On Some Properties of Tensor Product of Operators

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Abstract

In this article we introduced some properties of operators on Hilbert spaces, specially the tensor product of two operators on bounded Hilbert space. Moreover we studied both interior and exterior tensor product of two operators on the Hilbert $A$-module for given C*-algebra $A$.

1. INTRODUCTION

The approach in article is to study definition of the tensor product the Hilbert C*-module over the tensor products of C*-algebras. We have used these definitions to calculate the effects of linear numerical range in Hilbert C*-modules. We concluded the relationship between the tensor product in this and the classical tensor product in Hilbert spaces and we proved that in the case of complex numbers the tensor product of operators on Hilbert C*-modules coincide with the tensor product of operators on the Hilbert, we introduced a brief summary of the tensor product of operators on bounded operator on Hilbert spaces, one can found a rich material of this definition in [4,6]. Moreover the interior and the exterior tensor product of an operators on the bounded linear operators on Hilbert C*-modules was summarized and we refer for this definitions and other results in [2,3,5,7].

2. TENSOR PRODUCT OF BOUNDED OPERATORS ON HILBERT SPACES

Let $H_1$ and $H_2$ be two vector spaces over the field $K$. denote by $F(H_1, H_2)$ the finite linear combinations such that
\[
F (H_1, H_2) = \left\{ \sum_{i=1}^{n} C_i (f_i, g_i) : C_i \in k, f_i \in H_1, g_i \in H_2, i = 1, 2, \ldots, n \right\}
\]

Is a vector space. Let \( N \) be the subspace of \( F (H_1, H_2) \) spanned by the elements of the form
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_i f_i \otimes g_i - \left( \sum_{i=1}^{n} a_i f_i \right) \otimes \left( \sum_{j=1}^{m} b_i g_i \right)
\]

Is the quotient space. Then \( H_1 \otimes H_2 = F (H_1, H_2)/N \). Is called the algebraic tensor product of the two vector spaces \( H_1 \) and \( H_2 \).

And the pair \( (f_i, g_i) \) is denoted by \( f_i \otimes g_j \) The linear combination of elements of the form
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_i f_i \otimes g_i = \left( \sum_{i=1}^{n} a_i f_i \right) \otimes \left( \sum_{j=2}^{m} b_i g_i \right) = 0
\]

So, In particular, we have
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_i f_i \otimes g_i = \left( \sum_{i=1}^{n} a_i f_i \right) \otimes \left( \sum_{j=2}^{m} b_i g_i \right) = 0
\]

**Definition 2.1:** Let \( (H_1, <, , >_1) \) and \( (H_2, <, , >_2) \) Be Hilbert spaces, then we can define the inner product in the form
\[
\left\langle \sum_{i=1}^{n} C_i (f_i, g_i), \sum_{j=1}^{m} C_j' (f_j', g_j') \right\rangle
\]

It is a pre-Hilbert space. The completion of this pre-Hilbert space is denoted by \( H_1 \hat{\otimes} H_2 \) and called the tensor product of the Hilbert spaces \( H_1 \) and \( H_2 \).

Denoted it by \( x \otimes y \in H_1 \otimes H_2 \) \( x \in H_1, \ y \in H_2 \) for typical elements in \( H_1 \otimes H_2 \).

**Definition 2.2:** An operator \( T \) between two Hilbert spaces \( H_1 \) and \( H_2 \)

Is called bounded if there is \( K \geq 0 \) Such that \( \|Tx\| \leq K \|x\| \) for all \( x \in H_1 \).

That smallest such \( K \) is the norm of \( T \) and denoted \( \|T\| \), such that
\[
\|T\| = \text{Sup}\{|\|Tx\||, \|x\|\| = 1, x \in H_1\}
\]

The set of all bounded operators on \( H \) is denoted by \( \text{B}(H) \). \( \text{B}(H) \) is completed with the operator norm and satisfying
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\[ \|ST\| \leq \|S\| \cdot \|T\| \text{ for any } T, S \in (H). \]

Thus \( B(\mathcal{H}) \) is Banach algebra, and will denote by \( I \) the unit in \( B(\mathcal{H}) \).

Moreover if \( T \) and \( S \) are two operators on \( B(\mathcal{H}) \) there is an involution \( * \) satisfying the following:

\[ \begin{align*}
&i) \quad \langle T^*, y \rangle = \langle x, Ty \rangle. \\
&ii) (S + T)^* = S^* + T^*, \quad (\alpha T)^* = \bar{\alpha}T^* \text{ and } (T^*)^* = T \\
&iii) \quad \|T^*\| = \|T\| \text{ and } \|T^* T\| = \|T\|^2, \text{ for } T \in B(\mathcal{H}), \alpha \in \mathbb{C}.
\end{align*} \]

In fact \( B(\mathcal{H}) \) is defined \( C^* \)-algebra.

If \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are Hilbert spaces, there are natural extension of operators in \( B(\mathcal{H}_1), I = 1, 2 \), to the operators on \( \mathcal{H}_1 \otimes \mathcal{H}_2 : \) if \( T \in B(\mathcal{H}_1), \) define \( T \otimes I \in B(\mathcal{H}_1 \otimes \mathcal{H}_2) \) by

\[ (T \otimes I)(x \otimes y) = T x \otimes y \]

Similarly, for \( S \in B(\mathcal{H}_2), I \otimes S \) is defined by

\[ (I \otimes S)(x \otimes y) = x \otimes Sy \]

And \( T \mapsto T \otimes I \) and \( S \mapsto I \otimes S \) are isometries. The images of \( B(\mathcal{H}_1) \) and \( B(\mathcal{H}_2) \) is denoted by \( B(\mathcal{H}_1) \otimes I \) and \( I \otimes B(\mathcal{H}_2) \) respectively and so we get in general

If \( S_1, T_1 \in B(\mathcal{H}_1) \) and \( S_2, T_2 \in B(\mathcal{H}_2) \) we can define \( S_1 \otimes S_2 \in B(\mathcal{H}_1 \otimes \mathcal{H}_2) \) by \( (S_1 \otimes S_2)(x \otimes y) = S_1 x \otimes S_2 y \)

Then

\[ \begin{align*}
S_1 \otimes S_2 &= (S_1 \otimes I)(I \otimes S_2) \\
&= (I \otimes S_2)(S_1 \otimes I), \\
S_1 \otimes (S_2 + T_2) &= (S_1 \otimes S_2) + (S_1 \otimes T_2), \\
(S_2 \otimes S_2)(T_1 \otimes T_2) &= S_1 T_1 \otimes S_2 T_2, \\
(S_1 \otimes S_2) &= S_1^* \otimes S_2^* \text{ and} \\
\|S_1 \otimes S_2\| &= \|S_1\| \cdot \|S_2\|.
\end{align*} \]

where \( * \) is defined by

\[ \langle T^* x, y \rangle = \langle x, Ty \rangle. \]
**Theorem 3.1**: If \( T_1, T_2, \ldots, T_n \) are linearly independent in \( B(H_1) \) and \( S_1, S_2, \ldots, S_n \) are linearly independent in \( B(H_2) \). Then

\[
T_1 \otimes S_1 + T_2 \otimes S_2 + \cdots + T_n \otimes S_n = 0 \quad \text{if and only if} \quad T_1 = T_2 = \cdots = T_n = 0.
\]

**Proof**: Since \( T_1, T_2, \ldots, T_n \) are linearly independent.

This gives for some vectors \( \eta_i, \zeta_i \in H_i \),

Then for arbitrary vectors \( x \) and \( y \) in \( H_2 \), Then we have

\[
\langle S_1 x, y \rangle = \sum_{j=1}^{m} \sum_{i=1}^{r} \langle T_j \zeta_i, \eta_i \rangle \langle S_j x, y \rangle
\]

\[
= \sum_{j=1}^{m} \{ \sum_{i=1}^{r} T_j \otimes S_j, (\zeta_i \otimes x), (\zeta_i \otimes xy) \} = 0
\]

So \( \langle S_1 x, y \rangle = 0 \) and this give \( S_1 = 0 \)

Similarly we can get \( S_2 = S_2 = \cdots = S_n = 0 \).

**Theorem 4.2**: let \( T_i \in B(H_i), i = 1, 2, \ldots, n \). And \( T_1 \otimes T_2 \otimes \cdots \cdots \otimes T_n \)

Is an operator in \( H_1 \otimes H_2 \otimes \cdots \cdots \otimes T_n \) then \( T_1 \otimes T_2 \otimes \cdots \cdots \otimes T_n \)

is normal operator if \( T_i \) are normal operators.

**Proof**: It is suffices to prove that for

\[
T_1 \otimes T_2 \quad \text{let} \quad T_1 \otimes T_2 \quad \text{is normal operator in} \quad B \left( H_1 \otimes H_2 \right) \quad \text{then}
\]

\[
(T_1 \otimes T_2)^* \cdot (T_1 \otimes T_2) = (T_1 \otimes T_2) \cdot (T_1 \otimes T_2)^* \quad \text{ie} \quad T_1^* T_2 T_2^* T_1 - T_1 T_1^* T_2 T_2^* = 0
\]

if \( T_1 T_1^* \) and \( T_1^* T_1 \) are linearly independent.

then \( T_2 T_2^* = T_2^* T_2 = 0 \), so \( T_2 = 0 \)

\[\text{Contraduction, So } T_1 T_1^* = = \alpha T_1^* T_1 \quad \text{and} \]

similarly \( T_2 T_2^* = \beta T_2^* T_2 \) and \( ||T_1^*||^2 = \text{Sup}\{\langle T_1^* x, T_1^* x \rangle, ||x|| = 1\} \)

\[= \text{Sup}\{\langle r T_1^* x, T_1^* x \rangle, ||x|| = 1\} \]

\[= \text{Sup}\{r \langle T_1^* x, T_1^* x \rangle, ||x|| = 1\} \]

\[= r ||T_1||^2 \]

Since \( ||T_1|| = ||T_1^*|| \quad \text{therefore} \quad r = 1. \)

And similarly \( \alpha = 1 \). So \( T_1^* T_1 = T_1 T_1^* \) and \( T_2^* T_2 = T_2^* T_2 \)

Then \( T_1 \) and \( T_2 \) are normal operators, the other hand is easy to prove.
3. **C*-ALGEBRAS AND HILBERT C*-MODULES**

**Definition 3.1**: A *-algebra is an algebra $A$ together with an involution. A Banach *-algebra is a Banach algebra $A$ together with an isometric involution. That is, we insist that $\|a^*\| = \|a\|$ for all $a \in A$.) An algebra homomorphism $\varphi: A \rightarrow B$ between *-algebras is called a *-homomorphism if $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$.

**Definition 3.2**: A Banach algebra $A$ is a Banach space together with a norm compatible algebra structure, namely: forall $x, y \in A$, $\|xy\| \leq \|x\| \|y\|$. If $A$ has a multiplicative unit (denoted by $e$ or $1$), we say that the algebra $A$ is unital. In this case, $\|e\| \leq \|e^2\| \leq \|e\|^2$, and so $\|e\| = 0$ implies that $\|e\| \geq 1$. By scaling the norm, we assume that $\|e\| = 1$.

**Example 3.3**: The set $(C(X), \|\cdot\|_\infty)$ of continuous functions on a compact Hausdorff space $X$ introduced in Example 1.1.4 becomes a Banach algebra under pointwise multiplication of functions. That is, for $f, g \in C(X)$, $\|(fg)(x)\| = \|f(x)g(x)\|$ for all $x \in X$.

**Example 3.4**: Let $X$ be a Banach space, then the Banach space $B(X)$ with the operator norm $\|A\| = \text{Sup} \{\|Ax\| : \|x\| = 1\}$ defined as a Banach Algebra.

**Definition 3.5**: A Banach *-algebra is called a $C^*$-algebra if $\|a^*a\| = \|a\|^2$ for all $a \in A$.

**Example 3.6**: If $X$ is a locally compact Hausdorff space, then $A = C_0(X)$ is a $C^*$-algebra when equipped with the involution given by complex conjugation.

As we shall see, every commutative $C^*$-algebra arises as in Example 2.9. For the canonical noncommutative example, we need the following.

**Example 3.7**: Complex numbers are with the norm $\|y\| = |y|$ is $C^*$-algebra more details about $C^*$-algebras in [1,4,6].

**Example 3.8**: $B(H) = \{T: H \rightarrow H \ | \ T \text{ is bounded}\}$ with the operator norm $\|T\| = \text{Sup} \{\|Tx\| : \|x\| = 1\}$ defined as a $C^*$-algebra.

**Remark 3.9**: In general, if a $C^*$-algebra $A$ does not have an identity element, it is possible to append one as follows: Consider the algebra $\tilde{A} = A \oplus \mathbb{C}1$ with multiplication given by $(a, \lambda)(b, \nu) = (ab + a\nu + b\lambda, \lambda\nu)$.

We define a norm on $\tilde{A}$ via $\|(a, \lambda)\|_1 = \|a\| + |\lambda|$. Then we have $\|a^*(b, \nu)(a, \lambda)\| = \|ab + a\nu + b\lambda\| + |\lambda\| + |\nu| \leq \|a\| \|b\| + \|a\| |\nu| + \|b\| |\lambda| + |\lambda| |\nu| = (\|a\| + |\lambda|)(\|b\| + |\nu|)$.
\[ \| (a, \lambda) \| \| (b, \nu) \| \] It is clear that the embedding of \( A \) into \( \tilde{A} \) is linear and isometric, and that \( A \) sits inside of \( \tilde{A} \) as a closed ideal.

**Definition 3.10**: let \( A \) be a \( C^* \text{- algebra} \). A pre Hilbert - module is a complex vector space \( E \) which a right \( A \)-module with the map \( <, > : E \times E \rightarrow A \).

Which is linear in the second variable and satisfy the following relations,

\[
\text{for all } a \in A \text{ and } x, y \in E, \alpha, \beta \in \mathbb{C}
\]

\[ i) \langle x, x \rangle \geq 0 \text{ and if } \langle x, x \rangle = 0 \text{ implies } x = 0 \\
ii) \langle x, y \rangle^* = \langle y, x \rangle \\
iii) \langle x, ya \rangle^* = \langle y, x \rangle a \\
iv) \langle x, \alpha y + \beta \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle
\]

It is easy to check that the multiplication and the right \( A \) - module structure are compatible in the sense that

\[ \lambda (xa) = (\lambda x)a = x (\lambda a) \text{ for } x \in E, \alpha \in A, \lambda \in \mathbb{C} \]

\( E \) is called right pre- \( A \) – module.

**Definition 3.11**: The norm of an element \( x \in E \) is defined as \( \| x \| : = \sqrt{\langle x, x \rangle} \).

The norm on \( E \) satisfy

\[ i) \| \langle x, y \rangle \| \leq \| x \| \| y \| \\
ii) \| x + y \| \leq \| x \| + \| y \|, \\
iii) \| \langle x, y \rangle \|^2 \leq \| \langle x, x \rangle \| \| \langle y, y \rangle \| \text{ and} \\
vi) \| xa \| \leq \| x \| \| a \| \text{ for } x, y \in E \text{ and } a \in A
\]

If the pre-Hilbert module is complete with the respect norm above it is said to be Hilbert module over \( A \). In fact in the case right action the pre- Hilbert module is called Right Hilbert module.

**Example 3.12**: \( C^* \text{- algebra} \) \( A \) itself is a Hilbert \( A \)- module with the inner product \( \langle a, b \rangle := a^* b \)

Let \( E_1 \) and \( E_2 \) are Hilbert \( A \)-modules denoted by \( L_A(E_1, E_2) \)

The space of all adjointable operators \( T : E_1 \rightarrow E_2 \)

Satisfy \( \langle Tx, y \rangle = \langle x, T^* y \rangle \) for all \( x \in E_1 \) and \( y \in E_2 \). These operators are linear bounded operators from \( E_1 \) to \( E_2 \) and have a norm \( \| T \| = \text{sup} \{ \| Tx \| : \| x \| = 1 \} \).
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**Theorem 3.13:** \( L_A(E) \) the set of all bounded operator on the Hilbert \( A \)-module \( E \) is \( C^* \)-algebra.

**Proof:** it is straightforward to check that \( L_A(E) \) is \( \ast \)-algebra and \( ||ST|| \leq ||S|| \cdot ||T|| \) holds, and one can show that \( ||T^*|| = ||T|| \),

And \( ||\langle TX, Tx \rangle|| = ||\langle x, T^*Tx \rangle|| \leq ||T^*Tx|| \cdot ||T^*T|| \cdot ||x|| = 1 \)

This yields \( ||T||^2 \leq ||T^*T|| \) and it is clear that \( ||T^*T|| \leq ||T||^2 \) and so \( L_A(E) \)

So \( L_A(E) \) is \( C^* \)-algebra.

**Definition:** let \( L_A(E) \) be the set of all bounded operator on \( E \). Let \( T \in L_A(E) \)

Denote by \( W_A(T) = \{ \langle Tx, x \rangle_A : ||x|| = 1 \} \), the numerical range of \( T \) in \( E \).

**Lemma 3.14:** i) \( W_A(T^*) = W_A(T)^* \)

ii) \( W_A(I) = 1 \)

**Proof:** i) \( W_A(T^*) = \{ \langle T^*x, x \rangle ||x|| = 1 \}

\[ = \{ \langle x, Tx \rangle, ||x|| = 1 \} \]

\[ = \{ \langle Tx, x \rangle^* : ||x|| = 1 \} = W(T)^*. \]

ii) \( W_A(I) = \{ \langle Ix, x \rangle : ||x|| = 1 \} = \{ \langle x, x \rangle = 1 = ||x||^2 = 1 \}. \)

**Definition 3.15:** Interior tensor product .

Let \( E \) be a Hilbert \( A \)-module , \( F \) a hilbert \( B \)-module

Let \( \varphi : A \to L_B(F) \) be a \( \ast \)-homorphism . Then \( \varphi \) makes \( F \) is a left \( A \)-module , and the action given by

\[ \langle a, y \rangle \mapsto \varphi(a)y \quad (a \in A, y \in F) \]

And we can form the algebraic tensor product of \( E \) and \( F \) over \( A , E \otimes_A F \) which is a right \( B \)-module .

\[ <, > : E \otimes_A F \times E \otimes_A F \to B \] satisfies

\[ \langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle y_1, \varphi((x_1, x_2))y_2 \rangle. \]

We can set \( N = \{ z \in E \otimes_A F : \langle z, z \rangle = 0 \} \)

Then \( N \) is \( B \)-submodule and we consider the quistion

\[ E \otimes_A F /N \] and the quotient map

\[ q : E \otimes_A F \to E \otimes_A F /N. \]
Then \( E \otimes_A F / N \) is a right \( B \) – module. And the inner product
\[
\langle q(x), q(y) \rangle = \langle x, y \rangle.
\]

**Definition 3.15: Exterior in tensor product**.

Let \( E \) be a Hilbert \( A \) – module and \( F \) a Hilbert \( B \)-module. the algebraic tensor product \( E \otimes_C F \) is a right module over the algebraic tensor product \( A \otimes C B \) such that
\[
(x \otimes y)(a \otimes b) = x a \otimes y b, \quad x \in E, y \in F, \ a \in A, b \in B.
\]
We can defined \( A \otimes B \) valued inner product \( E \otimes F \) by the map
\[
\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle \otimes \langle y_1, y_2 \rangle.
\]
Thus \( E \otimes F \) is a pre-\( A \otimes B \) – module.

For more details see [2,3].

**Definition 3.16**: let \( T \in L_A(E) \) and \( S \in L_A(F) \), then \( T \otimes S \in L_A(E \otimes F) \) where \( E \otimes F \) is the exterior tensor product of \( E \) and \( F \), then we define the numerical range of the tensor product of \( T \otimes S \in L_A(E \otimes F) \) as
\[
W_A(T \otimes S) = \{ \langle (T \otimes S)(x \otimes y), x \otimes y \rangle : \| x \otimes y \| = 1 \}.
\]

**Theorem 3.17**: Let \( T \) and \( S \) be two operators on \( L_A(E) \), then
\[
W_A(T + S) \subseteq W_A(T) + W_A(S)
\]

**Proof**: It is straightforward since the numerical range is linear.

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