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Topological properties of semi-linear uniform spaces

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Abstract

In this paper we shall give the most importent topological properties of semi-linear uniform space (X, Γ), also we shall show that every semi-linear uniform space (X, Γ), generate a Tychonof topology on X, namely τ_Γ, beside we shall prove that "For every sequence $V_0, V_1, V_2, \ldots$ of members of Γ satisfies $V_0 = X \times X$ and $2V_{i+1} \subseteq V_i$ for $i = 1, 2, \ldots,$ there exist a pseudometric p on the set X which is uniform with respect to Γ such that for every $i \geq 1$, $\left\{(x, y) : \rho(x, y) < \frac{1}{2^i}\right\} \subseteq V_i \subseteq \left\{(x, y) : \rho(x, y) \leq \frac{1}{2^i}\right\}$ spaces. Finally we ask the following open question. "For $x, y \in X$. Is $\delta(x, y)$ is open with respect to the topology induced by Γ on $X \times X$?.

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1. Introduction

A uniform space is a set with a uniform structure. Uniform spaces are topological spaces with additional structure that is used to define uniform properties such as completeness, uniform continuity and uniform convergence. The notion of uniformity has been investigated by several mathematician as Weil [13], [14], and [15]. L.W. Cohen [3], and [4].

Graves [6] . The theory of uniform spaces was given by Burbaki in [2] , Also Wiel´s in his booklet [14] define uniformly continuous mapping.

In 2009 Tallafha, A. and Khalil, R. defined a new type of uniform space namely, semi-linear uniform space [8] . The authors studied some cases of best approximation in such spaces, and gave some open problems in approximation theory in uniform spaces. Besides they defined a set valued map ρ, called metric type, on semi-linear uniform spaces that enables one to study analytical concepts on semi-linear uniformspaces.

In [9], [10] and [11] . Tallafha, A. answered the question "Is there a semi-linear uniform space which is not metrizable?". Besides he solved question 1 and 2 in [8] . Also he defined another set valued map called δ on $X \times X$, which is used with ρ to give more properties of semi-linear uniform spaces. Finally he studied the relation between ρ and δ, and he showed that, $\rho(x, y) = \rho(s, t)$ if and only if $\delta(x, y) = \delta(s, t)$. Also he defined Lipschitz condition, and contraction mapping on semi-linear uniform spaces, which enables one to study fixed point for such functions.

Since Lipschitz condition, and contractions are usually discussed in metric and normed spaces, and never been studied in other weaker spaces. We believe that the structure of semi-linear uniform spaces is very rich and all the known results on fixed point theory can be generalized.

In [1] , Alhihi,S. Gave mor properties of semi-linear uniform space. Also Tallafha, A. and Alhihi, S. [12] gave another properties of semi-linear uniform spaces.

In the next sections of this paper we shall give the most important topological properties of uniform spaces which still valid in semi-linear uniform spaces.

2. Uniform Spaces

In this section we shall give the definitions and notation of uniform spaces, besides the most important topological properties of Uniform spaces which is still valid in semi-linear uniform spaces.

Let X be a non empty set and let A, B be subsets of $X \times X$, i.e., A, B are relations on$X$. The invers relation of A will be denoted by $A^{-1} = \{(y, x) : (x, y) \in V\}$ and the composition of A, B will be denoted by $A \circ B$. Clearly the composition is associated but not commutative. For a relation A on X and a natural number n the relation nA is defined inductively by $1A = A$ and $nA = (n - 1) A \circ A$. The identity relation will be denoted by $\Delta = \{(x, x) : x \in X\}$. Let D_X be the set of all relations on X such that each element V of D_X contains the diagonal Δ and $V = V^{-1}$, D_X is called the family of all entourages of the diagonal. If $(x, y) \in A \in D_X$, we say the distance between x and y is less than A and we write $|x - y| < A$; otherwise we write $|x - y| \geq A$. For $E \subseteq X$, $A \in D_X$ if $E \times E \subset A$, we say that the diameter of E is less than A and we write $d(E) < A$. It is easy to check that for any $x, y, z \in X$ and $V, U, W \in D_X$ the following conditions hold:

(1) $|x - x| < V$.

(2) $|x - y| < V$ if and only if $|y - x| < V$.

(3) If $|x - y| < V_1$ and $|y - z| < V_2$, then $|x - z| < V_1 \circ V_2$.

Let $x_0 \in X$ and $A \in D_X$, the ball with center x_0 and radius A is defined by $B(x_0, A) = \{x \in X : |x_0 - x| < A\}$. It follows immediately that the diameter of $B(x_0, A)$ is $2A$. For $E \subseteq X$, $A \in D_X$, by the A ball about E we mean $B(E, A) = \bigcup_{x \in E} B(x, A)$.

Let Fourier be a sub-collection of D_X. Then we have the following definition.

Definition 2.1. [3] The pair $(X, Fourier)$ is called a **uniform space** if:

(i) $V_1 \cap V_2 \in Fourier$ for all V_1, V_2 in $Fourier$.

(ii) For every $V \in Fourier$, there exists $U \in Fourier$ such that $U \circ U \subseteq V$.

(iii) $\bigcap \{V : V \in Fourier\} = \Delta$.

(iv) If $V \in Fourier$ and $V \subseteq W \in D_X$, then $W \in Fourier$.

If the condition $V = V^{-1}$ is omitted then the space is quasiuniform space.

Condition (iv) is a very strong condition, using this condition, we may conclude that. For all $V \in Fourier$ and all $x, y \in X$, we have $V' = V \cup \{(x, y), (y, x)\} \in Fourier$.

Definition 2.2. Let $(X, Fourier)$ be a uniform space, a collection $\beta \subseteq Fourier$ is called a base for the uniformity $Fourier$ if for every $V \in Fourier$ there exist a $U \in \beta$ such that $U \subseteq V$.

The smallest cardinal number of β is called the weight of the uniformity and denoted by $\omega(\beta)$.

Uniform space induce a topological space, the induced topology satisfies a lot of important topological proprieties. We refer to [5] for the following results and definitions.

Proposition 2.3. Every uniform space $(X, Fourier)$ induce a T_1 topological space (X, τ), $\tau = \{G \subseteq X :$ for every $x \in G$ there exists a $V \in Fourier$ such that $B(x, V) \subseteq G\}$.

The topology τ is called the topology induced by the uniformity $Fourier$ and denoted by $\tau_{Fourier}$. The Tychonoff topology on $X \times X$ is called the topology induced by the uniformity $Fourier$ on $X \times X$.

Proposition 2.4. The interior of a set $A \subseteq X$ with respect to $\tau_{Fourier}$ is coincides with the set $A^\circ = \{x \in X :$ there exists a $V \in Fourier$ such that $B(x, V) \subseteq A\}$.

Corollary 2.5. Consider the topological space $(X, \tau_{Fourier})$ then,

1. For every $x \in X$ and any $V \in Fourier$ the set $\mathrm{Int}(B(x, V))$ is a neighborhood of x.

2. For every $x \in X$ and any $A \subseteq X$ we have $x \in \overline{A}$ if and only if $A \cap B(x, V)$ is not empty for all $V \in Fourier$.

Definition 2.6. [6] A topological space is called uniformizable if there is a uniform structure compatible with the topology.

Theorem 2.7. Let $Fourier$ be a uniformity on a set X, then $(X, \tau_{Fourier})$ is a T_3 space.

In particular, a compact Hausdorff space is uniformizable. In fact, for a compact Hausdorff space X the set of all neighborhoods of the diagonal in $X \times X$ form the unique uniformity compatible with the topology.

A Hausdorff uniform space is metrizable if its uniformity can be defined by a countable family of pseudometrics. Indeed, as discussed above, such a uniformity can be defined by a single pseudometric, which is necessarily a metric if the space is Hausdorff. In particular, if the topology of a vector space is Hausdorff and definable by a countable family of semi-norms, it is metrizable.

Similar to continuous functions between topological spaces, which preserve topological properties, are the uniform continuous functions between uniform spaces, which preserve uniform properties. Uniform spaces with uniform maps form a category. An isomorphism between uniform spaces is called a uniform isomorphism.

Definition 2.8. Consider a uniform space (X, ϕ) and a pseudometric ρ on the set X; we say that the pseudometric ρ is uniform with respect to ϕ if for every $\epsilon > 0$, there exist a $V \in \phi$ such that $\rho(x, y) < \epsilon$ whenever$|x - y| < V$.

Theorem 2.9. If a pseudometric ρ on a set X is uniform with respect to a uniformity $Fourier$ on X, then $\rho : X \times X \to \mathbb{R}$ is continuous.

One of the important results of the theory of uniform spaces is the following theorem.

Theorem 2.10. Let $Fourier$ be a uniformity on a set X. For every sequence $V_0, V_1, \ldots$ of members of ϕ where $V_0 = X \times X$ and $3V_{i+1} \subseteq V_i$ for $i = 1, 2, \ldots,$ there exist a pseudometric ρ on a set X such that for every $i \geq 1$,

$$\left\{(x, y) : \rho(x, y) < \frac{1}{2^i}\right\} \subseteq V_i \subseteq \left\{(x, y) : \rho(x, y) \leq \frac{1}{2^i}\right\}.$$

Corollary 2.11. For every uniformity $Fourier$ on a set X and any $V \in Fourier$, there exist a pseudometric ρ on the set X which is uniform with respect to $Fourier$ and satisfies the condition $\{(x, y) : \rho(x, y) < 1\} \subseteq V$.

Corollary 2.12. For every uniformity $Fourier$ on a set X, the family of all members of $Fourier$ which are open with respect to the topology induced by $Fourier$ on $X \times X$ and the family of all members of $Fourier$ which are closed with respect to the topology induced by $Fourier$ on $X \times X$ are bases for $Fourier$.

Corollary 2.13. For every uniformity $Fourier$ on a set X, the set X with the topology induced by $Fourier$ is Tychonoff space.

3. Semi-linear uniform spaces

In 2003 Tallafha, A. and Khalil, R. [8] defined a new type of uniform spaces namely, semi-linear uniform spaces, by omitting the strong condition "If $V \in Fourier$ and $V \subseteq W \in D_X$, then $W \in Fourier$" and replaced it by the chain condition. They defined also a set valued map $\rho : X \times X \to X \times X$, which played an important rule in the theory of semi-linear uniform spaces. For more properties of semi-linear uniform spaces one may refer to [1] , [8] , [9] , [10] , [11] , and [12] .

In this section we shall give the required definitions and notations of semi-linear uniform spaces.

Definition 3.1. [8] Let Γ be a sub collection of D_X, the pair (X, Γ) is called a **semi-linear uniform space** if,

(i) Γ is a chain.

(ii) For every $V \in \Gamma$, there exists $U \in \Gamma$ such that $U \circ U \subseteq V$.

(iii) $\bigcap\limits_{V \in \Gamma} V = \Delta$.

(iv) $\bigcup\limits_{V \in \Gamma} V = X \times X$.

Definition 3.2. Let (X, Γ) be a semi-linear uniform space, a collection $\beta \subseteq \Gamma$ is called a base for the uniformity $Fourier$ if, for every $V \in \Gamma$ there exist a $U \in \beta$ such that $U \subseteq V$.

The smallest cardinal number of β is called the weight of the semi-linear uniform space and denoted by $\omega(\beta)$.

Definition 3.3. [8] Let (X , Γ) be a semi-linear uniform space, for $(x, y) \in X \times X$, let $\Gamma_{(x,y)} = \{V \in \Gamma : (x, y) \in V\}$. Then, the set valued map ρ on $X \times X$ is defined by $\rho(x, y) = \bigcap \left\{V : V \in \Gamma_{(x,y)}\right\}$.

Clearly from the above definition for all $(x, y) \in X \times X$, we have $\rho(x, y) = \rho(y, x)$ and $\Delta \subseteq \rho(x, y)$.

In [10] Tallafha A. define anew set valued map δ by. Let $\Gamma \backslash \Gamma_{(x,y)} = \left(\Gamma_{(x,y)}\right)^c = \{V \in \Gamma : (x, y) \notin V \}$, from now on, we shall denote $\Gamma \backslash \Gamma_{(x,y)}$ by $\Gamma^c_{(x,y)}$.

Definition 3.4. [10] Let (X, Γ) be a semi-linear uniform space. Then, the set valued map δ on $X \times X$ is defined by,

$$\delta(x, y) = \left\{ \begin{array}{ll} \bigcup\{V : V \in \Gamma^{c}_{(x,y)}\} & if\ x \neq y \\ \phi & if\ x = y \end{array} \right\},$$

where $\Gamma^{c}_{(x,y)} = \Gamma \backslash \Gamma_{(x,y)} = \{V \in \Gamma : (x, y) \notin V\}$.

By Proposition 2.3 uniform spaces is stronger than topological spaces. Now we shall show that semi-linear uniform spaces is also stronger than topological spaces.

Definition 3.5. Let $(X\ , \Gamma)$ be a semi-linear uniform space, for $x \in X$ and $V \in \Gamma$ the open ball of center x and radius V is defined by $B(x, V) = \{y : (x, y) \in V\}$.

The following Proposition is an immediate consequence of the above definition and the properties of Γ.

Proposition 3.6. Let $(X\ , \Gamma)$ be a semi-linear uniform space, then

1. $\bigcap\limits_{V\in\Gamma} B(x, V) = \{x\}$, for all $x \in X$.
2. For all $V_1, V_2 \in \Gamma$, there exist a $V_3 \in \Gamma$ such that $B(x, V_3) \subseteq B(x, V_1) \cap B(x, V_2)$.

4. Topological properties of semi-linear uniform spaces

In this section we shall state and prove the most important topological properties of semi-linear uniform spaces.

The topology indused by a semi-linear uniform space is a T_1 topology, the prove is obviuos by Proposition 3.6.

Proposition 4.1. Let $(X\ , \Gamma)$ be a semi-linear uniform space, for $x \in X$, $B = \{B(x, V) : x \in X, V \in \Gamma\}$ a base for some T_1 topology on X.

Now the proof of the following theorem is completed.

Theorem 4.2. Every semi-linear uniform space $(X\ , \Gamma)$ induce a T_1 topological space (X, τ_Γ) ,whose base is $B = \{B(x, V) : x \in X, V \in \Gamma\}$. More precisely, the topology $\tau_\Gamma = \{G \subseteq X :$ for every $x \in G$ there is a $V \in \Gamma$ such that $B\ (x, V) \subseteq G\}$.

Remark 4.3.

1. We are interested in semi-linear uniform spaces which is not metrizable. In Theorem 4.2 it is shown that the topology on the set X is a T_1, so if X is finite then we have the discreet topology which is metrizable, therefore the interesting examples are when X is infinite. Also, if X is infinite then Γ should be infinite, other wise $\Delta \in \Gamma$ which implies that, the topology is the discrete one (τ_{dis}) .

2. It is also known that, every topological space (X, τ) , whose topology induced by a metric or a norm on X, can be generated by a uniform space [12].

If (X , Γ) is a semi-linear uniform space, then Theorem 4.2 show that (X , Γ) induce a T_1 topological space (X, τ_Γ) . In the next Theorem we shall show that there are more than one Γ which induced the same topology on X.

Theorem 4.4. Let (X , Γ) be a semi-linear uniform space, and $\beta \subseteq \Gamma$ a base for Γ. Then (X, β) is a semi linear uniform space and $\tau_\Gamma = \tau_\beta$.

Proof. Is an immediate consequences of the properties of β, Γ and definitions of τ_Γ, and τ_β. ■

Now we shall give an example of a semi-linear uniform space which is not metrizable.

Example 4.5. Let $(X, \|.\|)$ be any norm space. Define u_ϵ by,

$$u_\epsilon = \{(x, y) \in X \times X : \|x.\| + \|y\| < \epsilon\} \cup \Delta.$$

Let $\Gamma = \{u_\epsilon : \epsilon > 0\}$, then (X, Γ) is a semi-linear uniform space which is not metrizable.

Alhihi, S. in [1] gave more properties of semi-linear uniform spaces using the set valued maps ρ and δ..Also Tallafha, A. in [10], gave some important properties of semi-linear uniform spaces, using the set valued maps ρ and δ, he showed that if (X, Γ) is a semi-linear uniform space, then $(X, \boldsymbol{\delta})$, $(X, \boldsymbol{\rho})$ and $(X, \Gamma \cup \boldsymbol{\rho} \cup \delta)$ are semi- linear uniform spaces, where

$$\boldsymbol{\delta} = \{\boldsymbol{\delta}(x, y) : (x, y) \in X \times X\},$$

$$\boldsymbol{\rho} = \{\boldsymbol{\rho}(x, y) : (x, y) \in X \times X\}.$$

So (X , Γ), $(X, \boldsymbol{\delta})$, $(X, \boldsymbol{\rho})$ and $(X, \Gamma \cup \boldsymbol{\rho} \cup \delta)$ induced the topology τ_Γ, τ_δ, τ_ρ and $\tau_{\Gamma \cup \boldsymbol{\rho} \cup \delta}$ respectively on the set X. Now we want to show that $\tau_\delta \subseteq \tau_\Gamma \subseteq \tau_\rho \subseteq \tau_{\Gamma \cup \boldsymbol{\rho} \cup \delta}$.

Theorem 4.6. Let (X , Γ) be a semi-linear uniform space, then $\tau_\delta \subseteq \tau_\Gamma \subseteq \tau_\rho \subseteq \tau_{\Gamma \cup \boldsymbol{\rho} \cup \delta}$.

Proof. Let (X , Γ) be a semi-linear uniform space. Clearly $\tau_\rho \subseteq \tau_{\Gamma \cup \boldsymbol{\rho} \cup \delta}$. To complete the prove, we shall show that $\tau_\delta \subseteq \tau_\Gamma$ and $\tau_\Gamma \subseteq \tau_\rho$.

1. Let $x_0 \in G \in \tau_\delta$, then there exist $(x, y) \in X \times X$ such that $B(x_0, \boldsymbol{\delta}(x, y)) \subseteq G$, for $V \in \Gamma^c_{(x,y)}$ we have $B(x_0, V) \subseteq G$, so $G \in \tau_\Gamma$, hence $\tau_\delta \subseteq \tau_\Gamma$.

2. Let $x_0 \in G \in \tau_\Gamma$, then there exist $V \in \Gamma$ such that $B(x_0, V) \subseteq G$, since $\Delta \notin \Gamma$ we can find a point $(x, y) \in V, x \neq y$, then $B(x_0, \boldsymbol{\rho}(x, y)) \subseteq B(x_0, V) \subseteq G$, so $G \in \tau_\rho$, hence $\tau_\Gamma \subseteq \tau_\rho$. ■

Proposition 4.7. Let $(X\ ,\Gamma)$ be a semi-linear uniform space, the interior of a set $A \subseteq X$ with respect to τ_Γ is coincides with the set

$$A^\circ = \{x \in X : \text{ there exists a } V \in \Gamma \text{ such that } B(x, V) \subseteq A\}.$$

Proof. Every open set $G \subseteq A$ is contained in A°, it is suffices to show that A° is open set. For any $x \in A^\circ$ there exists a $V \in \Gamma$ such that $B(x, V) \subseteq A$; take $W \in \Gamma$ satisfying $2W \subseteq V$. It follows that for every $y \in B(x, W)$ we have $B(y, W) \subseteq B(x, V) \subseteq A$; hence $y \in A^\circ$, which implies $B(y, W) \subseteq A^\circ$. Therefore A° is open. ■

Corollary 4.8. If a topology of a space X is induced by a semi-linear uniform space $(X\ ,\Gamma)$ then,

1. For every $x \in X$ and any $V \in \Gamma$ the set $int\,(B(x, V))$ is a neighborhood of x.
2. For every $x \in X$ and any $A \subseteq X$ we have $x \in \overline{A}$ (the topological closer of A) if and only if $A \cap B(x, V)$ non empty for all $V \in \Gamma$.
3. For every $A \subseteq X$ and any $V \in \Gamma$ we have $d\left(\overline{A}\right) < 3V$ whenever $d(A) < V$.

Proof.

1. As $B(x, V) \subseteq B(x, V)$, the point x belongs to the interior of $B(x, V)$.
2. Clear from the definition of $\overline{A}$.
3. Assume $d(A) < V$, for any $x, y \in \overline{A}$, there exist $s, t \in A$ such that $s \in A \cap B(x, V)$ and $t \in A \cap B(y, V)$. Hence $(x, y) \in V \circ V \circ V$, so $|x - y| < 3V$. ■

Theorem 4.9. Let $(X\ ,\Gamma)$ be a semi-linear uniform space, then (X, τ_Γ) is T_3 space.

Proof. By Theorem 4.2 (X, τ_Γ) is T_1.To show regularity, let $F \subseteq X$ be any closed set and $x \notin F$, then there exist $V \in \Gamma$such that $B(x, V) \cap F = \phi$. Let $W \in \Gamma$ be such that $2W \subseteq V$, then $\overline{B(x, W)} \subseteq B(x, 2W) \subseteq B(x, V)$, therefor $\overline{int\,(B(x, W))} \cap F = \phi$. ■

As in uniform spaces, consider a semi-linear uniform space $(X\ ,\Gamma)$ and a pseudometric p on X; we say that the pseudometric p is uniform with respect to Γ if for every $\epsilon > 0$ there exists a $V \in \Gamma$ such that $p(x, y) < \epsilon$ whenever $|x - y| < V$. Clearly if p is uniform with respect to Γ, then and multiple of p is uniform with respect to Γ

From now on, if the topology on the set of real numbers $\mathbb{R}$ not mentioned then its the usual topology, also the topology on $X \times X$ is the Tychonoff topology induced by Γ on $X \times X$.

Proposition 4.10. Let (X, Γ) be a semi-linear uniform space and a pseudometric p which is uniform with respect to Γ, then $\rho : X \times X \to \mathbb{R}$ is a continuous function.

Proof. Let (x_0, y_0) be any point of $X \times X$; and $\epsilon > 0$. Since p is uniform with respect to Γ, then there exists a $V \in \Gamma$ such that $p(x, y) < \frac{\epsilon}{2}$ whenever $|x - y| < V$. Let $(x, y) \in B(x_0, V) \times B(y_0, V)$ then by triangle inequality $|p(x_0, y_0) - p(x, y)| \leq p(x_0, x) + p(y_0, y) < \epsilon$. Since $int(B(x_0, V)) \times int(B(y_0, V))$ is a neighborhood of the point (x_0, y_0) the result follows. ■

The following theorem is proved in [5] with the assumption $3V_{i+1} \subseteq V_i$ for $i = 1, 2, \ldots,$ see Theorem 2.10. Now we shall give a modified result which is the most important result in this paper.

Theorem 4.11. Let (X, Γ) be a semi-linear uniform space. For every sequence $V_0, V_1, V_2, \ldots$ of members of Γ satisfies $V_0 = X \times X$ and $2V_{i+1} \subseteq V_i$ for $i = 1, 2, \ldots,$ there exist a pseudometric p on the set X which is uniform with respect to Γ such that for every $i \geq 1$, $\left\{(x, y) : \rho(x, y) < \frac{1}{2^i}\right\} \subseteq V_i \subseteq \left\{(x, y) : \rho(x, y) \leq \frac{1}{2^i}\right\}$.

Proof. For any pair of points x, y of X, there exist a sequence $x_0, x_1, \ldots, x_k$ of points such that $x_0 = x$, $x_k = y$ and $\left(x_{j-1}, x_j\right) \in V_{i_j}$, $j = 1, 2, \ldots, k$. The sequence which satisfies the above conditions not unique, let $A_{x,y}$ be the set of all such sequences. For every sequence in $A_{x,y}$ assign the number $r_{x,y} = \frac{1}{2^{i_1}} + \frac{1}{2^{i_2}} + \cdots + \frac{1}{2^{i_k}}$. Define $\rho(x, y)$ by taking the infimum of all $r_{x,y}$, over the set of all sequences in $A_{x,y}$, more precisely $\rho(x, y) = \inf_{x_0, x_1, \ldots, x_k \in A_{x,y}} r_{x,y}$. Clearly $\rho(x, y) \geq 0$ for all $x, y \in X$, $\rho(x, x) = 0$ and $\rho(x, y) = \rho(y, x)$. Now let x, y, z be any three points in X, since $A_{x,y} \cup A_{y,z} \subseteq A_{x,z}$, the triangle inequality follows.

From the definition of $\rho(x, y)$ it follows that $V_i \subseteq \left\{(x, y) : \rho(x, y) \leq \frac{1}{2^i}\right\}$. It remains to show that if $\rho(x, y) < \frac{1}{2^i}$ then $(x, y) \in V_i$. If $\rho(x, y) < \frac{1}{2^i}$ then there exist a sequence $x_0, x_1, \ldots, x_k$ of points such that $x_0 = x$, $x_k = y$, $\left(x_{j-1}, x_j\right) \in V_{i_j}$, $j = 1, 2, \ldots, k$. and $r_{x,y} < \frac{1}{2^i}$, so $\frac{1}{2^{i_1}} + \frac{1}{2^{i_2}} + \cdots + \frac{1}{2^{i_k}} < \frac{1}{2^i}$. To complete the prove we shall show that

$$\text{if } \frac{1}{2^{i_1}} + \frac{1}{2^{i_2}} + \cdots + \frac{1}{2^{i_k}} < \frac{1}{2^i},$$

then $(x_0, x_k) \in V_i$. $(*)$

To Prove $(*)$ we use induction over k.

1. If $k = 1$, then $\frac{1}{2^{i_1}} < \frac{1}{2^i}$, i.e., $i_1 > i$ hence $V_{i_1} \subseteq V_i$ and $(x, y) = (x_0, x_1) \in V_{i_1} \subseteq V_i$.
2. If $k = 2$, then $\frac{1}{2^{i_1}} + \frac{1}{2^{i_2}} < \frac{1}{2^i}$, i.e., $s = \min\{i_1, i_2\} > i$ hence $(x, y) = (x_0, x_2) \in V_{i_1} \circ V_{i_2} \subseteq 2V_s \subseteq V_i$.

3. If $k = 3$, then $\frac{1}{2^{i_1}} + \frac{1}{2^{i_2}} + \frac{1}{2^{i_3}} < \frac{1}{2^i}$, and $s = \min\{i_1, i_2\} > i, i_3 > i$. Hence booth s, and i_3 are greater than or equal $i+1$, therefore $(x_0, x_2) \in V_{i_1} \circ V_{i_2} \subseteq 2V_s \subseteq V_{i+1}$ and $(x_2, x_3) \in V_{i_3} \subseteq V_{i+1}$, so $(x, y) = (x_0, x_3) \in 2V_{i+1} \subseteq V_i$.

4. Assume $m \geq 4$ and $(*)$ holds for all $k < m$. Suppose

$$\frac{1}{2^{i_1}} + \frac{1}{2^{i_2}} + \cdots + \frac{1}{2^{i_m}} < \frac{1}{2^i}$$

then

$$\frac{1}{2^{i_1}} < \frac{1}{2^{i+2}}$$

or

$$\frac{1}{2^{i_2}} < \frac{1}{2^{i+2}}$$

or $\frac{1}{2^{i_{m-1}}} < \frac{1}{2^{i+2}}$ or $\frac{1}{2^{i_m}} < \frac{1}{2^{i+2}}$ by symmetry of the assumptions we may assume

$$\frac{1}{2^{i_1}} < \frac{1}{2^{i+2}}$$

or

$$\frac{1}{2^{i_2}} < \frac{1}{2^{i+2}}.$$

now we have two cases,

(a) If $\frac{1}{2^{i_1}} < \frac{1}{2^{i+2}}$, let n be the largest integer $\leq m-1$ such that

$$\frac{1}{2^{i_1}} + \frac{1}{2^{i_2}} + \cdots + \frac{1}{2^{i_n}} < \frac{1}{2^{i+2}}.$$

If $n < m-1$, then

$$\frac{1}{2^{i_1}} + \frac{1}{2^{i_2}} + \cdots + \frac{1}{2^{i_n+1}} \geq \frac{1}{2^{i+2}},$$

hence

$$\frac{1}{2^{i_n+2}} + \frac{1}{2^{ii_n+3}} + \cdots + \frac{1}{2^{im}} < \frac{1}{2^{i+2}}.$$

By the inductive assumption $(x_0, x_n) \in V_{i+2}$, $(x_{n+1}, x_m) \in V_{i+2}$ and $(x_n, x_{n+1}) \in V_{i_n+1} \subseteq V_{i+2}$, so $(x_0, x_m) \in 3V_{i+2} \subseteq 4V_{i+2} \subseteq V_i$. If $n = m-1$, by the inductive assumption we have $(x_0, x_{m-1}) \in V_{i+1}$ and $\left(x_{m-1}, x_m\right) \in V_{i_m} \subseteq V_{i+1}$, so $(x_0, x_m) \in 2V_{i+1} \subseteq V_i$.

(b) If $\frac{1}{2^{i_2}} < \frac{1}{2^{i+2}}$, and $\frac{1}{2^{i_1}} \geq \frac{1}{2^{i+2}}$, then $i_1 = i + 1$ or $i_1 = i + 2$. If $i_1 = i + 1$, then $(*)$ implies $\frac{1}{2^{i_2}} + \cdots + \frac{1}{2^{i_m}} < \frac{1}{2^{i+1}}$, so by inductive assumption $(x_1, x_m) \in V_{i+1}$ and $(x_0, x_1) \in V_{i+1}$, so $(x_0, x_m) \in 2V_{i+1} \subseteq V_i$. If $i_1 = i+2$, then $(x_0, x_1) \in V_{i+2}$. Let n be the largest integer $\leq m - 1$ such that

$$\frac{1}{2^{i_2}} + \cdots + \frac{1}{2^{i_n}} < \frac{1}{2^{i+2}}.$$

If $n < m - 1$, then

$$\frac{1}{2^{i_2}} + \cdots + \frac{1}{2^{i_n+1}} \geq \frac{1}{2^{i+2}},$$

hence

$$\frac{1}{2^{i_n+2}} + \frac{1}{2^{i i_n+3}} + \cdots + \frac{1}{2^{im}} < \frac{1}{2^{i+2}}.$$

By the inductive assumption

$$(x_1, x_n) \in V_{i+2}, (x_{n+1}, x_m) \in V_{i+2}$$

and

$$(x_n, x_{n+1}) \in V_{i_n+1} \subseteq V_{i+2}.$$

So $(x_0, x_n) \in 4V_{i+2} \subseteq V_i$.

To complete the proof, we want to show that p is uniform with respect to Γ. Let $\epsilon > 0$, and j_0 be such that $\frac{1}{2^{j_0}} < \epsilon$, then

$$V_{i_0} \subseteq \left\{(x, y) : \rho(x, y) \leq \frac{1}{2^{j_0}}\right\} \subseteq \{(x, y) : \rho(x, y) < \epsilon\}.$$

■

In uniform spaces $Fourier$ we know that, the family of all members of $Fourier$ which are open with respect to the topology induced by $Fourier$ on $X \times X$ and the family of all members of $Fourier$which are closed with respect to the topology induced by $Fourier$ on $X \times X$ are bases for $Fourier$. The proof depend on the property (iv) in Definition 2.1 which is not true in semi-linear uniform spaces. So we have the following question.

Question Let (X, Γ) be a semi-linear uniform space. For $x, y \in X$. Is $\delta(x, y)$ is open with respect to the topology induced by Γ on $X \times X$?.

In the following corollary we shall show that for every elements $V \in \Gamma$ there exist a pseudometric p on the set X which is uniform with respect to $\Gamma$and satisfies the condition $\{(x, y) : \rho(x, y) < 1\} \subseteq V$. So the interior of V with respect to the topology induced by Γ on $X \times X$ is non empty for all $V \in \Gamma$ since $\{(x, y) : \rho(x, y) < 1\}$ is open by Proposition 4.10.

Corollary 4.12. Let (X, Γ) be a semi-linear uniform space. Then for every $V \in \Gamma$ there exist a pseudometric p on the set X which is uniform with respect to $\Gamma$and satisfies the condition $\{(x, y) : \rho(x, y) < 1\} \subseteq V$.

Proof. For $V \in \Gamma$, choose the sequence $V_0 = X \times X$, $V_1 = V$ and $2V_{i+1} \subseteq V_i$ for $i = 1, 2, \ldots$ By Theorem 4.11 there exist a pseudometric p^* on the set X which is uniform with respect to Γ, such that for every $\left\{(x, y) : \rho^*(x, y) < \frac{1}{2}\right\} \subseteq V$, therefore $2p^*$is the required pseudometric. ■

Corollary 4.13. Let (X, Γ) be a semi-linear uniform space, then (X, τ_Γ) is a Tychonof space.

Proof. For every $x \in X$ and every closed subset F of X not contained x, x belong to the complement of F which is open, therefor there exist a $V \in \Gamma$ such that $B(x, V) \cap F = \phi$. Now as in Corollary 4.12 choose the sequence $V_0 = X \times X$, $V_1 = V$ and $3V_{i+1} \subseteq V_i$ for $i = 1, 2, \ldots$ By Theorem 4.11 again there exist a pseudometric p on the set X which is uniform with respect to Γ, and $\{(x, y) : \rho(x, y) < 1\} \subseteq V$, Let $f : X \to [0, 1]$ be defined by $f(t) = \min\{1, \rho(x, t)\}$. Clearly f is continuos and $f(x) = 0$ and $f(t) = 1$ for every $t \in F$. ■

Conversely, each Tychonof space (completely regular space) is uniformizable. A uniformity compatible with the topology of a completely regular space X can be defined as the coarsest uniformity that makes all continuous real-valued functions on X uniformly continuous. A fundamental system of entourages for this uniformity is provided by all finite intersections of sets $(f \times f)^{-1}(V)$, where f is a continuous real-valued function on X and V is an entourage of the uniform space X. This uniformity defines a topology, which is clearly coarser than the original topology of X; that it is also finer than the original topology (hence coincides with it) is a simple consequence of complete regularity: for any $x \in X$ and a neighborhood V of x, there is a continuous real-valued function $f : X \to \mathbb{R}$ with $f(x) = 0$ and $f\left(v^c\right) = 1$, where v^c is the complement of v.

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