [11] The minimal rank preserving direct product \( H_1 \bowtie H_2 \) of two hypergraphs \( H_1 = (V_1, E_1) \) and \( H_2 = (V_2, E_2) \) is a hypergraph with vertex set \( V_1 \times V_2 \).

A subset \( e = \{(u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)\} \) of \( V_1 \times V_2 \) is an edge of \( H_1 \bowtie H_2 \) if
Theorem 3.4. Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two hypergraphs. If the degree of at least one vertex of every hyperedge of degree 2 of the hypergraph $H_1 (or \ H_2)$ is different from 1, then the minimal rank preserving direct product $H_1 \times H_2$ of $H_1$ and $H_2$ is $T_0$.

Proof. Suppose the degree of at least one vertex of every hyperedge of degree 2 of the hypergraph $H_1$ is different from 1. Let $(u_1, v_1)$ and $(u_2, v_2)$ be two distinct vertices of $H_1 \times H_2$.

We consider the following two cases to show that there exists a hyperedge of $H_1 \times H_2$ of containing $(u_1, v_1)$ but not $(u_2, v_2)$.

Case 1. $u_1 = u_2, v_1 \neq v_2$
Let $e = \{u_1, u_3\}$ be a hyperedge of $H_1$ and $f = \{w_1, w_2\}$, where $w_1 = v_1$ be a hyperedge of $H_2$. Then $\{(u_1, w_1), (u_3, w_2)\}$ is a hyperedge of $H_1 \times H_2$ of containing $(u_1, v_1)$ but not $(u_2, v_2)$.

Case 2. $u_1 \neq u_2, v_1 \neq v_2$
Suppose there exist two distinct hyperedges $f_1$ and $f_2$ of $H_1$ containing $u_1$ and $u_2$ respectively.

Let $f_1 = \{w_1, w_2\}$ and $f_2 = \{x_1, x_2\}$, where $w_1 = u_1$ and $x_1 = u_2$. Let $g = \{y_1, y_2\}$, where $y_1 = v_1$ be a hyperedge of $H_2$ containing $v_1$. Then $\{(w_1, y_1), (w_2, y_2)\}$ is a hyperedge of $H_1 \times H_2$ of containing $(u_1, v_1)$ but not $(u_2, v_2)$.

Suppose that all the hyperedges of $H_1$ containing $u_1$ contains $u_2$. Then $\{u_1, u_2\}$ is a hyperedge of $H_1 \times H_2$. Suppose $f = \{w_1, w_2\}$, where $w_1 = v_1$ is hyperedge of $H_2$ containing $v_1$ but not $v_2$. Then $\{(u_1, w_1), (u_2, w_2)\}$ is a hyperedge of $H_1 \times H_2$ of containing $(u_1, v_1)$ but not $(u_2, v_2)$. If no such hyperedge $f$ exist, then $\{v_1, v_2\}$ is a hyperedge of $H_2$. By the hypothesis degree of either $u_1$ or $u_2$ is different from 1. Without loss of generality assume that degree of $u_1$ is different from 1. Let $h$ be a hyperedge of $H_1$ containing $u_1$ and a vertex $x$ different from $u_2$. Then $\{(u_1, v_1), (x, v_2)\}$ is a hyperedge of $H_1 \times H_2$ of containing $(u_1, v_1)$ but not $(u_2, v_2)$.

Hence the theorem.

Now, we discuss the $T_0$ property of the normal product of two hypergraphs. The restriction of the normal product to simple graphs coincides with the usual strong product of graphs.
Definition 3.5. [11] The normal product $H_1 \boxdot H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$ and a subset $e = \{(u_1, v_1), (u_2, v_2), (u_3, v_3), \ldots, (u_n, v_n)\}$ of $V_1 \times V_2$ is an edge of $H_1 \boxdot H_2$ if,

1. $\{u_1, u_2, \ldots, u_n\}$ is an edge of $H_1$ and $v_1 = v_2 = \ldots = v_n \in V_2$, or
2. $\{v_1, v_2, \ldots, v_n\}$ is an edge of $H_2$ and $u_1 = u_2 = \ldots = u_n \in V_1$, or
3. $\{u_1, u_2, \ldots, u_n\}$ is an edge of $H_1$ and $\{v_1, v_2, \ldots, v_n\}$ is a subset of an edge of $H_2$, or
4. $\{v_1, v_2, \ldots, v_n\}$ is an edge of $H_2$ and $\{u_1, u_2, \ldots, u_n\}$ is a subset of an edge of $H_1$.

Remark 3.6. Cartesian product $H_1 \Box H_2$ of two hypergraphs $H_1$ and $H_2$ is a partial subhypergraph of their normal product $H_1 \boxdot H_2$.

Theorem 3.7. Let $H_1$ and $H_2$ be two hypergraphs. Then the normal product $H_1 \boxdot H_2$ of $H_1$ and $H_2$ is $T_0$.

3.3. Maximal Rank Preserving Direct Product

This section discusses the $T_0$ property of maximal rank preserving direct product of two hypergraphs. As in the case of minimal rank preserving direct product here also the restriction of this product to simple graphs coincides with the direct product of graphs.

Definition 3.8. [11] The maximal rank preserving direct product $H_1 \bar{\times} H_2$ of two hypergraphs $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ is a hypergraph with vertex set $V_1 \times V_2$. A subset $e = \{(u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)\}$ of $V_1 \times V_2$ is an edge of $H_1 \bar{\times} H_2$ if

1. $\{u_1, u_2, \ldots, u_r\}$ is an edge of $H_1$ and there is an edge $f \in \mathcal{E}_2$ of $H_2$ such that $\{v_1, v_2, \ldots, v_r\}$ is a multiset$^1$ of elements of $f$, and $f \subseteq \{v_1, v_2, \ldots, v_r\}$, or
2. $\{v_1, v_2, \ldots, v_r\}$ is an edge of $H_2$ and there is an edge $e \in \mathcal{E}_1$ of $H_1$ such that $\{u_1, u_2, \ldots, u_r\}$ is a multiset of elements of $e$, and $e \subseteq \{u_1, u_2, \ldots, u_r\}$.

If $e$ is the edge of $H_1 \bar{\times} H_2$ determined by the edges $e_1$ of $H_1$ and $e_2$ of $H_2$, then $|e| = \max\{|e_1|, |e_2|\}$. Therefore, $r(H_1 \bar{\times} H_2) = \max\{r(H_1), r(H_2)\}$ [11].

Let $H_1 = (V_1, \mathcal{E}_1)$ and $H_2 = (V_2, \mathcal{E}_2)$ be two hypergraphs with no loops. The edges of $H_1 \bar{\times} H_2$ and $H_1 \times H_2$ corresponding to the edges of degree 2 in $H_1$ and $H_2$ are same. Hence a similar result of Theorem 3.4 also holds in the case of maximal rank preserving direct product.

$^1$A multiset is an unordered collection of objects (called the elements) in which, unlike a standard (Cantorian) set, elements are allowed to repeat. In other words a multiset is a set in which elements may belong more than once. \{1, 1, 1, 2, 3, 3\} is a multiset.
Theorem 3.9. Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two hypergraphs with no loops. If the degree of at least one vertex of every hyperedge of degree 2 of the hypergraph $H_1$ (or $H_2$) is different from 1, then the maximal rank preserving direct product $H_1 \times H_2$ of $H_1$ and $H_2$ is $T_0$.

Next we consider the $T_0$ property of the strong product of two hypergraphs. As in the case of normal product of two hypergraphs, the restriction of the strong product to simple graphs coincides with the usual strong product of graphs.

Definition 3.10. [11] The strong product $H_1 \boxtimes H_2$ of two hypergraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ is a hypergraph with vertex set $V_1 \times V_2$ and a subset $e = \{(u_1, v_1), (u_2, v_2), (u_3, v_3), \ldots, (u_n, v_n)\}$ of $V_1 \times V_2$ is an edge of $H_1 \boxtimes H_2$ if,

1. $\{u_1, u_2, \ldots, u_n\}$ is an edge of $H_1$ and $v_1 = v_2 = \ldots = v_r \in V_2$, or
2. $\{v_1, v_2, \ldots, v_n\}$ is an edge of $H_2$ and $u_1 = u_2 = \ldots = u_r \in V_1$, or
3. $\{u_1, u_2, \ldots, u_r\}$ is an edge of $H_1$ and there is an edge $f \in E_2$ of $H_2$ such that $\{v_1, v_2, \ldots, v_r\}$ is a multiset of elements of $f$, and $f \subseteq \{v_1, v_2, \ldots, v_r\}$, or
4. $\{v_1, v_2, \ldots, v_r\}$ is an edge of $H_2$ and there is an edge $f \in E_1$ of $H_1$ such that $\{u_1, u_2, \ldots, u_r\}$ is a multiset of elements of $f$, and $f \subseteq \{u_1, u_2, \ldots, u_r\}$.

Remark 3.11. $E(H_1 \boxtimes H_2) = E(H_1 \boxtimes H_1) \cup E(H_1 \times H_2)$. Thus it is immediate that if $H_1$ and $H_2$ are two any hypergraphs, then their strong product is $T_0$.

3.4. Non-rank Preserving Direct Product

This section is devoted to study the $T_0$ property of non-rank-preserving direct product $H_1 \times H_2$ of two hypergraphs $H_1$ and $H_2$. This concept was first introduced by M.Hellmuth et al. in their paper entitled “A survey on Hypergraph Products” in the year 2012 [11]. Further he mentioned in the same paper that $r(H_1 \times H_2) = (r(H_1) - 1)(r(H_2) - 1) + 1$ and therefore, the rank of the product will not be the rank of one of its factors. Also if we restrict the definition of this product to simple graphs, then this is exactly the definition of the direct product of graphs.

Definition 3.12. [11] The non-rank-preserving direct product $H_1 \mathcal{\times} H_2$ of two hypergraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ is a hypergraph with vertex set $V_1 \times V_2$ and edge set $\{(\{u, v\}) \cup (\{f - \{v\} : u \in e \in E_1, v \in f \in E_2\}) : u \in e \in E_1, v \in f \in E_2\}$.

Remark 3.13. If $H_1$ is a hypergraph with all of its edges are loops then for any hypergraph $H_2$, the edges of $H_1 \times H_2$ are loops. Hence it is $T_0$.

Theorem 3.14. Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two hypergraphs. Then their non-rank-preserving direct product $H_1 \mathcal{\times} H_2$ is $T_0$ provided one of them is $T_0$. 
Theorem 3.16. Let one of the following conditions hold.

1. $u_1 = u_2$.

2. $u = \{u_1, u_2, \ldots, u_{m+1}\}$ be a hyperedge of $H_1$ and $f = \{w_1, w_2, \ldots, w_n\}$, where $w_1 = v_1$ be a hyperedge of $H_2$. Then $\\{(u_1, w_1)\} \cup \{(u_3, u_4, \ldots, u_{m+1}) \times (w_2, w_3, \ldots, w_\eta)\}$ is a hyperedge of $H_1 \times H_2$ containing $(u_1, v_1)$ but not $(u_2, v_2)$.

Suppose that $u_1 \neq u_2$.

As $H_1$ is $T_0$, there exists a hyperedge of $H_1$ containing $u_1$ but not $u_2$. Let $g = \{x_1, x_2, \ldots, x_p\}$, where $x_1 = u_1$ be a hyperedge of $H_1$ containing $u_1$ but not $u_2$. Let $h = \{y_1, y_2, \ldots, y_m\}$, where $y_1 = v_1$ be hyperedge of $H_2$ containing $v_1$. Then $\\{(x_1, y_1)\} \cup \{(x_2, x_3, \ldots, x_p) \times (y_2, y_3, \ldots, y_m)\}$ is a hyperedge of $H_1 \times H_2$ containing $(u_1, v_1)$ but not $(u_2, v_2)$.

3.5. Wreath Product

In [8], G. Hahn introduced the definition of wreath product of two hypergraphs as a generalisation of that of graphs. He used the following notations while defining the same.

Let $A$ and $B$ be two sets. For each $e \in P(A \times B)$, we denote the set of first coordinates of elements of $e$ by $e^1$ and the set of second coordinates of elements of $e$ by $e^2$.

Definition 3.15. [8] Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two hypergraphs with disjoint vertex sets. The wreath product $H = (V, E) = H_1[H_2]$ of $H_1$ and $H_2$ is a hypergraph with vertex set $V = V_1 \times V_2$ and a subset $e$ of $V \times V$ belongs to $E$ if at least one of the following conditions holds.

1. $e^1 \in E_1$ and $|e \cap (\{u\} \times V_2)| \leq 1$ for each $u \in V_1$, or
2. $|e^1| = 1$ and $e^2 \in E_2$.

Theorem 3.16. Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two hypergraphs. Then their wreath product $H_1[H_2]$ is $T_0$.

Proof. Consider two distinct vertices $(u_1, v_1)$ and $(u_2, v_2)$ of $H_1[H_2]$.

Case 1. $u_1 = u_2$ and $v_1 \neq v_2$

Let $e = \{u_1, u_3, u_4, \ldots, u_{m+1}\}$ be an edge of $H_1$. Then $\{(u_1, v_1), (u_3, v_1), (u_4, v_1), \ldots, (u_{m+1}, v_1)\}$ is a hyperedge of $H_1[H_2]$ containing $(u_1, v_2)$ but not $(u_2, v_2)$.

Case 2. $u_1 \neq u_2$ and $v_1 = v_2$

Let $f = \{v_1, v_3, v_4, \ldots, v_{p+1}\}$ be an edge of $H_2$. Then the hyperedge $\{(u_1, v_1), (u_1, v_3), (u_1, v_4), \ldots, (u_1, v_{p+1})\}$ is a hyperedge of $H_1[H_2]$ containing $(u_1, v_2)$ but not $(u_2, v_2)$.

Case 3. $u_1 \neq u_2$ and $v_1 \neq v_2$

Subcase 1. There exists an edge of $H_2$ containing both $v_1$ and $v_2$.

Let $f = \{v_1, v_2, \ldots, v_p\}$ be an edge of $H_2$. Then the hyperedge $\{(u_1, v_1), (u_1, v_2), (u_1, v_3), \ldots, (u_1, v_p)\}$ is a hyperedge of $H_1[H_2]$ containing $(u_1, v_2)$ but not $(u_2, v_2)$.
Subcase 2. There exist no edge of $H_2$ containing both $v_1$ and $v_2$.
Let $f_1 = \{v_1, w_2, w_3, \ldots, w_p\}$ and $f_2 = \{v_2, z_2, z_3, \ldots, z_q\}$ be edges $H_2$ containing $v_1$ and $v_2$ respectively. Then the hyperedge $\{(u_1, v_1), (u_1, w_2), (u_1, w_3), \ldots, (u_1, w_p)\}$ is a hyperedge of $H_1[H_2]$ containing $(u_1, v_2)$ but not $(u_2, v_2)$.

4. Conclusion

In this paper $T_0$ hypergraphs have been discussed with examples. Sufficient conditions for dual, incidence graph, middle graph, and total graph of hypergraph to be $T_0$ is derived. Further more, sufficient condition for join and corona of two hypergraphs to be $T_0$ is derived. We also discussed conditions under which minimal rank, maximal rank, non-rank, preserving direct product of two hypergraphs to be $T_0$. It is proved that Cartesian, normal, strong and wreath product of any two hypergraphs is always $T_0$.

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