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A study of exponential stability for a flexible Euler-Bernoulli beam with variable coefficients under a force control in rotation and velocity rotation

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Abstract

In this article, we examine the exponential stability for a flexible Euler-Bernoulli beam with variable coefficients clamped at one end and is free at the other. In order to stabilize the system, we apply a linear boundary control force in rotation and velocity rotation. By adopting the Riesz basis approach, it is shown that the closed-loop system is a Riesz spectral system. Consequently, the spectrum-determined growth condition and the exponential stability are obtained.

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1. Introduction

In this paper, we study the Riesz basis property and the exponential stability for a flexible Euler-Bernoulli beams with variable coefficients under a force control in rotation and velocity rotation. The equations of motion of the system are described as follows

$$m(x)w_{tt}(x,t) + (EI(x)w_{xx}(x,t))_{xx} = 0, \quad 0 < x < 1,\ t > 0, \tag{1.1}$$

$$w(0,t) = w_x(0,t) = 0, \quad t > 0, \tag{1.2}$$

$$(EI(.)w_{xx})_x(1,t) = 0, \quad t > 0, \tag{1.3}$$

$$-EI(1)w_{xx}(1,t) = \alpha w_{xt}(1,t) + \beta w_x(1,t), \quad t > 0, \tag{1.4}$$

where α and β are two given positive constants, $w(x,t)$ stands for the transverse displacement of the beam at the position x and time t. The subscripts t and x denote derivatives with respect to the time t and the position x respectively. Without loss of generality, the length of the beam is chosen to be unity. $EI(x)$ is the flexural rigidity function and $m(x)$ is the mass density function of the beam. $-(EI(x)w_{xx}(x,t))_{xx}$ is the total lateral force acting on a slice of the beam of length dx, located at the position x and the time t. $(EI(.)w_{xx})_x(1,t)$ and $-EI(1)w_{xx}(1,t)$ are the force and the torque acting on the rigid body from the beam at the time t. Throughout the paper, we always assume that:

$$m(x), \quad EI(x) \in C^4(0,1), \quad m(x), EI(x) > 0$$

for any $x \in [0,1]$. Likewise, the coefficients are supposed to be variable because it is common, in engineering, to adopt problems with nonhomogeneous materials such as smart materials ([9]).

Exponential stability is the most desirable kind of stability. More precisely, the problem (1.1)-(1.4) is the nonuniform version to ([15], case $\beta \neq 0$) and ([13], case $\beta = 0$). In their papers, the authors have proved that the uniform closed-loop system is well-posed in the sense of C_0-semigroup of contractions theory. From Shkalikov's method [14], a spectral analysis of the operator and the property of the Riesz basis were studied to derive the exponential stability of the system. Moreover, there are two steps usually found in the study of linear systems with variable coefficients (see e.g. [6]): the first is to transform the dominant term of the system under study into a uniform dominant equation by space scaling and state transformation where no variable coefficient is involved any longer, while the second is to approximate the eigenfunctions of the system by those of the uniform dominant equation. This fundamental idea comes essentially from Birkhoff's works [2] and from [11]. This approach was used to study the Euler-Bernoulli equations with variable coefficients (see [6], [7], [17], [18]). Also, another method which is used for the verification of the Riesz basis is the Bari's Theorem. In our case, we use a result due to Wang et al. (see e.g. [17]) in order to study the problem with eigenvalues related to problem (1.1)-(1.4) in the form of an ordinary differential equation $L(f) = \lambda f$ with λ-polynomials boundary conditions (see [14]; [16]). We establish conditions on the two feedbacks parameters at the boundary α and β in order to obtain the property of the Riesz basis and the exponential stability of the system (1.1)-(1.4).

The rest of the paper is organized as follows. In section 2, the system (1.1)-(1.4) is formulated as an evolution problem and studied in semigroup framework. In order to examine if the system is exponentially stable, in section 3, the spectrum of the system operator is analyzed and it is demonstrated that the generalized eigenvalues of the operator form an Riesz basis in the corresponding state space.

2. Semigroup formulation

The semigroups generated by the system operator of an abstract Cauchy problem, can be used to completely characterize the well-posedness and the stability of its solution. Hence, the following formulation provides an efficient tool for the discussion on asymptotic and exponential stability. In the following, the notation $v = w_t$ (velocity of the beam) is used. Let us introduce the following spaces:

$$H_E^2(0,1) = \left\{ w \in H^2(0,1) \,|\, w(0) = w_x(0) = 0 \right\}. \tag{2.1}$$

The Hilbert space is defined by:

$$\mathbb{H} = H_E^2(0,1) \times L^2(0,1), \tag{2.2}$$

with the inner product

$$\langle w, v\rangle_{\mathbb{H}} = \int_0^1 \left(m(x) f_2(x)\, \overline{g_2(x)} + EI(x)\, f_1''(x)\, \overline{g_1''(x)} \right) dx + \beta f_1'(1)\, \overline{g_1'(1)}, \tag{2.3}$$

where $w = (f_1, f_2)^T \in \mathbb{H}$, $v = (g_1, g_2)^T \in \mathbb{H}$ and we denote by $\|.\|_{\mathbb{H}}$ the corresponding norm. The superscript T stands for the transpose and the spaces $L^2(0,1)$ and $H^k(0,1)$ are defined as

$$L^2(0,1) = \left\{ w : [0,1] \to \mathbb{C} \Big| \int_0^1 |w|^2\, dx < \infty \right\} \tag{2.4}$$

$$H^k(0,1) = \left\{ w : [0,1] \to \mathbb{C} \big| w, w^{(1)}, \ldots, w^{(k)} \in L^2(0,1) \right\}. \tag{2.5}$$

Let $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ an unbounded linear operator with the domain

$$D(A) = \Big\{ (f, g)^T \in (H^4(0,1) \cap H_E^2(0,1)) \times H_E^2(0,1) \,\Big|\, \left(EI(.)\, f''(.)\right)'(1) = 0,$$
$$-EI(1) f''(1) = \alpha g_x(1) + \beta f_x(1) \Big\} \tag{2.6}$$

defined by

$$A(f, g)^T = \left(g(x), -\frac{1}{m(x)} \left(EI(x) f''(x) \right)'' \right)^T. \tag{2.7}$$

Now (1.1)−(1.4) can be written formally as a first order evolution problem

$$\begin{cases} \dfrac{d}{dt}y(t) = Ay(t) \\ y(0) = y_0 \in \mathbb{H}, \end{cases} \tag{2.8}$$

where $y(t) = (w(.,t), w_t(.,t))^T$, $y(0) = (w_0, v_0)^T$ for all $t > 0$.

We demonstrate now the following fundamental well-posedness result for the system (1.1)−(1.4).

Theorem 2.1. The operator A defined by (2.6) and (2.7) is a closed, densely defined, dissipative operator with compact resolvents. Furthermore, A is invertible with A^{-1} being compact and A generates a C_0-semigroup of contractions on $\mathbb{H}$ denoted by $\{S(t)\}_{t\geq 0}$.

Proof. When the functions $\mu = EI = 1$, this result is obtained in [15]. We will use again here the well known Lumer-Phillips theorem (see, e.g., [12]). First, we will show that the operator A is dissipative. For any $w = (f, g)^T \in D(A)$,

$$< Aw, w >= \left\langle \left(g(x), -\frac{1}{\mu(x)}\big(EI(x)f''(x)\big)'' \right)^T, (f, g)^T \right\rangle$$

$$< Aw, w >= -\int_0^1 \big(EI(x)f''(x)\big)''\overline{g(x)}\,dx + \int_0^1 EI(x)g''(x)\overline{f''(x)}\,dx + \beta g'(1)\overline{f'(1)}$$

Integrating twice by parts and using the boundary conditions (1.2)−(1.4), we have

$$\begin{aligned} < Aw, w > = & -\int_0^1 EI(x)\big[g''(x)\overline{f''(x)} - f''(x)\overline{g''(x)}\big]\,dx \\ & + \beta\big(g'(1)\overline{f'(1)} - f'(1)\overline{g'(1)}\big) - \alpha|g'(1)|^2. \end{aligned}$$

Taking real parts, we obtain $Re < Aw, w >= -\alpha|g'(1)|^2 < 0$. Thus, A is a dissipative operator.

Next, the domain $D(A)$ is clearly dense in $\mathbb{H}$ and the operator is closed. Finally, we prove that A^{-1} exists. For any $\Psi = (g_1,\ g_2)^T \in \mathbb{H}$, we need to find a unique $\Phi = (f_1,\ f_2)^T \in D(A)$ such that $A\Phi = \Psi$ which yields

$$\begin{aligned} f_2(x) &= g_1(x), \quad g_1 \in H_E^2(0,1) \\ (\big(EI(x)\, f_1''(x)\big)'' &= -m(x)g_2(x), \quad g_2 \in L^2(0,1) \\ f_1(0) &= f_1'(0) = (EI(.)f_1''(.))'(1) = 0 \\ -EI(1)\, f_1''(1) &= \alpha f_2'(1) + \beta f_1'(1) = \alpha g_1'(1) + \beta f_1'(1). \end{aligned}$$

A direct computation shows that the above solution is given by

$$\begin{cases} f_2(x) = g_1(x) \\ f_1(x) = -\displaystyle\int_0^x \int_0^s \left[\frac{\beta f_1'(1) + \alpha g_1'(1)}{EI(\xi)} + \frac{1}{EI(\xi)}\int_\eta^1 \int_x^1 m(r)\, g_2(r)\, dr d\eta\right] d\xi ds. \end{cases}$$

Thus, A^{-1} exists and is bounded on $\mathbb{H}$. Furthermore, the Sobolev embedding Theorem implies that A^{-1} is compact on $\mathbb{H}$. Now according to Lumer-Phillips Theorem, the statement of the Theorem follows. ■

3. Riesz basis and Exponential stability

3.1. Spectral Analysis of operator A

In this subsection, we study the eigenvalue problem of A. Our work shall make use of the following result from [18], which deals with the eigenvalue problem of beams in the form of an ordinary differential equation $L(f) = \lambda f$ with λ-polynomial boundary conditions (see [14]; [16]). To begin, we recall some notations and definitions. Let $L(f)$ be an ordinary differential operator of order $n = 2m \in \mathbb{N}$,

$$L(f) = f^{(n)}(x) + \sum_{\nu=1}^{n} f_\nu(x) f^{(n-\nu)}(x), \ 0 < x < 1, \tag{3.1}$$

and let the boundary conditions defined at the two points $x = 0$, and $x = 1$ be

$$B_j(f) = \sum_{\nu=0}^{k_j} \left(\alpha_{j_\nu} f^{(k_j-\nu)}(0) + \beta_{j_\nu} f^{(k_j-\nu)}(1) \right), \ 1 \le j \le n, \tag{3.2}$$

where $k_j \in \mathbb{N}$, $1 \le k_j \le n-1$ and $\alpha_{j_\nu}, \beta_{j_\nu} \in \mathbb{C}$, $\left|\alpha_{j0}\right| + \left|\beta_{j0}\right| > 0$. Suppose that the coefficient functions $f_\nu(x)$ $(1 \le \nu \le n)$ in (3.1) are sufficiently smooth in $(0, 1)$, and that the boundary conditions are normalized in the sense that $\kappa = \sum_{j=1}^{n} k_j$ is minimal with respect to all equivalent boundary conditions (see M. A. Naimark [11]).

Let $f_k(x, \rho)$ $(k = 1, 2, \ldots, n)$ be the fundamental solutions for the equation:

$$L(f) + \rho^n f + \rho^m \mu(x) f(x) = 0, \ \rho \in \mathbb{C} \tag{3.3}$$

where $\mu(x)$ being continuous in $[0, 1]$, and let ω_k $(k = 1, 2, \ldots, n)$ be the n-th roots of $\omega^n + 1 = 0$. If we denote by $\Delta(\rho)$ the characteristic determinant of (3.3) with respect to (3.2)

$$\Delta(\rho) = \det\Big[B_j(f_k(., \rho))\Big]_{j,k=1,2,\ldots,n},$$

then $\Delta(\rho)$ can be expressed asymptotically in the form for $(r \ge 1)$

$$\Delta(\rho) = \rho^k \sum_{\mathbb{k}_k} e^{\rho\mu_{\mathbb{k}_k}} \left[F^{\mathbb{k}_k}\right]_r, \tag{3.4}$$

whenever ρ is large enough (see A. A. Shkalikov [14] and M. A. Naimark [11]). Here, $\mathbb{k}_k$ is a k-elements subset of $\{1, 2, \ldots, n\}$, $\mu_{\mathbb{k}_k} = \sum_{j \in \mathbb{k}_k} \omega_j$,

$$\left[F^{\mathbb{k}_k}\right]_r = F_0^{\mathbb{k}_k} + \rho^{-1} F_1^{\mathbb{k}_k} + \ldots + \rho^{-r+1} F_{r-1}^{\mathbb{k}_k} + \mathcal{O}\left(\rho^{-r}\right),$$

and the sum runs over all possible selections of $\Bbbk_k$. Here and henceforth, $\mathcal{O}\left(\rho^{-r}\right)$ means that $\left|\rho^r \times \mathcal{O}\left(\rho^{-r}\right)\right|$ is bounded as $|\rho| \to \infty$.

Definition 3.1. ([18] p. 461) The boundary problem (3.3) with (3.2) is said to be regular if the coefficients $F_0^{\Bbbk_k}$ in (3.4) are nonzero. Furthermore, the regular boundary problem (3.3) with (3.2) is said to be strongly regular if the zeros of $\Delta(\rho)$ are asymptotically simple and isolated one from another.

Let $W_2^m(0, 1)$ be the usual Sobolev space of order m and let

$$V_E^m(0, 1) = \left\{ f(x) \in W_2^m(0, 1) \mid B_j(f) = 0, \quad k_j < m \right\}.$$

Define a Hilbert space

$$H = V_E^m(0, 1) \times L^2(0, 1),$$

with the norm

$$\|(f, g)\|_H^2 = \|f\|_{W_2^m}^2 + \|g\|_2^2$$

and define the operator $\mathbb{A}$ in H by

$$\begin{cases} \mathbb{A}(f, g) = (g, -L(f) - \mu(x) g) \\ D(\mathbb{A}) = \{(f, g) \in H \mid \mathbb{A}(f, g) \in H, \ B_j(f) = 0, \ k_j \geq m\}. \end{cases} \tag{3.5}$$

The following result used in [18] was presented in [19]. The reader can be also referred to ([17], chapter 3).

Theorem 3.2. ([18] p. 461) If the ordinary differential system with parameter $\lambda = \rho^m$

$$\begin{cases} L(f, \lambda) = L(f) + \lambda^2 f + \lambda\mu(x) f \\ \qquad B_j(f) = 0, \quad 1 \leq j \leq 2m \end{cases} \tag{3.6}$$

has strongly regular boundary conditions, then the generalized eigenfunction system of $\mathbb{A}$ form a Riesz basis in the Hilbert space H.

According to Theorem 2.1, A has a compact resolvent then $\sigma(A)$, the spectrum of A consists only of isolated eigenvalues, which distribute in conjugate pairs on the complex plane.

Let $\lambda \in \sigma(A)$ and $\Phi = (\phi, \Psi)$ be an eigenfunction of A corresponding to λ. Then we have $\Psi = \lambda\phi$ and ϕ satisfies the following equations:

$$\begin{cases} \lambda^2 m(x)\phi(x) + \left(EI(x)\phi''(x)\right)'' = 0, \quad 0 < x < 1, \\ \phi(0) = \phi'(0) = \left(EI(.)\phi''(.)\right)'(1) = 0 \\ \phi''(1) = -\dfrac{1}{EI(1)}(\alpha\lambda + \beta)\phi'(1). \end{cases} \tag{3.7}$$

In order to solve (3.7), spatial transformations as introduced in [6] are performed, which convert the first equation of (3.7) into a more convenient form. For this reason, (3.7) is firstly rewritten as:

$$\begin{cases} \phi^{(4)}(x) + \dfrac{2EI'(x)}{EI(x)}\phi'''(x) + \dfrac{EI''(x)}{EI(x)}\phi''(x) + \dfrac{\lambda^2 m(x)}{EI(x)}\phi(x) = 0, \quad 0 < x < 1, \\ \phi(0) = \phi'(0) = \left(EI(.)\phi''(.)\right)'(1) = 0 \\ \phi''(1) = -\dfrac{1}{EI(1)}(\alpha\lambda + \beta)\phi'(1). \end{cases} \tag{3.8}$$

In order to transform the coefficient function appearing with ϕ in the first expression of (3.8) into a constant, a space transformation is introduced. Let

$$f(z) = \phi(x), \quad z = z(x) = \frac{1}{h}\int_0^x \left(\frac{m(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta \tag{3.9}$$

where

$$h = \int_0^1 \left(\frac{m(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta.$$

Thus, (3.8) can be transformed using again its boundary conditions

$$\begin{cases} f^{(4)}(z) + a(z) f'''(z) + b(z) f''(z) + c(z) f'(z) + \lambda^2 h^4 f(z) = 0, \quad 0 < z < 1, \\ f(0) = f'(0) = 0, \\ EI(1)z_x^3(1) f'''(1) + \left[EI'(1)z_x^2(1) + 3EI(1)z_{xx}(1) z_x(1)\right] f''(1) \\ +\left[EI(1)z_{xxx}(1) + EI'(1)z_{xx}(1)\right] f'(1) = 0, \\ f''(1) + \left[\dfrac{z_{xx}(1)}{z_x^2(1)} + \dfrac{\alpha\lambda + \beta}{EI(1) z_x(1)}\right] f'(1) = 0, \end{cases} \tag{3.10}$$

with

$$a(z) = \frac{6z_{xx}}{z_x^2} + \frac{2EI'(x)}{z_x EI(x)} \tag{3.11}$$

$$b(z) = \frac{3z_{xx}^2}{z_x^4} + \frac{6z_{xx}EI'(x)}{z_x^3 EI(x)} + \frac{EI''(x)}{z_x^2 EI(x)} + \frac{4z_{xxx}}{z_x^3} \tag{3.12}$$

$$c(z) = \frac{z_{xxxx}}{z_x^4} + \frac{2z_{xxx}EI'(x)}{z_x^4 EI(x)} + \frac{z_{xx}EI''(x)}{z_x^4 EI(x)} \tag{3.13}$$

$$z_x = \frac{1}{h}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}}, \quad z_x^4 = \frac{1}{h^4}\frac{m(x)}{EI(x)} \tag{3.14}$$

and

$$z_{xx} = \frac{1}{4h}\left(\frac{m(x)}{EI(x)}\right)^{-\frac{3}{4}} \frac{d}{dx}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}}. \tag{3.15}$$

In order to solve (3.10), the strategy as in Chapter 2, Section 4 of [11] is used. Hence, in order to eliminate the third derivative term $a(z)\, f'''(z)$ in (3.10), a new invertible space transformation is introduced:

$$g(z) = \exp\left(\frac{1}{4}\int_0^z a(\zeta)\, d\zeta\right) f(z), \quad 0 < z < 1.$$

Boundary value problem (3.10) can be written as:

$$\begin{cases} g^{(4)}(z) + b_1(z)\, g''(z) + c_1(z)\, g'(z) + d_1(z)\, g(z) + \lambda^2 h^4 g(z) = 0, \quad 0 < z < 1, \\ g(0) = g'(0) = 0 \\ g''(1) + b_{11} g'(1) + b_{12} g(1) = 0 \\ g'''(1) + b_{21} g''(1) + b_{22} g'(1) + b_{23} g(1) = 0, \end{cases} \tag{3.16}$$

where

$$b_1(z) = -\frac{3}{2}a'(z) - \frac{3}{8}a^2(z) + b(z) \tag{3.17}$$

$$c_1(z) = \frac{1}{8}a^3(z) - \frac{1}{2}a(z)b(z) - a''(z) + c(z) \tag{3.18}$$

$$\begin{aligned} d_1(z) &= \frac{3}{16}a'^2(z) - \frac{1}{4}a'''(z) + \frac{3}{32}a'(z)a^2(z) - \frac{3}{256}a^4(z) \\ &\quad + b(z)\left(\frac{1}{16}a^2(z) - \frac{1}{4}a'(z)\right) - \frac{a(z)c(z)}{4} \end{aligned} \tag{3.19}$$

$$b_{11} = -\frac{1}{2}a(1) + \frac{z_{xx}(1)}{z_x^2(1)} + \frac{\alpha\lambda}{EI(1)z_x(1)} + \frac{\beta}{EI(1)z_x(1)} \tag{3.20}$$

$$b_{12} = -\frac{1}{4}a'(1) + \frac{1}{16}a^2(1) - \frac{1}{4}a(1)\left(\frac{z_{xx}(1)}{z_x^2(1)} + \frac{\alpha\lambda}{EI(1)z_x(1)} + \frac{\beta}{EI(1)z_x(1)}\right) \tag{3.21}$$

$$b_{21} = -\frac{3}{4}a(1) + \frac{3z_{xx}(1)}{z_x^2(1)} + \frac{EI'(1)}{EI(1)z_x(1)} \tag{3.22}$$

$$\begin{aligned} b_{22} &= -\frac{3}{4}a'(1) + \frac{3}{16}a^2(1) - \frac{EI'(1)a(1)}{2EI(1)z_x(1)} - \frac{3z_{xx}(1)a(1)}{2z_x^2(1)} + \frac{z_{xxx}(1)}{z_x^3(1)} \\ &\quad + \frac{EI'(1)z_{xx}(1)}{EI(1)z_x^3(1)} \end{aligned} \tag{3.23}$$

$$\begin{aligned} b_{23} &= -\frac{1}{4}a''(1) + \frac{3}{16}a'(1)a(1) - \frac{1}{64}a^3(1) - \frac{a'(1)EI'(1)}{4EI(1)z_x(1)} \\ &\quad - \frac{3a'(1)z_{xx}(1)}{4z_x^2(1)} + \frac{EI'(1)a^2(1)}{16EI(1)z_x(1)} + \frac{3z_{xx}(1)a^2(1)}{16z_x^2(1)} \\ &\quad - \frac{a(1)z_{xxx}(1)}{4z_x^3(1)}. \end{aligned} \tag{3.24}$$

Due to the invertibility of the above transformations, the obtained problem (3.16) is equivalent to the original problem (3.7).

To further solve the eigenvalue problem (3.16), we follow the procedure in G. D. Birkhoff [2], [3] and M. A. Naimark [11] and divide the complex plane into eight distinct sectors,

$$S_k = \left\{ z \in \mathbb{C} : \frac{k\pi}{4} \leq \arg z \leq \frac{(k+1)\pi}{4} \right\}, \ k = 0, 1, 2, \ldots, 7 \tag{3.25}$$

and let $\omega_1, \omega_2, \omega_3, \omega_4$ be the roots of equation $\theta^4 + 1 = 0$ that are arranged so that

$$\operatorname{Re}(\rho\omega_1) \leq \operatorname{Re}(\rho\omega_2) \leq \operatorname{Re}(\rho\omega_3) \leq \operatorname{Re}(\rho\omega_4), \quad \forall \rho \in S_k. \tag{3.26}$$

Obviously, in sector S_1, we name the roots of -1 as

$$\omega_1 = \exp\left(i\frac{3}{4}\pi\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,$$

$$\omega_2 = \exp\left(i\frac{1}{4}\pi\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,$$

$$\omega_3 = \exp\left(i\frac{5}{4}\pi\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i,$$

$$\omega_4 = \exp\left(i\frac{7}{4}\pi\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i,$$

which satisfy the inequalities in (3.26) and choices can also be made for other sectors. In the rest of this section, we shall derive the asymptotic behavior of the eigenvalue of the sectors S_1 and S_2 because the same will hold for the other sectors with similar proofs.

Set $\lambda = \dfrac{\rho^2}{h^2}$, in each sector S_k. In order to analyse the asymptotic fundamental solutions of system (3.16), we need the following result(see [11] and [18]):

Lemma 3.3. For $\rho \in S_k$ with ρ large enough, the equation:

$$g^{(4)}(z) + b_1(z)\, g''(z) + c_1(z)\, g'(z) + d_1(z)\, g(z) + \rho^4 g(z) = 0, \quad 0 < z < 1,$$

has four linearly independent asymptotic fundamental solutions,

$$\Phi_s(z, \rho) = e^{\rho\omega_s z}\left(1 + \frac{\Phi_{s,1}(z)}{\rho} + \mathcal{O}\left(\rho^{-2}\right)\right), \quad s = 1, 2, 3, 4$$

and hence their derivatives for $s = 1, 2, 3, 4$ and $j = 1, 2, 3$ are given by

$$\frac{d^j}{dz^j}\Phi_s(z, \rho) = (\rho\omega_s)^j\, e^{\rho\omega_s z}\left(1 + \frac{\Phi_{s,1}(z)}{\rho} + \mathcal{O}\left(\rho^{-2}\right)\right)$$

where

$$\Phi_{s,1}(z) = -\frac{1}{4\omega_s}\int_0^z b_1(\zeta)\,d\zeta.$$

Hence, for $s = 1, 2, 3, 4$,

$$\Phi_{s,1}(0) = 0, \quad \Phi_{s,1}(1) = -\frac{1}{4\omega_s}\int_0^1 b_1(\zeta)\,d\zeta = \frac{\mu_1}{\omega_s}, \qquad \text{with } \mu_1 = -\frac{1}{4}\int_0^1 b_1(\zeta)\,d\zeta.$$

For convenience, we introduce the notation $[a]_2 = a + \mathcal{O}\left(\rho^{-2}\right)$.

Lemma 3.4. For $\rho \in S_1$, if we set $\delta = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, then we have following inequalities

$$\operatorname{Re}(\rho\omega_1) \le -|\rho|\,\delta, \ \operatorname{Re}(\rho\omega_4) \ge |\rho|\,\delta \text{ and } e^{\rho\omega_1} = \mathcal{O}\left(\rho^{-2}\right) \text{ when } |\rho| \to \infty.$$

Using Lemma 3.3, we obtain asymptotic expressions for the boundary conditions for large enough $|\rho|$, for $s = 1, 2, 3, 4$,

$$U_4(\Phi_s, \rho) = \Phi_s(0, \rho) = 1 + \mathcal{O}\left(\rho^{-2}\right) = [1]_2,$$

$$U_3(\Phi_s, \rho) = \Phi_s'(0, \rho) = \rho\omega_s\left(1 + \mathcal{O}\left(\rho^{-2}\right)\right) = \rho\omega_s[1]_2,$$

$$U_2(\Phi_s, \rho) = \Phi_s''(1, \rho) + b_{11}\Phi_s'(1, \rho) + b_{12}\Phi_s(1, \rho),$$

$$U_2(\Phi_s, \rho) = (\rho\omega_s)^2 e^{\rho\omega_s}\left(1 + \gamma\omega_s^{-2} + \mu_1\rho^{-1}\omega_s^{-1} + \gamma\mu_1\rho^{-1}\omega_s^{-3} + \mathcal{O}\left(\rho^{-2}\right)\right)$$

where

$$\gamma = -\frac{\alpha\, a(1)}{4EI(1)z_x(1)h^2}.$$

Then,

$$U_2(\Phi_s, \rho) = (\rho\omega_s)^2 e^{\rho\omega_s}\left[1 + \gamma\omega_s^{-2} + \mu_1\rho^{-1}\omega_s^{-1} + \gamma\mu_1\rho^{-1}\omega_s^{-3}\right]_2.$$

Similary,

$$U_1(\Phi_s, \rho) = \Phi_s'''(1, \rho) + b_{21}\Phi_s''(1, \rho) + b_{22}\Phi_s'(1, \rho) + b_{23}\Phi_s(1, \rho)$$

$$U_1(\Phi_s, \rho) = (\rho\omega_s)^3 e^{\rho\omega_s}(1 + (\mu_1 + b_{21})\rho^{-1}\omega_s^{-1} + \mathcal{O}\left(\rho^{-2}\right))$$

$$U_1(\Phi_s, \rho) = (\rho\omega_s)^3 e^{\rho\omega_s}\left[1 + (\mu_1 + b_{21})\rho^{-1}\omega_s^{-1}\right]_2.$$

Note that $\lambda = \frac{\rho^2}{h^2} \neq 0$ is the eigenvalue of (3.16) if and only if ρ satisfies the characteristic equation

$$\Delta(\rho) = \begin{vmatrix} U_4(\Phi_1, \rho) & U_4(\Phi_2, \rho) & U_4(\Phi_3, \rho) & U_4(\Phi_4, \rho) \\ U_3(\Phi_1, \rho) & U_3(\Phi_2, \rho) & U_3(\Phi_3, \rho) & U_3(\Phi_4, \rho) \\ U_2(\Phi_1, \rho) & U_2(\Phi_2, \rho) & U_2(\Phi_3, \rho) & U_2(\Phi_4, \rho) \\ U_1(\Phi_1, \rho) & U_1(\Phi_2, \rho) & U_1(\Phi_3, \rho) & U_1(\Phi_4, \rho) \end{vmatrix} = 0. \tag{3.27}$$

By substitution, we obtain

$$\Delta(\rho) = \left| \begin{matrix} [1]_2 & [1]_2 \\ \rho\omega_1 [1]_2 & \rho\omega_2 [1]_2 \\ 0 & (\rho\omega_2)^2 e^{\rho\omega_2}\left[1+\gamma\omega_2^{-2}+\mu_1\rho^{-1}\omega_2^{-1}+\gamma\mu_1\rho^{-1}\omega_2^{-3}\right]_2 \\ 0 & (\rho\omega_2)^3 e^{\rho\omega_2}\left[1+(\mu_1+b_{21})\rho^{-1}\omega_2^{-1}\right]_2 \end{matrix} \right.$$

$$\begin{matrix} [1]_2 \\ \rho\omega_3 [1]_2 \\ (\rho\omega_3)^2 e^{\rho\omega_3}\left[1+\gamma\omega_3^{-2}+\mu_1\rho^{-1}\omega_3^{-1}+\gamma\mu_1\rho^{-1}\omega_3^{-3}\right]_2 \\ (\rho\omega_3)^3 e^{\rho\omega_3}\left[1+(\mu_1+b_{21})\rho^{-1}\omega_3^{-1}\right]_2 \end{matrix}$$

$$\left. \begin{matrix} 0 \\ 0 \\ (\rho\omega_4)^2 e^{\rho\omega_4}\left[1+\gamma\omega_4^{-2}+\mu_1\rho^{-1}\omega_4^{-1}+\gamma\mu_1\rho^{-1}\omega_4^{-3}\right]_2 \\ (\rho\omega_4)^3 e^{\rho\omega_4}\left[1+(\mu_1+b_{21})\rho^{-1}\omega_4^{-1}\right]_2 \end{matrix} \right|.$$

Expanding the above determinant, we obtain the following expression:

$$\begin{aligned} \Delta(\rho) = \rho^6 e^{\rho\omega_4}\{&(-1)(\omega_3-\omega_1)\big[(\omega_2^{-2}-\omega_4^{-2})\gamma+(\mu_1+b_{21})(\omega_4^{-1}-\omega_2^{-1})\rho^{-1} \\ &+\gamma(\mu_1+b_{21})(\omega_2^{-2}\omega_4^{-1}-\omega_4^{-2}\omega_2^{-1})\rho^{-1} \\ &+\mu_1(\omega_2^{-1}-\omega_4^{-1})\rho^{-1}+\gamma\mu_1(\omega_2^{-3}-\omega_4^{-3})\rho^{-1}\big]e^{\rho\omega_2} \\ &+(\omega_2-\omega_1)\big[(\omega_3^{-2}-\omega_4^{-2})\gamma+(\mu_1+b_{21})(\omega_4^{-1}-\omega_3^{-1})\rho^{-1} \\ &+\gamma(\mu_1+b_{21})(\omega_3^{-2}\omega_4^{-1}-\omega_4^{-2}\omega_3^{-1})\rho^{-1} \\ &+\mu_1(\omega_3^{-1}-\omega_4^{-1})\rho^{-1}+\gamma\mu_1(\omega_3^{-3}-\omega_4^{-3})\rho^{-1}\big]e^{-\rho\omega_2}+\mathcal{O}\left(\rho^{-2}\right)\}. \end{aligned}$$

In sector S_1, the choices are such that:

$\omega_1^2=-i,\ \ \omega_2^2=i,\ \ \omega_3^2=i,\ \ \omega_4^2=-i,\ \ \omega_3^{-1}\omega_4=i,\ \ \omega_2^{-1}\omega_4=-i,\ \ \omega_3=-\omega_2,$

$\omega_4-\omega_3=\sqrt{2},\ \ \omega_1-\omega_3=\sqrt{2}i,\ \ \omega_2-\omega_1=\sqrt{2},\ \ \omega_4-\omega_2=-i\sqrt{2},$

$\omega_2^{-2}-\omega_4^{-2}=-2i,\ \ \omega_3^{-2}-\omega_4^{-2}=-2i,\ \ \omega_3^2\omega_4^2=1,\ \ \omega_2^2\omega_4^2=1,$
$\omega_2^{-3}-\omega_4^{-3}=-(1+i)\omega_2,\ \ \omega_3^{-3}-\omega_4^{-3}=(1-i)\omega_2.$

Putting them into $\Delta(\rho)$, we obtain:

$$\Delta(\rho)=2\sqrt{2}\gamma\rho^6 e^{\rho\omega_4}\{e^{\rho\omega_2}-i\,e^{-\rho\omega_2}+[\mu_2 e^{\rho\omega_2}+\mu_3 e^{-\rho\omega_2}]\rho^{-1}+\mathcal{O}\left(\rho^{-2}\right)\}, \tag{3.28}$$

where

$$\begin{cases} \mu_2=\dfrac{\sqrt{2}}{2}\left[2\mu_1+b_{21}-\dfrac{b_{21}}{\gamma}\right] \\ \mu_3=\dfrac{\sqrt{2}}{2}\left[2\mu_1+b_{21}+\dfrac{b_{21}}{\gamma}\right] \end{cases} \tag{3.29}$$

Theorem 3.5. If $\gamma \neq 0$, the boundary eigenvalue problem (3.16) is strongly regular.

Proof. Since

$$\Theta_{-1,0} = -2\sqrt{2}i\gamma \quad \Theta_{1,0} = 2\sqrt{2}\gamma \quad \Theta_{0,0} = 0$$

so the eigenvalue problem (3.16) is strongly regular. ∎

For the definition of *strongly regular* boundary problem, the reader is referred to (pp. 43 Definition 3.2.5 in [17]).

Now, we compute the asymptotic behavior of λ_n. The equation $\Delta(\rho) = 0$ and (3.28) imply that

$$e^{\rho\omega_2} - i\, e^{-\rho\omega_2} + \mu_2\rho^{-1}e^{\rho\omega_2} + \mu_3\rho^{-1}e^{-\rho\omega_2} + \mathcal{O}\left(\rho^{-2}\right) = 0 \tag{3.30}$$

which can be rewritten as

$$e^{\rho\omega_2} - ie^{-\rho\omega_2} + \mathcal{O}\left(\rho^{-1}\right) = 0. \tag{3.31}$$

Ignoring the higher order terms, the following equation

$$e^{\rho\omega_2} - ie^{-\rho\omega_2} = 0$$

has solutions

$$\rho_n = \left(\frac{1}{4} - n\right)\frac{\pi i}{\omega_2},\ n = 1, 2, \ldots \tag{3.32}$$

Let $\widetilde{\rho_n}$ be the solutions of (3.31). So by Rouche's theorem (see M. A. Naimark, p.70 in [11]) to (3.31), we get the following expression:

$$\widetilde{\rho_n} = \rho_n + \alpha_n = \left(\frac{1}{4} - n\right)\frac{\pi i}{\omega_2} + \alpha_n, \quad \alpha_n = \mathcal{O}\left(n^{-1}\right),\ n = N, N+1, \ldots, \tag{3.33}$$

where N is a large positive integer. Substituting $\widetilde{\rho_n}$ into (3.30), and using the fact that $e^{\rho\omega_2} = ie^{-\rho\omega_2}$, we obtain

$$e^{\alpha_n\omega_2} - e^{-\alpha_n\omega_2} + \mu_2\widetilde{\rho_n}^{-1}e^{\alpha_n\omega_2} - i\mu_3\widetilde{\rho_n}^{-1}e^{-\alpha_n\omega_2} + \mathcal{O}\left(\widetilde{\rho_n}^{-2}\right) = 0.$$

Expanding the exponential function according to its Taylor series, we get

$$\alpha_n = -\frac{\mu_2}{2\omega_2\rho_n} + \frac{\mu_3}{2\omega_2\rho_n}i + \mathcal{O}\left(n^{-2}\right),\ n = N, N+1, \ldots$$

Therefore, we have

$$\widetilde{\rho_n} = \left(\frac{1}{4} - n\right)\frac{\pi i}{\omega_2} + \frac{\mu_2}{2\left(\frac{1}{4} - n\right)\pi}i + \frac{\mu_3}{2\left(\frac{1}{4} - n\right)\pi} + \mathcal{O}\left(n^{-2}\right),\ n = N, N+1, \ldots$$

Note that $\lambda_n = \dfrac{\widetilde{\rho_n}^2}{h^2} \neq 0$, $\omega_2 = e^{i\frac{\pi}{4}}$ and $\omega_2^2 = i$. So we have

$$\lambda_n = \frac{\sqrt{2}}{2h^2}\left(\mu_3 - \mu_2\right) + \frac{1}{h^2}\left[\frac{\sqrt{2}}{2}\left(\mu_3 + \mu_2\right) + \left(\frac{1}{4} - n\right)^2 \pi^2\right] i + \mathcal{O}\left(n^{-1}\right), \quad (3.34)$$

where $n = N, N+1, \ldots$ with N large enough.

The same proof can be applied to sector S_2 because the eigenvalues of the problem (3.16) can be obtained by a similar calculation with the choices

$$\omega_1 = \exp\left(i\frac{1}{4}\pi\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,$$

$$\omega_2 = \exp\left(i\frac{3}{4}\pi\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,$$

$$\omega_3 = \exp\left(i\frac{7}{4}\pi\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i,$$

$$\omega_4 = \exp\left(i\frac{5}{4}\pi\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i,$$

so that they satisfy the inequality

$$\operatorname{Re}\left(\rho\omega_1\right) \leq \operatorname{Re}\left(\rho\omega_2\right) \leq \operatorname{Re}\left(\rho\omega_3\right) \leq \operatorname{Re}\left(\rho\omega_4\right), \ \forall \rho \in S_2.$$

Then, in sector S_2, the characteristic determinant $\Delta\left(\rho\right)$ of (3.7) :

$$\Delta\left(\rho\right) = 2\sqrt{2}\gamma\rho^6 e^{\rho\omega_4}\{e^{\rho\omega_2} + i\, e^{-\rho\omega_2} - [\mu_2 e^{\rho\omega_2} + \mu_3 e^{-\rho\omega_2}]\rho^{-1} + \mathcal{O}\left(\rho^{-2}\right)\}.$$

By a direct calculation, we have

$$\widetilde{\rho_n} = \left(\frac{1}{4} - n\right)\frac{\pi i}{\omega_2} - \frac{\mu_2}{2\left(\frac{1}{4} - n\right)\pi} i + \frac{\mu_3}{2\left(\frac{1}{4} - n\right)\pi} + \mathcal{O}\left(n^{-2}\right), \ n = N, N+1, \ldots \quad (3.35)$$

with N large enough. Again, using $\lambda_n = \dfrac{\rho_n^2}{h^2} \neq 0$, $\omega_2 = e^{i\frac{3\pi}{4}}$ and $\omega_2^2 = -i$.

$$\lambda_n = \frac{\sqrt{2}}{2h^2}\left(\mu_3 - \mu_2\right) - \frac{1}{h^2}\left[\frac{\sqrt{2}}{2}\left(\mu_3 + \mu_2\right) + \left(\frac{1}{4} - n\right)^2 \pi^2\right] i + \mathcal{O}\left(n^{-1}\right), \quad (3.36)$$

where $n = N, N+1, \ldots$ with N large enough.

Here we should point out that the eigenvalues generated from the other sectors S_k coincide with those from S_1 and S_2. The detailed argument can be found in M. A.

Naimark [11]. Combining with (3.34) and (3.36), we obtain the following result on the eigenvalues.

Theorem 3.6. Let A be defined by (2.6) and (2.7). If $\gamma \neq 0$, then an asymptotic expression of the eigenvalues of the problem (3.16) is given by

$$\lambda_n = \frac{\sqrt{2}}{2h^2}(\mu_3 - \mu_2) \pm \frac{1}{h^2}\left[\frac{\sqrt{2}}{2}(\mu_3 + \mu_2) + \left(\frac{1}{4} - n\right)^2 \pi^2\right] i + \mathcal{O}\left(n^{-1}\right), \quad (3.37)$$

where $n = N, N+1, \ldots$ with N large enough, and

$$\mu_3 - \mu_2 = \frac{\sqrt{2}b_{21}}{\gamma} = -\frac{1}{\alpha}\left(\sqrt{2}h(m(1))^{\frac{1}{4}}(EI(1))^{\frac{3}{4}}\right) \quad (3.38)$$

$$\mu_3 + \mu_2 = \sqrt{2}(2\mu_1 + b_{21}). \quad (3.39)$$

Moreover, λ_n $(n = N, N+1, \ldots)$ with sufficiently large modulus are simple and distinct except for finitely many of them, and satisfy

$$\lim_{n\to+\infty} \mathrm{Re}\lambda_n = -\frac{1}{\alpha h}\left((m(1))^{\frac{1}{4}}(EI(1))^{\frac{3}{4}}\right). \quad (3.40)$$

Notice that in the uniform case with $m(x) = EI(x) = 1$, we find the following result which was obtained in [15]:

$$\lim_{n\to+\infty} \mathrm{Re}\lambda_n = -\frac{1}{\alpha}.$$

3.2. Riesz basis property of eigenfunctions of A

In this subsection, we discuss the Riesz basis property of the eigenfunctions of operator A of the system (2.8). We follow an idea due to Wang (see [18] pp. 473–475). We begin with showing that the generalized eigenfunctions of A form an unconditional basis in Hilbert state space H.

For this task, we introduce a transformation $\mathcal{L}$ via

$$\mathcal{L}(f, g) = (\phi, \psi)$$

where

$$\phi(x) = f(z), \ \psi(x) = g(z), \ z = \frac{1}{h}\int_0^x \left(\frac{m(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta, \quad (3.41)$$

with

$$h = \int_0^1 \left(\frac{m(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta. \quad (3.42)$$

It is easily seen that $\mathcal{L}$ is a bounded invertible operator on $\mathbb{H}$. Next define the following ordinary differential operator:

$$\begin{cases} L(f) = f^{(4)}(z) + a(z) f'''(z) + b(z) f''(z) + c(z) f'(z), \\ B_1(f) = f(0) = 0, \quad B_2(f) = f'(0) = 0, \\ B_3(f) = f''(1) + \left[\dfrac{z_{xx}(1)}{z_x^2(1)} + \dfrac{(\alpha\lambda + \beta)}{EI(1) z_x(1)}\right] f'(1) = 0 \\ B_4(f) = EI(1) z_x^3(1) f'''(1) + \left[EI'(1) z_x^2 + 3EI(1) z_{xx}(1) z_x(1)\right] f''(1) \\ \qquad + \left[EI(1) z_{xxx}(1) + EI'(1) z_{xx}(1)\right] f'(1) = 0, \end{cases} \tag{3.43}$$

where the coefficients are given by (3.11)-(3.15). Let $\mathbb{A}$ be defined as in (2.7), $\eta \in \sigma(\mathbb{A})$ be an eigenvalue of $\mathbb{A}$ and (f, g) be an eigenfunction corresponding to η, then we have $g = \eta f$ and f will satisfy the following equation:

$$f^{(4)}(z) + a(z) f'''(z) + b(z) f''(z) + c(z) f'(z) + \eta^2 f(z) = 0,$$

with boundary conditions $B_j(f) = 0, \ j = 1, 2, 3, 4$. Now by taking $\lambda = \dfrac{\eta}{h^2}$ and

$$\mathcal{L}(f, g) = (\phi(x), \psi(x))$$

we see that $\psi = \lambda\phi$ and ϕ satisfies the equation

$$\begin{cases} \phi^{(4)}(x) + \dfrac{2EI'(x)}{EI(x)} \phi'''(x) + \dfrac{EI''(x)}{EI(x)} \phi''(x) + \dfrac{\lambda^2 m(x)}{EI(x)} \phi(x) = 0, \quad 0 < x < 1, \\ \phi(0) = \phi'(0) = (EI(.)\phi'')'(1) = 0 \\ \phi''(1) = -\dfrac{1}{EI(1)} (\alpha\lambda + \beta) \phi'(1). \end{cases} \tag{3.44}$$

Hence we have the following result:

$$\eta \in \sigma(\mathbb{A}) \Leftrightarrow \lambda \in \sigma(A).$$

Theorem 3.7. Let operator A of the system (2.8). Then the eigenvalues of operator A are all simple except for finitely many of them, and the generalized eigenfunctions of operator A form a Riesz basis for the Hilbert state space $\mathbb{H}$.

Proof. According to Theorem 3.5, the boundary problem (3.16) is strongly regular. Therefore the eigenvalues are separated and simple except for finitely many of them. Also, the strongly regular boundary conditions ensure that the generalized eigenfunction sequence $F_n = \left(f_n, \eta_n f_n\right)$ of operator $\mathbb{A}$ forms a Riesz basis for $\mathbb{H}$. Since $\mathcal{L}$ is bounded and invertible on $\mathbb{H}$, it follows that $\Psi_n = \left(\phi_n, \lambda_n \phi_n\right) = \mathcal{L} F_n$ also forms a Riesz basis on $\mathbb{H}$. ■

We are now in a position to investigate the exponential stability of system (2.8). Since the Riesz basis property implies the spectrum-determined growth condition (see

Curtain and Zwart [5]) and (3.40) describes the asymptote of $\sigma(A)$, for any small $\varepsilon > 0$ there are only finitely many eigenvalues of A in the following half-plane:

$$\Sigma : \mathrm{Re}\lambda > -\frac{1}{\alpha h}\left((m(1))^{\frac{1}{4}}(EI(1))^{\frac{3}{4}}\right) + \varepsilon. \tag{3.45}$$

3.3. Exponential stability

The Theorem 3.7 is one of the fundamental properties of the evolutive system (1.1)−(1.4). Many other important properties of this system can be concluded from Theorem 3.7. The exponential stability stated below is one of such important property.

Theorem 3.8. System (1.1)−(1.4) is exponential stable for any $\beta > 0$ and $\alpha > 0$. That is, there are nonnegative constants M, ω such that the energy $E(t)$ of system (1.1)−(1.4) satisfies

$$E(t) = \frac{1}{2}\int_0^1 EIw_{xx}^2\,dx + \frac{1}{2}\int_0^1 mw_t^2\,dx + \frac{\beta}{2}(w_x(1,t))^2 \leq ME(0)\,e^{-\omega t}, \ \forall t \geq 0,$$

for any initial condition $(w(x,0), w_t(x,0)) \in \mathbb{H}$.

Proof. According to Theorem 2.1, A is dissipative and e^{At} is contraction semigroup on $\mathbb{H}$. If we can show that there is no eigenvalue on the imaginary axis, then the exponential stability holds. Let $\lambda = ir$ with $r \in \mathbb{R}$ be an eigenvalue of operator A on the imaginary axis and $\Psi = (\phi, \psi)^T$ be the corresponding eigenfunction, then $\psi = \lambda\phi$. Then we have

$$\mathrm{Re}(\langle A\Psi, \Psi\rangle_{\mathbb{H}}) = -\alpha\left|\psi'(1)\right|^2$$

$$0 = \|\Psi\|_{\mathbb{H}}^2\,\mathrm{Re}(\lambda) = \mathrm{Re}(\langle A\Psi, \Psi\rangle_{\mathbb{H}}) = -\alpha\left|\psi'(1)\right|^2,$$

since $\alpha > 0$, we get

$$\psi'(1) = 0.$$

Then $\phi'(1) = 0$. $\phi(x)$ has to obey the uniqueness theorem of the differential equation:

$$\begin{cases} \lambda^2 m(x)\phi(x) + \left(EI(x)\,\phi''(x)\right)'' = 0, \quad 0 < x < 1, \\ \phi(0) = \phi'(0) = \phi(1) = \phi'(1) = \phi''(1) = \phi'''(1) = 0. \end{cases} \tag{3.46}$$

The above equation has a zero solution only (see [8]). However, $\Psi = 0$ contradicts Ψ being an eigenfunction and so there is no eigenvalue on the imaginary axis. Therefore, we get $\mathrm{Re}(\lambda) < 0$.

From Theorem 3.7 and the spectrum-determined growth condition, the system is exponentially stable. ■

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