Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 6 (2016), pp. 5185–5204

http://www.ripublication.com/gjpam.htm

A Modified Reich-Sabach Iteration Scheme for Approximating a Common Element of the Solutions of Equilibrium Problems and Fixed Point Problems in Hilbert Spaces

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Abstract

A modified Reich and Sabach iteration scheme in [5 *Contemporary Mathematics*, **568**, (2012), 225–240] is introduced in a real Hilbert space H for the approximation of a common element of the fixed point sets of a finite family of multi-valued nonexpansive-type mappings and the set of solutions of a finite family of equilibrium problems. Furthermore, the numerical computations of the algorithm with (respectively without) the strict fixed point set conditions are presented. These computaions are positive steps towards the resolution of the controversy over the computability of algorithms for approximating the solutions of equilibrium problems for bifunctions involving the construction of the sequences $\{K_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$, from an arbitrary $x_0 \in H$, where $K_{n+1} = \{z \in K_n : \|z - u_n\|^2 \leq \|z - x_n\|^2\}$, $x_{n+1} = P_{K_{n+1}}x_0$, while P_{K_n} is the projection map and $\{u_n\}_{n=1}^{\infty}$ is the sequence of the resolvent of the bifunctions.

AMS subject classification: 47H10, 54H25.
Keywords: Hilbert spaces, nonexpansive-type mappings, strong convergence, strict fixed point sets, equilibrium problems, finite families.

1. Introduction

Let H be a real Hilbert space with an inner product $\langle ., . \rangle$ and a norm $\|.\|$, respectively and let K be a nonempty closed convex subset of H. Let $A : H \to H$ be an operator on H and $F : K \times K \to \mathbb{R}$ be a bifunction on K, where $\mathbb{R}$ is the set of real numbers. The variational inequality problem of A in K denoted by $VIP(A, K)$ is to find an $x^* \in K$ such that

$$\langle x - x^*, A(x^*) \rangle \geq 0, \quad \forall x \in K, \tag{1}$$

while the equilibrium problem for F is to find $x^* \in K$ such that

$$F(x^*, x) \geq 0, \quad \forall x \in K. \tag{2}$$

The set of solutions of (2) is denoted by $EP(F)$. Suppose $F(x, y) = \langle y - x, Ax \rangle$ for all $x, y \in K$, then $w \in EP(F)$ if and only if w is a solution of (1). Many problems in optimization, economics and physics reduce to finding a solution of (1), (see for examples, [4], [5], [7]) and the references therein. The following conditions are assumed for solving the equilibrium problems for a bifunction $F : K \times K \to \mathbb{R}$,

(A1) $F(x, x) = 0$ for all $x \in K$.

(A2) F is monotone, that is , $F(x, y) + F(y, x) \leq 0$, for all $x, y \in K$.

(A3) For each $x, y, z \in K$, $\lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$.

(A4) For each $x \in K$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Let X be a nonempty set and let $T : X \to X$ be a map. A point $x \in X$ is called a fixed point of T if $x = Tx$. If $T : X \to 2^X$ is a multi-valued map from X into the family of nonempty subsets of X, then x is a fixed point of T if $x \in Tx$. If $Tx = \{x\}$, x is called a strict fixed point of T. The set $F(T) = \{x \in D(T) : x \in Tx\}$(respectively $F(T) = \{x \in D(T) : x = Tx\}$) is called the fixed point set of multi-valued(respectively single-valued) map T while the set $F_s(T) = \{x \in D(T) : Tx = \{x\}\}$ is called the strict fixed point set of T.

Several algorithms have been introduced by authors for approximating solutions of equilibrium problems for a bifunction (or finite family of bifunctions) as well as a common element of the fixed point sets of finite family of multi-valued (or single-valued) mappings and the set of solutions of finite family of equilibrium problems (see for example [8], [9], [10], [11], [12] and references therein).

In [8], Reich and Sabach proposed three algorithms for solving (common) equilibrium problems for bifunction(s) g in general reflexive Banach spaces, using the well chosen convex function f, the Bregman distance and the projection associated with it. They proposed the algorithms as follows:

Let X be a reflexive Banach space, $\{K_i\}_{i=1}^N$ a finite family of nonempty, closed and convex subsets of X. Let $\{\lambda^i\}_{i=1}^N$ be a finite family of positive real numbers and $\{g_i\}_{i=1}^N$ a finite family of bifunctions with $g_i : K_i \times K_i \to \mathbb{R}$ for each $i = 1, 2, \ldots, N$. Suppose $f : X \to \mathbb{R}$ is a coercive Legendre function which is bounded, uniformly *Frèchet* differentiable and totally convex on bounded subsets of X, $Res^f_{\lambda^i g_i}$ is the resolvent of g_i with respect to λ^i and f for each $i = 1, 2, \ldots, N$ and D_f is the Bregman distance on X. If $E = \bigcap_{i=1}^{N} EP(g_i) \neq \emptyset$, then the sequences $\{x_n\}_{n=1}^{\infty}$ were generated from an arbitrary $x_0 \in X$ as follows.

Algorithm 1 (Algorithm I [8]).

$$\begin{cases} x_0 \in X, \\ y_n^i = Res^f_{\lambda_n^i g_i}(x_n + e_n^i), \\ K_n^i = \{z \in X : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\}, \\ K_n = \bigcap_{i=1}^{N} K_n^i, \\ Q_n = \{z \in X : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\}, \\ x_{n+1} = proj^f_{K_n \cap Q_n}(x_0), \quad n = 0, 1, 2, \ldots \end{cases} \tag{3}$$

Algorithm 2 (Algorithm II [8]).

$$\begin{cases} x_0 \in X, \\ K_0^i = X, \quad i = 1, 2, \ldots, N \\ y_n^i = Res^f_{\lambda_n^i g_i}(x_n + e_n^i), \\ K_{n+1}^i = \{z \in K_n^i : D_f(z, y_n^i) \le D_f(z, x_n + e_n^i)\}, \\ K_{n+1} = \bigcap_{i=1}^N K_{n+1}^i, \\ x_{n+1} = proj^f_{K_{n+1}}(x_0), \quad n = 0, 1, 2, \ldots \end{cases} \tag{4}$$

Algorithm 3 (Algorithm III [8]).

$$\begin{cases} x_0 \in X, \\ K_0^i = X, \quad i = 1, 2, \ldots, N \\ y_n^i = Res^f_{\lambda_n^i g_i}(x_n + e_n^i), \\ K_{n+1}^i = \{z \in K_n^i : \langle \nabla f(x_n + e_n^i) - \nabla f(y_n^i), z - y_n^i \rangle \le 0\}, \\ K_{n+1} = \bigcap_{i=1}^N K_{n+1}^i \\ x_{n+1} = proj^f_{K_{n+1}}(x_0), \quad n = 0, 1, 2, \ldots \end{cases} \tag{5}$$

Furthermore, they proved that if $\liminf_{n\to\infty} \lambda_n^i > 0$ and $\lim_{n\to\infty} e_n = 0$, the sequences converge strongly to $proj_E^f(x_0)$. We refer interested readers to Reich and Sabach [8] for definitions of important terms in the above algorithms.

It is interesting to note that in Algorithms 1, if $T : X \to X$ is the identity map on X and $\upsilon_n^i = \alpha_n x_n + (1 - \alpha_n)T x_n + e_n^i = x_n + e^i$, then we can rewrite Algorithm I in the following form:

Algorithm 4.

$$\begin{cases} x_0 \in X, \\ \upsilon_n^i = \alpha_n x_n + (1 - \alpha_n)T x_n + e_n^i, \\ y_n^i = Res^f_{\lambda_n^i g_i}(\upsilon_n^i), \\ K_n^i = \{z \in X : D_f(z, y_n^i) \le D_f(z, \upsilon_n^i)\}, \\ K_n = \bigcap_{i=1}^N K_n^i, \\ Q_n = \{z \in X : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \le 0\}, \\ x_{n+1} = proj^f_{K_n \cap Q_n}(x_0), \quad n = 0, 1, 2, \ldots \end{cases} \tag{6}$$

Also, in Algorithms 2 and 3, if $T^i : K^i \to K^i$ is the identity map on K^i, $i = 1, 2, \ldots, N$, respectively and set $\upsilon_n^i = \alpha_n x_n + (1 - \alpha_n)T^i x_n + e_n^i = x_n + e_n^i$, then we can rewrite Algorithm 2 and 3 in the following forms:

Algorithm 5.

$$\begin{cases} x_0 \in X, \\ K_0^i = X, \quad i = 1, 2, \ldots, N \\ \upsilon_n^i = \alpha_n x_n + (1-\alpha_n)T^i x_n + e_n^i, \\ y_n^i = Res^f_{\lambda_n^i g_i}(\upsilon_n^i), \\ K_{n+1}^i = \{z \in K_n^i : D_f(z, y_n^i) \le D_f(z, \upsilon_n^i)\}, \\ K_{n+1} = \bigcap_{i=1}^N K_{n+1}^i, \\ x_{n+1} = proj^f_{K_{n+1}}(x_0), \quad n = 0, 1, 2, \ldots \end{cases} \tag{7}$$

Algorithm 6.

$$\begin{cases} x_0 \in X, \\ K_0^i = X, \quad i = 1, 2, \ldots, N \\ \upsilon_n^i = \alpha_n x_n + (1-\alpha_n)T^i x_n + e_n^i, \\ y_n^i = Res^f_{\lambda_n^i g_i}(\upsilon_n^i), \\ K_{n+1}^i = \{z \in K_n^i : \langle \nabla f(\upsilon_n^i) - \nabla f(y_n^i), z - y_n^i \rangle \le 0\}, \\ K_{n+1} = \bigcap_{i=1}^N K_{n+1}^i \\ x_{n+1} = proj^f_{K_{n+1}}(x_0), \quad n = 0, 1, 2, \ldots \end{cases} \tag{8}$$

Recently, Isiogugu [9] obtained a strong convergence of a hybrid algorithm to a common element of the fixed point sets of two multi-valued strictly pseudocontractive-type mappings and the set of solutions of an equilibrium problem in Hilbert spaces, using a strict fixed point set condition. She proved the following theorem:

Theorem 1.1. ([9]) Let K be a nonempty closed convex subset of a real Hilbert space H, let $f : K \times K \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4) and let $S, T : K \to P(K)$ be two strictly pseudocontractive-type mappings with contractive coefficients λ_1 and λ_2 respectively such that $\mathbb{F} = F_s(T) \bigcap F_s(S) \bigcap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in K$ as follows:

$$\begin{cases} x_0 \in H, \\ K_1 = K, \\ x_1 = P_K x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)[\beta_n v_n + (1-\beta_n) z_n], \\ u_n \in K \text{ such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - y_n \rangle \ge 0, \quad \forall y \in K, \\ K_{n+1} = \{z \in K_n : \|z - u_n\|^2 \le \|z - x_n\|^2\}, \\ x_{n+1} = P_{K_{n+1}} x_0, \end{cases}$$

where $v_n \in T x_n$, $z_n \in S x_n$. $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ are sequences in [0,1] satisfying

(i) $\alpha_n \geq \max\{\lambda_1, \lambda_2\}$.

(ii) $\liminf_{n\to\infty}(1-\alpha_n)(1-\beta_n)(\alpha_n-\lambda_1) > 0, \liminf_{n\to\infty}(1-\alpha_n)(\alpha_n-\lambda_2)\beta_n > 0$.

(iii) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$ converges strongly to $p \in P_{\mathbb{F}}x_0$.

Motivated by the above observations in Algorithms I-III of Reich and Sabach [8] and the iteration scheme in Theorem 1.1 considered by Isiogugu in [9], a hybrid algorithm for approximating a common element of the fixed point sets of a finite family of multi-valued nonexpansive-type mappings and the set of solutions of a finite family of equilibrium problems is constructed in real Hilbert spaces, without error terms and with (respectively without) strict fixed point set conditions. Also, the numerical computations of the algorithms are presented.

The results of this research (that is, the numerical computations) are indeed positive steps towards the resolution of the controversy over the computability of algorithms for approximating the solutions of equilibrium problems for bifunctions involving the construction of the sequences $\{K_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$, from an arbitrary $x_0 \in H$, where $K_{n+1} = \{z \in K_n : \|z-u_n\|^2 \leq \|z-x_n\|^2\}$, $x_{n+1} = P_{K_{n+1}}x_0$, while P_{K_n} is the projection map and $\{u_n\}_{n=1}^{\infty}$ is the sequence of the resolvent of the bifunctions.

2. Preliminaries

In the sequel, we shall need the following definitions and lemmas.

Let X be a normed space. A subset K of X is called proximinal if for each $x \in X$ there exists $k \in K$ such that

$$||x-k|| = \inf\{||x-y|| : y \in K\} = d(x, K). \tag{9}$$

It is known that every closed convex subset of a uniformly convex Banach space is proximinal. We shall denote the family of all nonempty closed and bounded subsets of X by $CB(X)$, the family of all nonempty subsets of X by 2^X and the family of all proximinal subsets of X by $P(X)$, for a nonempty set X.

Let H denote the Hausdorff metric induced by the metric d on X, that is, for every $A, B \in CB(X)$,

$$H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}.$$

Let X be a normed space. Let $T : D(T) \subseteq X \to 2^X$ be a multi-valued mapping on X. A multi-valued mapping $T : D(T) \subseteq X \to 2^X$ is called $L - Lipschitzian$ if there exists $L \geq 0$ such that for all $x, y \in D(T)$

$$H(Tx, Ty) \leq L||x-y||. \tag{10}$$

In (10), if $L \in [0, 1)$ T is said to be a contraction while T is nonexpansive if $L = 1$. T is said to be nonexpansive-type of Isiogugu [1] if given any pair $x, y \in D(T)$ and $u \in Tx$, there exists $v \in Ty$ satisfying $||u - v|| \leq H(Tx, Ty)$ and

$$H(Tx, Ty) \leq ||x - y||.$$

Clearly, every multi-valued nonexpansive mapping $T : D(T) \subseteq X \to P(X)$ is nonexpansive-type.

Let X be a normed space and $T : D(T) \subseteq X \to 2^X$ be a multi-valued map. T is said to be of type-one (see for example [13], [14]) if given any pair $x, y \in D(T)$, then

$$||u - v|| \leq H(Tx, Ty), \; for \; all \; u \in P_T x, v \in P_T y,$$

where $P_T : D(T) \to P(X)$ is defined for each x by

$$P_T(x) = \{y \in Tx : d(x, Tx) = \|x - y\|\}.$$

Lemma 2.1. ([9]). Let K be a nonempty subset of a real Hilbert space H and let $T : K \to P(K)$ be a nonexpansive-type mapping such that $F_s(T)$ is nonempty. Then $F_s(T)$ is closed and convex.

Lemma 2.2. Let H be a real Hilbert space and let K be a nonempty closed convex subset of H and P_K the convex projection onto K, then, convex projection is characterized by the following relations;

(i) $x^* = P_K(x) \Leftrightarrow \langle x - x^*, y - x^* \rangle \leq 0$, for all $y \in K$.

(ii) $||x - P_K x||^2 \leq ||x - y||^2 - ||y - P_K x||^2$.

(iii) $||x - P_K y||^2 \leq ||x - y||^2 - ||P_K y - y||^2$.

Lemma 2.3. ([4]). Let K be a nonempty closed convex subset of a real Hilbert space H and $F : K \times K \to \mathbb{R}$ a bifunction satisfying (A1)–(A4). Let $r > 0$ and $x \in H$, then, there exists $z \in K$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in K.$$

Lemma 2.4. ([5]). Let K be a nonempty closed convex subset of a real Hilbert space H. Assume that $F : K \times K \to \mathbb{R}$ and satisfies (A1)–(A4). Let $r > 0$ and $x \in H$. Define $T_r : H \to 2^K$ by

$$T_r(x) = \{z \in K : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0\}, \quad \forall y \in K.$$

Then the following conditions hold:

(1) T_r is single valued.

(2) T_r is firmly nonexpansive, that is for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$.

(3) $F(T_r) = EP(F)$.

(4) $EP(F)$ is closed and convex.

Lemma 2.5. ([6]). Let K be a nonempty closed convex subset of a real Hilbert space H and $F : K \times K \to \mathbb{R}$ a bifunction satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then for all $x \in H$ and $p \in F(T_r)$

$$\|p - T_r x\|^2 + \|T_r x - x\|^2 \leq \|p - x\|^2.$$

3. Main Results

We now consider the following algorithm.

Let H be a real Hilbert space and $\{K^i\}_{i=1}^N$ a finite family of nonempty closed convex subsets of H. Let $\{f^i\}_{i=1}^N$ be a finite family of bifunctions and $\{T^i\}_{i=1}^N$ a finite family of nonexpansive-type mappings such that $f^i : K^i \times K^i \to \mathbb{R}$ and $T^i : K^i \to P(K^i)$ for all $i = 1, 2, \ldots, N$ respectively. Let $\{\alpha_n^i\}_{n=1}^\infty$ be sequences in [0,1] and $\{r_n^i\}_{n=1}^\infty \subset [a, \infty)$ for some $a > 0$ for all i=1, 2, …, N, then from an arbitrary $x_0 \in H$ we generate the sequence $\{x_n\}_{n=1}^\infty$ as follows.

Algorithm 7.

$$\begin{cases} x_0 \in H, \\ K_1^i = K^i, \quad \forall\, i = 1, 2, \ldots, N, \\ K_1 = \bigcap_{i=1}^N K_1^i, \\ x_1 = P_{K_1} x_0, \\ y_n^i = \alpha_n^i x_n + (1 - \alpha_n^i) v_n^i, \\ u_n^i \in K^i \text{ such that } f^i(u_n^i, y) + \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - y_n^i \rangle \geq 0, \quad \forall y \in K^i, \\ K_{n+1}^i = \{z \in K_n^i : \|z - u_n^i\|^2 \leq \|z - x_n\|^2\}, \\ K_{n+1} = \bigcap_{i=1}^N K_{n+1}^i, \\ x_{n+1} = P_{K_{n+1}} x_0, \end{cases}$$

where $v_n^i \in T^i x_n$.

Theorem 3.1. Let H, $\{K^i\}_{i=1}^N$, $\{T^i\}_{i=1}^N$, $\{f^i\}_{i=1}^N$, $\{\alpha_n^i\}_{n=1}^\infty$ and $\{r_n^i\}_{n=1}^\infty$ be as in algorithm 7. Suppose f^i satisfy (A1)-(A4) for all $i = 1, 2, \ldots, N$, $\mathbb{F} = \bigcap_{i=1}^N F_s T^i \bigcap \bigcap_{i=1}^N EP(f^i) \neq \emptyset$, then $\{x_n\}$ converges strongly to $p \in P_{\mathbb{F}} x_0$ if for each $i = 1, 2, \ldots, N$ and for all $n \geq 1$, $\liminf_{n\to\infty} \alpha_n^i(1 - \alpha_n^i) > 0$.

Proof. Since K_n^i is closed and convex for all $n \geq 1$ and for all $i = 1, 2, \ldots, N$, $K_n = \bigcap_{i=1}^{N} K_n^i$ is closed and convex and hence $P_{K_{n+1}}x_0$ is well defined, also, $u_n^i = T_{r_n^i} y_n^i$. Next, we show that $\mathbb{F} \subset K_n$, for all $n \geq 1$. $\mathbb{F} \subset K_1^i = K^i$ for all $i = 1, 2, \ldots, N$, therefore, $\mathbb{F} \subset \bigcap_{i=1}^{N} K_1^i = K_1$. Assume $\mathbb{F} \subset K_k = \bigcap_{i=1}^{N} K_k^i$. Using Lemma 4, for all $q \in \mathbb{F}$ we have

$$\begin{aligned}
\|q - u_k^i\|^2 &= \|q - T_{r_k^i} y_k^i\|^2 \\
&\leq \|q - y_k^i\|^2 \\
&= \alpha_k^i \|x_k - q\|^2 + (1 - \alpha_k^i)\|v_k^i - q\|^2 \\
&\quad -\alpha_k^i (1 - \alpha_k^i)\|x_k - v_k^i\|^2 \\
&\leq \alpha_k^i \|x_k - q\|^2 + (1 - \alpha_k^i) H^2(T^i x_k, T^i q) \\
&\quad -\alpha_k^i (1 - \alpha_k^i)\|x_k - v_k^i\|^2 \\
&\leq \alpha_k^i \|x_k - q\|^2 + (1 - \alpha_k^i)[\|x_k - q\|^2 + k^i \|x_k - v_k^i\|^2] \\
&\quad -\alpha_k^i (1 - \alpha_k^i)\|x_k - v_k^i\|^2 \\
&= \|x_k - q\|^2 - (1 - \alpha_k^i)\alpha_k^i \|x_k - v_k^i\|^2 \qquad (11) \\
&\leq \|x_k - q\|^2.
\end{aligned}$$

This shows that $q \in K_{k+1}^i$ for all $i = 1, 2, \ldots, N$, therefore, $q \in \bigcap_{i=1}^{N} K_{k+1}^i = K_{k+1}$ and hence, $\mathbb{F} \subseteq K_n$ for all $n \geq 1$. From $x_n = P_{K_n} x_0$ and Lemma 2(i), we obtain

$$\langle x_n - y, x_0 - x_n \rangle \geq 0, \quad \forall\, y \in K_n. \qquad (12)$$

and

$$\langle x_n - q, x_0 - x_n \rangle \geq 0, \quad \forall\, q \in \mathbb{F}. \qquad (13)$$

Using Lemma 2(ii), we have that

$$\begin{aligned}
\|x_n - x_0\|^2 &= \|P_{K_n} x_0 - x_0\|^2 \leq \|x_0 - q\|^2 - \|q - x_n\|^2 \\
&\leq \|x_0 - q\|^2,
\end{aligned}$$

for each $q \in \mathbb{F} \subset K_n$ and for all $n \geq 1$. Consequently, the sequences $\{x_n\}_{n=1}^{\infty}$, $\{v_n^i\}_{n=1}^{\infty}$, $i = 1, 2, \ldots, N$ are bounded. Furthermore, since $x_n = P_{K_n} x_0$, $x_{n+1} = P_{K_{n+1}} x_0 \in K_{n+1} \subset K_n$ then from definition of P_K we have $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$ for all $n \geq 1$. Therefore, the sequence $\{\|x_n - x_0\|\}_{n=1}^{\infty}$ is nondecreasing. Thus, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. From the construction of K_n, we have that $K_m \subset K_n$ and $x_m = P_{K_m} x_0 \in K_n$ for any integer $m \geq n$. Thus, from Lemma 2(iii)

$$\begin{aligned}
\|x_m - x_n\|^2 &= \|x_m - P_{K_n} x_0\|^2 \\
&\leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2. \qquad (14)
\end{aligned}$$

Letting $m, n \to \infty$ in (14), we have $\|x_m - x_n\| \to 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since H is Hilbert and K^i is closed and convex for all $i = 1, 2, \ldots, N$, we can assume that $x_n \to p \in K^i$, for all $i = 1, 2, \ldots, N$ as $n \to \infty$. We now show that $p \in F(T^i)$, for all $i = 1, 2, \ldots, N$. In particular, when $m = n + 1$ in (14), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{15}$$

Also, since $x_{n+1} \in K_{n+1}$, $\|x_{n+1} - u_n^i\| \le \|x_{n+1} - x_n\|$, using (15) gives

$$\lim_{n\to\infty} \|x_{n+1} - u_n^i\| = 0. \tag{16}$$

Combining (15) and (16) we have

$$\lim_{n\to\infty} \|x_n - u_n^i\| = 0. \tag{17}$$

It follows from $\lim_{n\to\infty} \|x_n - p\| = 0$ and (17) that

$$\lim_{n\to\infty} \|u_n^i - p\| = 0. \tag{18}$$

Setting $n = k$ in (11), we obtain

$$\|u_n^i - q\|^2 \le \qquad \|x_n - q\|^2 - (1 - \alpha_n^i)\alpha_n^i \|x_n - v_n^i\|^2 \tag{19}$$

Observe that

$$\begin{aligned} \|q - x_n\|^2 - \|q - u_n^i\|^2 &= \|x_n\|^2 - \|u_n^i\|^2 - 2\langle q, x_n - u_n^i\rangle \\ &\le \|x_n - u_n^i\|(\|x_n\| + \|u_n^i\|) + 2\|q\|\|x_n - u_n^i\|. \end{aligned}$$

It follows from (17) that

$$\lim_{n\to\infty} \|q - x_n\| - \|q - u_n^i\| = 0. \tag{20}$$

Using $\liminf_{n\to\infty} \alpha_n^i(1 - \alpha_n^i) > 0$, we deduce from (19) that $\lim_{n\to\infty} \|x_n - v_n^i\| = 0$. Hence, $p \in F(T^i)$ for each $i = 1, 2, \ldots, N$. It then follows that $p \in \bigcap_{i=1}^{N} F(T^i)$. It remains to show that p is in $EP(f^i)$ for all $i = 1, 2, \ldots, N$. Now from (19)

$$\|q - y_n^i\| \le \|q - x_n\|. \tag{21}$$

Also, using $u_n^i = T_{r_n^i} y_n^i$, Lemma 5 and (21) we have

$$\begin{aligned} \|u_n^i - y_n^i\|^2 &= \|T_{r_n^i} y_n^i - y_n^i\|^2 \\ &\le \|q - y_n^i\|^2 - \|q - T_{r_n^i} y_n^i\|^2 \\ &\le \|q - x_n\|^2 - \|q - T_{r_n^i} y_n^i\|^2 \\ &= \|q - x_n^i\|^2 - \|q - u_n^i\|^2. \end{aligned} \tag{22}$$

Therefore, from (20) and (22)

$$\lim_{n\to\infty} \|u_n^i - y_n^i\| = 0. \tag{23}$$

Consequently, from (18) and (23)

$$\lim_{n\to\infty} \|y_n^i - p\| = 0. \tag{24}$$

From the assumption that $r_n^i \geq a > 0$,

$$\lim_{n\to\infty} \frac{\|u_n^i - y_n^i\|}{r_n^i} = 0. \tag{25}$$

Since $u_n^i = T_{r_n^i} y_n^i$ implies

$$f(u_n^i, y) + \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - y_n^i\rangle \geq 0,$$

we deduce from (A2) that

$$\frac{\|u_n^i - y_n^i\|^2}{r_n^i} \geq \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - y_n^i\rangle \geq -f(u_n^i, y) \geq f(y, u_n^i).\ \forall y \in K^i$$

By taking limit as $n \to \infty$ of the above inequality and from (A4), (18) and (24), $f(y, p) \leq 0$, for all $y \in K^i$. Let $t \in (0, 1)$ and for all $y \in K^i$, since $p \in K^i$, $y_t = ty + (1-t)p \in K^i$. Hence, $f(y_t, p) \leq 0$. Therefore, from (A1),

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y),$$

that is, $f(y_t, y) \geq 0$. Letting $t \downarrow 0$, from (A3), we obtain $f(p, y) \geq 0$ for all $y \in K^i$ so that $p \in EP(f^i)$ for all $i = 1, 2, \ldots, N$. Hence $p \in \mathbb{F}$. Finally, we show that $p = P_{\mathbb{F}}x_0$. By taking the limits as $n \to \infty$ in (12) we have

$$\langle p - q, x_0 - p\rangle \geq 0, \quad \forall q \in \mathbb{F}.$$

Thus, from Lemma 2(i), $p = P_{\mathbb{F}}x_0$. This completes the proof. ■

Theorem 3.2. Let H be a real Hilbert space and $\{K^i\}_{i=1}^N$ a finite family of nonempty closed convex subsets of H. Let $\{f^i\}_{i=1}^N$ be a finite family of bifunctions and $\{T^i\}_{i=1}^N$ a finite family of type-one multi-valued nonexpansive-type mappings such that $f^i : K^i \times K^i \to \mathbb{R}$ and $T^i : K^i \to P(K^i)$ for all $i = 1, 2, \ldots, N$ respectively. Suppose f^i satisfy (A1)-(A4) for all $i = 1, 2, \ldots, N$, and $\mathbb{F} = \bigcap_{i=1}^N F(T^i) \bigcap \bigcap_{i=1}^N EP(f^i) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence in [0,1] and $\{r_n\}_{n=1}^\infty \subset [a, \infty)$ for some $a > 0$, then the sequence

$\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_0 \in H$ as follows,

$$\begin{cases} x_0 \in H, \\ K_1^i = K_i, \quad for\ all\ i = 1, 2, \ldots, N, \\ K_1 = \bigcap_{i=1}^{N} K_1^i, \\ x_1 = P_{K_1} x_0, \\ y_n^i = \alpha_n x_n + (1-\alpha_n) v_n^i, \\ u_n^i \in K^i \text{ such that } f^i(u_n^i, y) + \frac{1}{r_n}\langle y - u_n^i, u_n^i - y_n^i \rangle \geq 0, \quad \forall y \in K^i, \\ K_{n+1}^i = \{z \in K_n^i : \|z - u_n^i\|^2 \leq \|z - x_n\|^2\}, \\ K_{n+1} = \bigcap_{i=1}^{N} K_{n+1}^i, \\ x_{n+1} = P_{K_{n+1}} x_0, \end{cases}$$

where $v_n^i \in P_{T^i} x_n$, converges strongly to $p \in P_{\mathbb{F}} x_0$ if for each $i = 1, 2, \ldots, N$ and for all $n \geq 1$,

(i) $\liminf_{n\to\infty} \alpha_n^i (1 - \alpha_n^i) > 0$.

Proof. The proof is a similar procedure to the proof of Theorem 3.1, therefore, it is omitted. ■

Corollary 3.3. Let H be a real Hilbert space and $\{K^i\}_{i=1}^N$ a finite family of nonempty closed convex subsets of H. Let $\{f^i\}_{i=1}^N$ be a finite family of bifunctions and $\{e_n^i\}_{n=1}^{\infty}$ finite family of sequences of errors such that $f^i : K^i \times K^i \to \mathbb{R}$ for all $i = 1, 2, \ldots, N$ respectively. Suppose f^i satisfy (A1)-(A4) for all $i = 1, 2, \ldots, N$, and $\mathbb{F} = \bigcap_{i=1}^{N} EP(f^i) \neq \emptyset$. $\{r_n^i\}_{n=1}^{\infty} \subset [a, \infty)$ for some $a > 0$, then the sequence $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_0 \in H$ as follows,

$$\begin{cases} x_0 \in H, \\ K_1^i = K^i, \quad for\ all\ i = 1, 2, \ldots, N \\ K_1 = \bigcap_{i=1}^{N} K_1^i, \\ x_1 = P_{K_1} x_0, \\ y_n^i = x_n + e_i, \\ u_n^i \in K^i \text{ such that } f^i(u_n^i, y) + \frac{1}{r_n^i}\langle y - u_n^i, u_n^i - y_n^i \rangle \geq 0, \quad \forall y \in K^i, \\ K_{n+1}^i = \{z \in K_n^i : \|z - u_n^i\|^2 \leq \|z - x_n\|^2\}, \\ K_{n+1} = \bigcap_{i=1}^{N} K_{n+1}^i \\ x_{n+1} = P_{K_{n+1}} x_0, \end{cases}$$

converges strongly to $p \in P_{\mathbb{F}} x_0$ if $\lim_{n\to\infty} e_n^i = 0$.

Proof. The proof follows easily from Theorem 3.1 if T^i is the identity mapping on K^i, for each $i = 1, 2, \ldots, N$. ■

We now present the results of our numerical computations for two cases namely;

(i) the common element is a strict fixed point of each T^i.

(ii) the common element is not a strict fixed point for any T^i.

Case i.

Example 3.4. Let $H = \mathbb{R}$ (the reals with the usual metric and inner product), $i = 1, 2, \ldots, 8.$ and $K^i = \mathbb{R}$ for all i. Then for each i, we define:

(i) $T^i : \mathbb{R} \to P(\mathbb{R})$ by

$$T^i x = \begin{cases} [\frac{x}{i+1}, x], & x \in [0, \infty) \\ \{\frac{x}{i+1}\}, & x \in (-\infty, 0). \end{cases}$$

(ii) $f^i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f^i(x, y) = -ix^2 + iy^2.$$

(iii) $\{\alpha_n^i\}_{n=1}^{\infty} = \dfrac{n^2 + i^2 + 2(ni + 1)}{2(n^2 + i^2 + 2ni)}$.

(iv) $\{r_n^i\}_{n=1}^{\infty} = \dfrac{n + i}{ni}$.
We choose

(v) $v_n^i = \dfrac{x_n}{i + 1}$,
while our computation gives

(vi) $u_n^i = \dfrac{y_n^i}{2ir_n^i + 1}$.

It is easy to see that $F_s(T^i) = \{0\} \neq \emptyset$, $EPf^i = \{0\}$ for each i and

$$\mathbb{F} = \bigcap_{i=1}^{N} F_s(T^i) \bigcap \bigcap_{i=1}^{N} EP(f^i) = \{0\}.$$

The algorithm is computed with Microsoft word Excel 97-2003 Workbook. Table 1 below shows the sequences $\{x_n\}$ and $\{K_n\}$ generated from our computation using two different values of $x_0 = 10, -10$ while Table 2 shows different sequences generated

Table 1: The values of x_n and K_n for $x_0 = 10, -10$.

n	x_n	K_{n+1}	x_n	K_{n+1}
0	10	(-∞, ∞)	-10	(-∞, ∞)
1	10	(-∞, 5.149086708]	-10	[-5.149086708, ∞)
2	5.149086708	(-∞, 2.706651235]	-5.149086708	[-2.706651235, ∞)
3	2.706651235	(-∞, 1.444740339]	-2.706651235	[-1.444740339, ∞)
4	1.444740339	(-∞, 0.780338146]	-1.444740339	[-0.780338146, ∞)
5	0.780338146	(-∞, 0.425461454]	-0.780338146	[-0.425461454, ∞)
6	0.425461454	(-∞, 0.233756933]	-0.425461454	[-0.233756933, ∞)
7	0.233756933	(-∞, 0.129250346]	-0.233756933	[-0.129250346, ∞)
8	0.129250346	(-∞, 0.071850626]	-0.129250346	[-0.071850626, ∞)
9	0.071850626	(-∞, 0.040125802]	-0.071850626	[-0.040125802, ∞)
10	0.040125802	(-∞, 0.022497922]	-0.040125802	[-0.022497922, ∞)
11	0.022497922	(-∞, 0.01265811]	-0.022497922	[-0.01265811, ∞)
12	0.01265811	(-∞, 0.007143717]	-0.01265811	[-0.007143717, ∞)
13	0.007143717	(-∞, 0.004042592]	-0.007143717	[-0.004042592, ∞)
14	0.004042592	(-∞, 0.002293246]	-0.004042592	[-0.002293246, ∞)
15	0.002293246	(-∞, 0.001303739]	-0.002293246	[-0.001303739, ∞)
16	0.001303739	(-∞, 0.000742658]	-0.001303739	[-0.000742658, ∞)
17	0.000742658	(-∞ ,0.000423806]	-0.000742658	[-0.000423806, ∞)
18	0.000423806	(-∞, 0.000242246]	-0.000423806	[-0.000242246, ∞)
19	0.000242246	(-∞, 0.000138675]	-0.000242246	[-0.000138675, ∞)
20	0.000138675	(-∞, 0.000079495]	-0.000138675	[-0.000079495, ∞)
21	0.000079495	(-∞, 0.000045628]	-0.000079495	[-0.000045628, ∞)
22	0.000045628	(-∞, 0.00002622]	-0.000045628	[-0.00002622, ∞)
23	0.00002622	(-∞, 0.000015083]	-0.00002622	[-0.000015083, ∞)
24	0.000015083	(-∞, 0.000008685]	-0.000015083	[-0.000008685, ∞)
25	0.000008685	(-∞, 0.000005006]	-0.000008685	[-0.000005006, ∞)
26	0.000005006	(-∞, 0.000002888]	-0.000005006	[-0.000002888, ∞)
27	0.000002888	(-∞, 0.000001667]	-0.000002888	[-0.000001667, ∞)
28	0.000001667	(-∞, 0.000000963]	-0.000001667	[-0.000000963, ∞)
29	0.000000963	(-∞, 0.000000556]	-0.000000963	[-0.000000556, ∞)
30	0.000000556	(-∞, 0.000000321]	-0.000000556	[-0.000000321, ∞)
31	0.000000321	(-∞, 0.000000185]	-0.000000321	[-0.000000185, ∞)
32	0.000000185	(-∞, 0.000000107]	-0.000000185	[-0.000000107, ∞)
33	0.000000107	(-∞, 0.000000061]	-0.000000107	[-0.000000061, ∞)
34	0.000000061	(-∞, 0.000000035]	-0.000000061	[-0.000000035, ∞)
35	0.000000035	(-∞, 0.00000002]	-0.000000035	[-0.00000002, ∞)
36	0.00000002	(-∞, 0.000000011]	-0.00000002	[-0.000000011, ∞)
37	0.000000011	(-∞, 0.000000006]	-0.000000011	[-0.000000006, ∞)
38	0.000000006	(-∞, 0.000000003]	-0.000000006	[-0.000000003, ∞)
39	0.000000003	(-∞, 0.000000001]	-0.000000003	[-0.000000001, ∞)
40	0.000000001	(-∞, 0]	-0.000000001	[0, ∞)
41	0		0	0, ∞)

Table 2: The sequences generated for different values of x_0.

n	x_n	x_n	x_n	x_n	x_n	x_n
0	10	-10	50	-50	1000	-1000
1	10	-10	50	-50	1000	-1000
2	5.149086708	-5.149086708	25.74543354	-25.74543354	514.9086709	-514.9086709
3	2.706651235	-2.706651235	13.53325618	-13.53325618	270.6651235	-270.6651235
4	1.444740339	-1.444740339	7.223701698	-7.223701698	144.474034	-144.474034
5	0.780338146	-0.780338146	3.901690731	-3.901690731	78.03381465	-78.03381465
6	0.425461454	-0.425461454	2.127307274	-2.127307274	42.54614551	-42.54614551
7	0.233756933	-0.233756933	1.168784668	-1.168784668	23.37569339	-23.37569339
8	0.129250346	-0.129250346	0.646251734	-0.646251734	12.92503471	-12.92503471
9	0.071850626	-0.071850626	0.359253134	-0.359253134	7.185062699	-7.185062699
10	0.040125802	-0.040125802	0.200629013	-0.200629013	4.012580278	-4.012580278
11	0.022497922	-0.022497922	0.112489613	-0.112489613	2.249792281	-2.249792281
12	0.01265811	-0.01265811	0.063290554	-0.063290554	1.265811108	-1.265811108
13	0.007143717	-0.007143717	0.035718592	-0.035718592	0.714371858	-0.714371858
14	0.004042592	-0.004042592	0.020212964	-0.020212964	0.404259302	-0.404259302
15	0.002293246	-0.002293246	0.011466237	-0.011466237	0.229324764	-0.229324764
16	0.001303739	-0.001303739	0.0065187	-0.0065187	0.130374019	-0.130374019
17	0.000742658	-0.000742658	0.003713294	-0.003713294	0.07426591	-0.07426591
18	0.000423806	-0.000423806	0.002119033	-0.002119033	0.042380685	-0.042380685
19	0.000242246	-0.000242246	0.001211234	-0.001211234	0.024224699	-0.024224699
20	0.000138675	-0.000138675	0.000693379	-0.000693379	0.013867598	-0.013867598
21	0.000079495	-0.000079495	0.000397478	-0.000397478	0.00794958	-0.00794958
22	0.000045628	-0.000045628	0.000228144	-0.000228144	0.0045629	-0.0045629
23	0.00002622	-0.00002622	0.000131104	-0.000131104	0.002622107	-0.002622107
24	0.000015083	-0.000015083	0.000075422	-0.000075422	0.001508468	-0.001508468
25	0.000008685	-0.000008685	0.000043433	-0.000043433	0.00086869	-0.00086869
26	0.000005006	-0.000005006	0.000025035	-0.000025035	0.000500734	-0.000500734
27	0.000002888	-0.000002888	0.000014443	-0.000014443	0.000288892	-0.000288892
28	0.000001667	-0.000001667	0.000008339	-0.000008339	0.000166812	-0.000166812
29	0.000000963	-0.000000963	0.000004818	-0.000004818	0.000096396	-0.000096396
30	0.000000556	-0.000000556	0.000002786	-0.000002786	0.000055745	-0.000055745
31	0.000000321	-0.000000321	0.000001612	-0.000001612	0.000032259	-0.000032259
32	0.000000185	-0.000000185	0.000000933	-0.000000933	0.00001868	-0.00001868
33	0.000000107	-0.000000107	0.00000054	-0.00000054	0.000010823	-0.000010823
34	0.000000061	-0.000000061	0.000000313	-0.000000313	0.000006274	-0.000006274
35	0.000000035	-0.000000035	0.000000181	-0.000000181	0.000003639	-0.000003639
36	0.00000002	-0.00000002	0.000000104	-0.000000104	0.000002112	-0.000002112
37	0.000000011	-0.000000011	0.00000006	-0.00000006	0.000001226	-0.000001226
38	0.000000006	-0.000000006	0.000000034	-0.000000034	0.000000712	-0.000000712
39	0.000000003	-0.000000003	0.000000019	-0.000000019	0.000000413	-0.000000413
40	0.000000001	-0.000000001	0.00000001	-0.00000001	0.00000024	-0.00000024
41	0	0	0.000000005	-0.000000005	0.000000139	-0.000000139
42	0	0	0.000000002	-0.000000002	0.00000008	-0.00000008
43	0	0	0.000000001	-0.000000001	0.000000046	-0.000000046
44	0	0	0	0	0.000000026	-0.000000026
45	0	0	0	0	0.000000015	-0.000000015
46	0	0	0	0	0.000000008	-0.000000008
47	0	0	0	0	0.000000004	-0.000000004
48	0	0	0	0	0.000000002	-0.000000002
49	0	0	0	0	0.000000001	-0.000000001
50	0	0	0	0	0	0
51	0	0	0	0	0	0
52	0	0	0	0	0	0

Table 3: $v_n = x_n$ if $x_n \geq 0$, otherwise, $x_n/i + 1$.

n	x_n	K_{n+1}	x_n	K_{n+1}
0	1000	(-∞, ∞)	-1000	(-∞, ∞)
1	1000	(- ∞, 526.3157895]	-1000	[-514.9086709, ∞)
2	526.3157895	(- ∞, 287.0813397]	-514.9086709	[-270.6651235, ∞)
3	287.0813397	(- ∞, 160.7655502]	-270.6651235	[-144.474034, ∞)
4	160.7655502	(- ∞, 91.86602871]	-144.474034	[-78.03381465, ∞)
5	91.86602871	(- ∞, 53.34156506]	-78.03381465	[-42.54614551, ∞)
6	53.34156506	(- ∞, 31.37739122]	-42.54614551	[-23.37569339, ∞)
7	31.37739122	(- ∞, 18.65682722]	-23.37569339	[-12.92503471, ∞)
8	18.65682722	(- ∞, 11.19409633]	-12.92503471	[-7.185062699, ∞)
9	11.19409633	(- ∞, 6.768523363]	-7.185062699	[-4.012580278, ∞)
10	6.768523363	(- ∞, 4.119970742]	-4.012580278	[-2.249792281, ∞)
11	4.119970742	(- ∞, 2.522431068]	-2.249792281	[-1.265811108, ∞)
12	2.522431068	(- ∞, 1.552265272]	-1.265811108	[-0.714371858, ∞)
13	1.552265272	(- ∞, 0.959582168]	-0.714371858	[-0.404259302, ∞)
14	0.959582168	(- ∞, 0.595602725]	-0.404259302	[-0.229324764, ∞)
15	0.595602725	(- ∞, 0.371031205]	-0.229324764	[-0.130374019, ∞)
16	0.371031205	(- ∞, 0.231894503]	-0.130374019	[-0.07426591, ∞)
17	0.231894503	(- ∞, 0.145366703]	-0.07426591	[-0.042380685, ∞)
18	0.145366703	(- ∞, 0.091373356]	-0.042380685	[-0.024224699, ∞)
19	0.091373356	(- ∞, 0.057577731]	-0.024224699	[-0.013867598, ∞)
20	0.057577731	(- ∞, 0.036364882]	-0.013867598	[-0.00794958, ∞)
21	0.036364882	(- ∞, 0.023015748]	-0.00794958	[-0.0045629, ∞)
22	0.023015748	(- ∞, 0.014595352]	-0.0045629	[-0.002622107, ∞)
23	0.014595352	(- ∞, 0.009272341]	-0.002622107	[-0.001508468, ∞)
24	0.009272341	(- ∞, 0.00590058]	-0.001508468	[-0.00086869, ∞)
25	0.00590058	(- ∞, 0.003760809]	-0.00086869	[-0.000500734, ∞)
26	0.003760809	(- ∞, 0.002400516]	-0.000500734	[-0.000288892, ∞)
27	0.002400516	(- ∞, 0.00153435]	-0.000288892	[-0.000166812, ∞)
28	0.00153435	(- ∞, 0.000981984]	-0.000166812	[-0.000096396, ∞)
29	0.000981984	(- ∞, 0.000629232]	-0.000096396	[-0.000055745, ∞)
30	0.000629232	(- ∞, 0.000403658]	-0.000055745	[-0.000032259, ∞)
31	0.000403658	(- ∞, 0.000259229]	-0.000032259	[-0.00001868, ∞)
32	0.000259229	(- ∞, 0.000166647]	-0.00001868	[-0.000010823, ∞)
33	0.000166647	(- ∞, 0.000107233]	-0.000010823	[-0.000006274, ∞)
34	0.000107233	(- ∞, 0.000069065]	-0.000006274	[-0.000003639 , ∞)
35	0.000069065	(- ∞, 0.000044521]	-0.000003639	[-0.000002112, ∞)
36	0.000044521	(- ∞, 0.000028723]	-0.000002112	[-0.000001226, ∞)
37	0.000028723	(- ∞, 0.000018545]	-0.000001226	[-0.000000712, ∞)
38	0.000018545	(- ∞, 0.000011982]	-0.000000712	[-0.000000413, ∞)
39	0.000011982	(- ∞, 0.000007747]	-0.000000413	[-0.00000024, ∞)
40	0.000007747	(- ∞, 0.000005012]	-0.00000024	[-0.000000139, ∞)
41	0.000005012	(- ∞, 0.000003245]	-0.000000139	[-0.00000008, ∞)
42	0.000003245	(- ∞, 0.000002102]	-0.00000008	[-0.000000046, ∞)
43	0.000002102	(- ∞, 0.000001362]	-0.000000046	[-0.000000026, ∞)

Table 3: Continued.

n	x_n	K_{n+1}	x_n	K_{n+1}
44	0.000001362	(- ∞, 0.000000883]	-0.000000026	[-0.000000015, ∞)
45	0.000000883	(- ∞, 0.000000573]	-0.000000015	[-0.000000008, ∞)
46	0.000000573	(- ∞, 0.000000372]	-0.000000008	[-0.000000004, ∞)
47	0.000000372	(- ∞, 0.000000241]	-0.000000004	[-0.000000002, ∞)
48	0.000000241	(- ∞, 0.000000156]	-0.000000002	[-0.000000001, ∞)
49	0.000000156	(- ∞, 0.000000101]	-0.000000001	[0, ∞)
50	0.000000101	(- ∞, 0.000000065]	0	[0, ∞)
51	0.000000065	(- ∞, 0.000000042]		
52	0.000000042	(- ∞, 0.000000027]		
53	0.000000027	(- ∞, 0.000000017]		
54	0.000000017	(- ∞, 0.000000011]		
55	0.000000011	(- ∞, 0.000000007]		
56	0.000000007	(- ∞, 0.000000004]		
57	0.000000004	(- ∞, 0.000000002]		
58	0.000000002	(- ∞, 0.000000001]		
59	0.000000001	(- ∞, 0]		
60	0	(- ∞, 0]		

for different values of x_0. In particular, we considered without loss of generality $x_0 = 10, -10, 50, -50, 1000, 1000$.

In Table 3 below, we choose

$$v_n^i = \begin{cases} x_n, & x_n \in [0, \infty) \\ \left\{ \dfrac{x_n}{i+1} \right\}, & x_n \in (-\infty, 0). \end{cases}$$

to show that the convergence of the sequence is independent of the choice of $v_n^i \in T^i x_n$, if the common element is a strict fixed point for each T^i.

Case ii.

Example 3.5. Assume the sequences and bifunctions to be as in Example 1 and define $T^i : [0, \infty) \to P([0, \infty))$ by

$$T^i x = [0, x + i]$$

Clearly, $F_s(T^i) = \emptyset$ for each i but $\mathbb{F} = \bigcap_{i=1}^{N} F(T^i) \bigcap \bigcap_{i=1}^{N} EP(f^i) = \{0\} \neq \emptyset$. The numerical computation of the algorithm for the common element shows that the convergence of the sequence to a common element depends on the choice of $v_n^i \in T^i x_n$. Tables 4 and 5 are the results for $v_n^i \in P_{T^i} x_n$ and $v_n^i \notin P_{T^i} x_n$ respectively. We consider

(i) $v_n^i = x_n \in P_{T^i} x_n$

(ii) $v_n^i = x_n + i \notin P_{T^i} x_n$ for $x_n \in [0, \infty)$.

Table 4: $v_n^i = x_n \in P_{T^i}(x_n)$

n	x_n	K_{n+1}
0	10	$[0, \infty)$
1	10	[0,5.263157894]
2	5.263157894	[0,2.870813396]
3	2.870813396	[0, 1.607655501]
4	1.607655501	[0, 0.918660286]
5	0.918660286	[0, 0.533415649]
6	0.533415649	[0, 0.313773911]
7	0.313773911	[0, 0.186568271]
8	0.186568271	[0, 0.111940962]
9	0.111940962	[0, 0.067685232]
10	0.067685232	[0, 0.041199706]
11	0.041199706	[0, 0.025224309]
12	0.025224309	[0, 0.015522651]
13	0.015522651	[0, 0.00959582]
14	0.00959582	[0, 0.005956026]
15	0.005956026	[0, 0.003710311]
16	0.003710311	[0, 0.002318944]
17	0.002318944	[0, 0.001453666]
18	0.001453666	[0, 0.000913732]
19	0.000913732	[0, 0.000575776]
20	0.000575776	[0, 0.000363648]
21	0.000363648	[0, 0.000230156]
22	0.000230156	[0, 0.000145952]
23	0.000145952	[0, 0.000092722]
24	0.000092722	[0, 0.000059004]
25	0.000059004	[0, 0.000037606]
26	0.000037606	[0, 0.000024003]
27	0.000024003	[0, 0.000015342]
28	0.000015342	[0, 0.000009818]
29	0.000009818	[0, 0.000006291]
30	0.000006291	[0, 0.000004035]
31	0.000004035	[0, 0.000002591]
32	0.000002591	[0, 0.000001665]
33	0.000001665	[0, 0.000001071]
34	0.000001071	[0, 0.000000689]
35	0.000000689	[0, 0.000000444]
36	0.000000444	[0, 0.000000286]
37	0.000000286	[0, 0.000000184]
38	0.000000184	[0, 0.000000118]
39	0.000000118	[0, 0.000000076]
40	0.000000076	[0, 0.000000049]
41	0.000000049	[0, 0.000000031]
42	0.000000031	[0, 0.00000002]
43	0.00000002	[0, 0.000000012]

Table 4: $v_n^i = x_n \in P_{T^i}(x_n)$

n	x_n	K_{n+1}
44	0.000000012	[0, 0.000000007]
45	0.000000007	[0, 0.000000004]
46	0.000000004	[0, 0.000000002]
47	0.000000002	[0, 0.000000001]
48	0.000000001	{0}
49	0	

Table 5: v_n^i, *not in* P_{T^i}, the intersection of the $K_7^{i'}$s (that is, K_7 is an empty set). Therefore, $P_{K_7}(10)$ does not exist.

	$x_0 = 10$
i	K_7^i
1	(0, 0.516318211]
2	(0, 0.570890148]
3	(0, 0.617864464]
4	(0, 0.658631213]
5	[0.694295063, ∞)
6	[0.725729742, ∞)
7	[0.753628827, ∞)
8	[0.7785468, ∞)

Competing Interests: The Authors declare that there is no competing interest.

Acknowledgement

The third and fourth authors acknowledge with thanks the bursary and financial support from Department of Science and Technology and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DST-NRF COE-MaSS) Doctoral Bursary. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

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