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A wide class of Banach Spaces in which Grothendieck conjecture holds

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Abstract

In this work we introduce a wide class of infinite dimensional Banach spaces which we call spaces of type λ_p, $1 \leq p < \infty$. Being aware that Grothendiek conjecture about smallness of the ideal of nuclear operators was disproved in 1983 by G. Pisier, in general Banach spaces, we show that it is true in this class. In fact we search for a class as wide as possible in which Grothendiek conjecture holds. The class λ_2 contains every Banach space in which there exists such sequence of projections $\{P_n\}$ on subspaces ε- isometric to the Euclidean space ℓ_n^2, such that $\|P_n\| = o(\sqrt{n})$. In 1974 W.J.Davis and W.B.Johnson showed that this property is satisfied by any uniformly convex Banach space. We notice that in any Banach space an analogous sequence of projections satisfying $\|P_n\| = O(\sqrt{n})$, always exists. Moreover, we show that the class S_p, $1 \leq p < \infty$, introduced by Morell and Retherford in 1975 is contained in the class of type λ_2 which in tern, is contained in the class λ_1 in which the Grothendiek conjecture holds.

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1. Introduction

In 1955 Alexander Grothendieck [1] posed the following conjecture: Is it true that for any arbitrary infinite dimensional Banach spaces X and Y there always exists a bounded linear operator from X into Y which is not nuclear?. Many attempts were done from that time to solve this problem. In 1983 Gilles Pisier [2, 3] constructed infinite-dimensional Banach space X such that the projective and injective tensor products $X\widehat{\otimes}_\pi X$ and $X\widehat{\otimes}_\varepsilon X$ coincide. This disproved Grothendieck conjecture.

It is of interest for which Banach spaces this conjecture holds. In 1975 Morell and Retherford [4] introduced a class of Banach spaces called of type S_p, $1 \leq p < \infty$, in which this conjecture holds.

Our task here is to get a class of Banach spaces as wide as possible in which this conjecture holds. Of course such class must contain the class S_p and does not contain Pisier spaces.

For such class we introduce the class of Banach spaces of type λ_p, $1 \leq p < \infty$, which increases as λ decreases in which this conjecture holds. The class λ_2 contains every Banach space in which there exists such sequence of projections $\{P_n\}$ on subspaces ε-isometric to the Euclidean space ℓ_n^2, such that $\|P_n\| = o(\sqrt{n})$. In [5] W.J. Davis and W.B.Johnson showed that this property is satisfied by any uniformly convex Banach space. We notice that in any Banach space an analogous sequence of projections satisfying $\|P_n\| = O(\sqrt{n})$, always exists. This is a consequence of the famous result [6] of A.Dvoretzky in 1961. The class λ_2 contains S_p and is contained in the class λ_1 in which Grothendiek conjecture holds. We give several examples for such classes.

2. Basic Definitions and facts

By $L(E, F)$ we denote the Banach space of all linear continuous operators from a Banach space E into a Banach space F endowed with the usual norm. By ℓ^p is denoted the space of p-absolutely summable sequences of real numbers and by ℓ_n^p the Euclidean space equipped with the norm of ℓ^p. By L^p is denoted the space of p-Lebesgue integrable functions. A bounded linear operator P from a Banach space E into itself is called a projection on E if $P^2 = P$ (The idempotent property) [7]. The Banach-Mazur distance [8] of two isomorphic Banach spaces E and F, is defined by $d(E, F) = \inf \|T\|\|T^{-1}\|$, where the infimum is taken over all isomorphisms T from E onto F.

Definition 2.1. [Ideals of operators] [9, 10] An ideal of operators $\mathcal{A} = \{\mathcal{A}(X, Y) : X, Y \in \mathcal{X}\}$ where $\mathcal{X}$ is the category of all Banach spaces, is a collection of linear subsets $\mathcal{A}(X, Y)$ of $L(X, Y)$ satisfying the following conditions:

1. $\mathcal{A}(X, Y)$ contains the space $\mathcal{F}(X, Y)$ of finite rank operators.
2. $U \in L(Y, Y_0), T \in \mathcal{A}(X, Y), V \in L(X_0, X) \Longrightarrow UTV \in \mathcal{A}(X_0, Y_0)$.

If on each component $\mathcal{A}(X, Y)$ there is a norm ρ (or quasi norm) satisfying

$$\rho(UTV) \leq \|U\|\|V\|\rho(T) \quad \forall \quad U \in L(Y, Y_0), T \in \mathcal{A}(X, Y), V \in L(X_0, X)$$

then this ideal is called a normed (or quasi normed) operator ideal respectively.

Example 2.2.

1. The ideal $L_c = \{L_c(X, Y) : X, Y \in \mathcal{X}\}$ of compact operators [9] is a normed operator ideal with the usual operator norm.

2. The ideal $\mathcal{N}$ of nuclear operators [1, 9, 11] is a normed operator ideal.

3. The ideal Π_p of p-absolutely summing operators and the ideal $D\Pi_p$ of operators whose duals are p-absolutely summing [9, 10, 12] are normed operator ideals.

4. The ideal $\mathcal{N}_2^\infty$ [10] of all operators $T \in L(X, Y)$ which has a representation $T(x) = \sum_{n=1}^{\infty} \beta_n f_n(x) y_n$ where $\{f_n\}_{n=1}^\infty \subseteq X^*$, $\{y_n\}_{n=1}^\infty \subseteq Y$ and $\lim_n \beta_n = 0$, with $\sum_{n=1}^{\infty} |f_n(x)|^2 < +\infty$, $\sum_{n=1}^{\infty} |g(y_n)|^2 < +\infty$ for every $x \in X$, $g \in Y^*$.

5. The ideal $\mathcal{S}_{\ell p}^{app}$, $(p > 0)$ is a quasinormed operator ideal [9, 11], where

$$\mathcal{S}_{\ell p}^{app}(X, Y) = \{T \in L(X, Y) : \sum_{n=0}^{\infty} \alpha_n(T)^p < +\infty\},$$

   and $\alpha_n(T)$, is the nth approximation number of the operator T.

Remark 2.3.

1. It is well known [10], that if an operator A is p-absolutely summing, then $\sum_{n=1}^{\infty} \|A(x_n)\|^p < +\infty$ for any sequence $\{x_n\}_{n=1}^\infty$ in X for which $\sum_{n=1}^{\infty} |f(x_n)|^p < +\infty \quad \forall f \in X^*$.

2. Every operator $T \in \mathcal{N}_2^\infty(X, Y)$ is approximative. In fact, the sequence $\{T_k\}_{k=1}^\infty$ of finite rank operators, where $T_k(x) = \sum_{n=1}^{k} \beta_n f_n(x) y_n$ converges in norm to T. Consequently $\mathcal{N}_2^\infty(X, Y) \subseteq L_c(X, Y) \quad \forall X, Y \in \mathcal{X}$.

3. If X, Y are Hilbert spaces then

$$\mathcal{N}(X, Y) \subseteq \mathcal{N}_2^\infty(X, Y) = L_c(X, Y) \quad \forall X, Y \in \mathcal{X}.$$

   This is a consequence of spectral decomposition Theorem [11].

Definition 2.4. [3, 10] An operator ideal $\mathcal{A} = \{\mathcal{A}(X, Y) : X, Y \in \mathcal{X}\}$ is called small if

$$\{\mathcal{A}(X, Y) = L(X, Y)\} \Longrightarrow min\{dimX, dimY\} < \infty.$$

Example 2.5. The ideal of compact operators L_c and the ideal of absolutely summing operators Π_1 are not small ideals while $\mathcal{S}^{app}_{\ell^p}$, $(p > 0)$ is a small ideal. In fact, for the infinite dimensional sequence spaces ℓ^p we have,

1. $L_c(\ell^p, \ell^q) = L(\ell^p, \ell^q) \quad \forall \quad 1 \leq q < p$ [13, 14].
2. $\Pi_1(\ell^1, \ell^2) = L(\ell^1, \ell^2)$ [15].
3. In [16] it was proved that for any infinite dimensional Banach spaces X, Y and for any $0 < p < q$, it is true that:

$$\mathcal{S}^{app}_{\ell^p}(X, Y) \subsetneqq \mathcal{S}^{app}_{\ell^q}(X, Y) \subsetneqq L(X, Y).$$

3. Basic Theorems and Technical Lemmas:

In our work we will be in need to the following results:

Theorem 3.1. (Abel Dini) Let $\sum_{n=1}^{\infty} a_n$ be a divergent series with $a_n > 0$, and $S_n = \sum_{i=1}^{n} a_i, \quad n = 1, 2, \ldots$ its partial sums. The series $\sum_{n=1}^{\infty} \frac{a_n}{S_n^r}$ converges for $r > 1$ and diverges for $r \leq 1$.

Theorem 3.2. (Morell and Retherford [4]) Let X be an infinite dimensional Banach space, $\{\beta_n\}_{n=1}^{\infty}$ a convergent to zero sequence of positive numbers with $\beta_n < 1$, then there exists a basic sequence $\{x_n\}_{n=1}^{\infty}$ such that $\|x_n\| = \beta_n$ and satisfying:

$$\sup\left\{\Big(\sum_{n=1}^{\infty} |x^*(x_n)|^2\Big)^{\frac{1}{2}} : x^* \in X^*, \|x^*\| \leq 1\right\} \leq 1.$$

Corollary 3.3. Let Y be an infinite dimensional Banach space, $\{\beta_n\}_{n=1}^{\infty}$ a monotonically convergent to zero sequence of positive numbers. Then there exist a number $c > 0$ and biorthogonal system $\{y_n, g_n\}_{n=1}^{\infty}$ with the properties:

$$\sum_{n=1}^{\infty} | < y_n, g > |^2 < +\infty \quad \forall g \in Y^* \; and \; \|g_n\| \leq \frac{c}{\beta_n}, \quad (n = 1, 2, \ldots).$$

Proof. It is a direct consequence of Morell and Retherford Theorem if we remark that for any basic sequence $\{y_n\}$ there is a sequence of coefficient functionals $\{g_n\}$ such that $1 \leq \|y_n\| \, \|g_n\| \leq 2\gamma$ where $\gamma \geq 1$ is the basic constant.

Theorem 3.4. (Bessaga and Pełczyński [16]) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a Banach spaces X and let $\{\beta_n\}_{n=1}^{\infty}$ be any convergent to zero sequence of real numbers. If $\sum_{n=1}^{\infty} | < x_n, x^* > | < +\infty \quad \forall x^* \in X^*$, then the series $\sum_{n=1}^{\infty} \beta_n x_n$ unconditionally converges in norm.

Lemma 3.5. Let X be a Banach space and $\{x_n^*\}_{n=1}^{\infty}$ be a sequence of functionals in X^*. If $\sum_{n=1}^{\infty} | < x, x_n^* > | < +\infty$ for every $x \in X$, then there exists a constant $c > 0$ such that $\sum_{n=1}^{\infty} | < x, x_n^* > | \leq c\|x\| \quad \forall x \in X$, and in general,

$$\sum_{n=1}^{\infty} | < x_n^*, F > | \leq c\|F\| \quad \forall F \in X^{**}.$$

Proof. Let us define a sequence of seminorms $P_k(x) = \sum_{n=1}^{k} | < x, x_n^* > |$ for$(k = 1, 2, \dots)$. Since these seminorms are continuous then using Banach Steinhauss Theorem we get that $P(x) = \lim_k \sum_{n=1}^{k} | < x, x_n^* > |$ is a continuous seminorm. Hence $P(x) \leq c\|x\| \quad \forall x \in X$. Now, since any ball of center at zero in X is weak* dense in the ball of the same radius in X^{**} then for any $F \in X^{**}$ there is a generalized sequence $\{x_\alpha\}$ in X such that

$$\|x_\alpha\| \leq \|F\| \textit{ and } \quad \lim_\alpha < x_\alpha, x^* > = < x^*, F > \quad \forall x^* \in X^*$$

Since $\sum_{n=1}^{\infty} | < x_\alpha, x_n^* > | \leq c\|x_\alpha\| \leq c\|F\| \; \forall \alpha$, then taking limit by α we complete the proof. ■

The following Lemma is useful for showing that some operators are well defined.

Lemma 3.6. Let X, Y be Banach spaces, $\{f_n\}_{n=1}^{\infty}$ be a sequence of functionals in X^* and $\{z_n\}_{n=1}^{\infty}$ a sequence of elements in Y and let $\{\mu_n\}_{n=1}^{\infty}$ be a convergent to zero sequence of real numbers. If for any $x \in X$ and $g \in Y^*$,

$$\sum_{n=1}^{\infty} |f_n(x) g(z_n)| < +\infty, \text{ then the formula } A(x) = \sum_{n=1}^{\infty} \mu_n f_n(x) z_n$$

defines a compact operator from X into Y.

Proof. Applying Lemma (3.5) with $g(z_n)f_n$ instead of x_n^* we get for every $g \in Y^*$,

$$\sum_{n=1}^{\infty} | < x, g(z_n)f_n > | = \sum_{n=1}^{\infty} |f_n(x)g(z_n)| < +\infty \quad \forall x \in X.$$

Therefore, there exists $c_g > 0$ such that $\sum_{n=1}^{\infty} |f_n(x)g(z_n)| \leq c_g \|x\| \forall x \in X, g \in Y^*$. Applying Bessaga and Pełczyński Theorem for the weakly summable series $\sum_{n=1}^{\infty} f_n(x)z_n$ we get the convergence of the series $\sum_{n=1}^{\infty} \mu_n f_n(x)z_n$ and the operator A is well defined. The compactness of the operator A follows from being a limit, in norm, of the sequence $A_k(x) = \sum_{n=1}^{k} f_n(x)z_n \quad (k = 1, 2, \dots)$ of finite rank operators. ■

Definition 3.7. [4] A Banach space X is called of type $\mathcal{S}_{p,\lambda}$, $1 \leq p < \infty$ if there exist a number $\lambda \geq 1$ and sequences of operators $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$, such that

$$A_n : \ell_n^p \longrightarrow X, B_n : X \longrightarrow \ell_n^p, with \quad B_n A_n = I_{\ell_n^p}, and \quad \|A_n\| \|B_n\| \leq \lambda$$

A space X is called of type $\mathcal{S}_p$, if it is of type $\mathcal{S}_{p,\lambda}$ for some $\lambda \geq 1$.

Proposition 3.8. Let X be a space of type $\mathcal{S}_p$, $1 \leq p < \infty$. Then there exist sequences of operators $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$ such that:

1. $A_n : \ell_n^2 \longrightarrow X, \quad B_n : X \longrightarrow \ell_n^2$ and $B_n A_n = I_{\ell_n^2}$.
2. $inf \dfrac{\|A_n\| \|B_n\|}{\sqrt{n}} = 0$.

Proof. Let X be a Banach space of type $\mathcal{S}_{p,\lambda}$ for some $\lambda \geq 1$. Then there exist two sequences of operators $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$, such that

$$A_n : \ell_n^p \longrightarrow X, \ B_n : X \longrightarrow \ell_n^p, \ with \ B_n A_n = I_{\ell_n^p}, \ and \ \|A_n\| \|B_n\| \leq \lambda$$

Let $p \neq 2$ and let $\alpha_n : \ell_n^2 \longrightarrow \ell_n^p$ and $\beta_n : \ell_n^p :\longrightarrow \ell_n^2$ be the identity mappings. There are two possible cases:

Case I: $(p > 2)$ Let $x = \{x_i\}_{i=1}^n \in \ell_n^2$, then

$$\|\alpha_n(x)\|_p = \{\sum_{i=1}^{n} |x_i|^p\}^{\frac{1}{p}} \leq \{\sum_{i=1}^{n} |x_i|^2\}^{\frac{1}{2}} = \|x\|_2.$$

Since for $x_0 = (1, 0, 0, \dots, 0)$, we get $\|\alpha_n(x_0)\|_p = \|x_0\|_2 = 1$, then $\|\alpha_n\| = 1$. Let $y = (y_i)_{i=1}^n \in \ell_n^p, r = \dfrac{p}{2} > 1,\ r' = \dfrac{r}{r-1}$. Then using Hölder's inequality we get: $\sum_{i=1}^n |y_i|^2 \le \{\sum_{i=1}^n |y_i|^{2r}\}^{\frac{1}{r}} \{\sum_{i=1}^n 1^{r'}\}^{\frac{1}{r'}}$. Consequently,

$$\|\beta_n y\|_2 = \{\sum_{i=1}^n |y_i|^2\}^{\frac{1}{2}} \le n^{\frac{1}{2r'}} \left\{\sum_{i=1}^n |y_i|^p\right\}^{\frac{1}{p}} = n^{\frac{1}{2r'}} \|y\|_p.$$

Therefore we get $\|\beta_n\| \le n^{\frac{1}{2r'}} = n^{\frac{r-1}{2r}} = n^{\frac{p-2}{2p}}$. Taking $y_0 = (1, 1, 1, \dots, 1)$ we get $n^{\frac{1}{2r'}} \|y_0\| = n^{\frac{1}{2r'}} n^{\frac{1}{2r}} = \sqrt{n} = \|\beta_n(y_0)\|_2$. Therefore, $\|\beta_n\| = n^{\frac{1}{2r'}} = n^{\frac{p-2}{2p}}$. Consequently, $\|\alpha_n\|\|\beta_n\| = n^{\frac{p-2}{2p}}$ $(1, 2)$.

Case II: $(p < 2)$ Taking $r = \dfrac{2}{p}$ and using Hölder's inequality with power r, we get:

$$\|\alpha_n(x)\|_p^p = \sum_{i=1}^n |x_i|^p \le \left\{\sum_{i=1}^n |x_i|^{pr}\right\}^{\frac{1}{r}} n^{\frac{1}{r'}}.$$

Consequently,

$$\|\alpha_n(x)\|_p \le n^{\frac{1}{pr'}} \left\{\sum_{i=1}^n |x_i|^2\right\}^{\frac{1}{2}} = n^{\frac{r-1}{pr}} \|x\|_2 = n^{\frac{2-p}{2p}} \|x\|_2$$

Moreover, for $y_0 = (1, 1, \dots, 1)$ we get:

$$\|y_0\|_2 n^{\frac{2-p}{2p}} = n^{\frac{1}{2}} n^{\frac{2-p}{2p}} = n^{\frac{1}{p}} = \|\alpha_n(y_0)\|_p$$

Therefore, $\|\alpha_n\| = n^{\frac{2-p}{2p}}$. It is clear that

$$\|\beta_n(x)\|_2 = \left\{\sum_{i=1}^n |x_i|^2\right\}^{\frac{1}{2}} \le \left\{\sum_{i=1}^n |x_i|^p\right\}^{\frac{1}{p}} = \|x\|_p;$$

$\|\beta_n(x_0)\|_2 = 1 = \|x_0\|_p$ where $x_0 = (1, 0, 0, \dots, 0)$. Therefore, $\|\beta_n\| = 1$. Consequently,

$$\|\alpha_n\|\|\beta_n\| = n^{\frac{2-p}{2p}} \qquad (2, 2)$$

From (1.2),(2.2) it follows that for any $p \in (1, +\infty)$ $\|\alpha_n\|\|\beta_n\| = n^{\frac{|2-p|}{2p}}$.
Taking $C_n = A_n\alpha_n$, $D_n = \beta_n B_n$, then we get:

$$C_n : \ell_n^2 \longrightarrow X, D_n : X \longrightarrow \ell_n^2,\ \textit{with}\ D_nC_n = I_{\ell_n^2},\ \textit{and}\ \|C_n\|\|D_n\| \le \lambda n^{\frac{|p-2|}{2p}}.$$

Since

$$\frac{|p-2|}{2p} = |\frac{1}{2} - \frac{1}{p}| < \frac{1}{2}$$

then

$$\lim_n \frac{\|C_n\|\|D_n\|}{\sqrt{n}} = 0.$$

■

Proposition 3.9. If in a Banach space X there exist a sequence of subspaces $\{L_n\}_{n=1}^{\infty}$, $dim L_n = n$, and a sequence of projections $\{P_n\}_{n=1}^{\infty}$, such that

$$P_n : X \longrightarrow L_n \ and \ \|P_n\| d(L_n, \ell_n^2) = o(\sqrt{n}),$$

then there exist sequences of operators $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$ such that:

1. $A_n : \ell_n^2 \longrightarrow X, \quad B_n : X \longrightarrow \ell_n^2$ and $B_n A_n = I_{\ell_n^2}$.
2. $inf \dfrac{\|A_n\|\|B_n\|}{\sqrt{n}} = 0.$

Proof. From the definition of Banach Mazur distance, for each $n \in \mathcal{N}$ there exist two isomorphisms T_n, T_n^{-1}, such that

$$T_n : L_n \to \ell_n^2 \quad T_n^{-1} : \ell_n^2 \to L_n, \|T_n\|\|T_n^{-1}\| \leq 2d(L_n, \ell_n^2)$$

Let φ_n be the canonical injection of L_n into$X$. Then we get the diagram

$$\ell_n^2 \xrightarrow{T_n^{-1}} L_n \xrightarrow{\varphi_n} X \xrightarrow{P_n} L_n \xrightarrow{T_n} \ell_n^2$$

Taking $A_n = \varphi_n T_n^{-1} \quad B_n = T_n P_n$, we get,$B_n A_n = id_{\ell_n^2}$. Moreover,

$$inf \frac{\|A_n\|\|B_n\|}{\sqrt{n}} \leq inf \frac{\|T_n^{-1}\|\|T_n\|\|P_n\|}{\sqrt{n}} \leq inf \frac{\|P_n\| 2d(L_n, \ell_n^2)}{\sqrt{n}} = 0.$$

■

Corollary 3.10. Any uniformly convex Banach space satisfies (1) and (2). See [5].

4. Banach spaces of type λ_p

In this section we introduce a wider class of Banach spaces which contains the class satisfying (1), (2) of Propositions (3.8) and (3.9) for $p = 2$.

Definition 4.1. A Banach space X is called of type λ_p, $1 \leq p < \infty$ if there exist a sequence of elements $\{x_n\}_{n=1}^{\infty}$ and a sequence of functionals $\{f_n\}_{n=1}^{\infty}$ such that:

i) $\sum_{n=1}^{\infty} | < x_n, f > |^p < +\infty \quad \forall f \in X^*$;

ii) $\sum_{n=1}^{\infty} | < x, f_n > |^2 < +\infty \quad \forall x \in X$;

iii) $\sum_{n=1}^{\infty} | < x_n, f_n > |^p = +\infty$.

Proposition 4.2. Let X be a space of type λ_p. Then

1. $dim X = \infty$;
2. If X is a complemented subspace of a space Z, then Z is a space of type λ_p;
3. If $1 \leq q < p$, then X is of type λ_q.

Proof. Let us consider that the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ satisfy the conditions i), ii), iii). In fact,

1) If $dim X < \infty$, then $dim X^* < \infty$. Let $dim X = m$ and let $\{e_i\}_{i=1}^{m}$ be a basis in X^* and $f_n = \sum_{i=1}^{m} \lambda_{in} e_i, \quad (n = 1, 2, \ldots)$. From condition ii) it follows that: $\lim_n < x, f_n >= 0 \quad \forall x \in X$. I.e the sequence $\{f_n\}_{n=1}^{\infty}$ converges weakly to zero in a finite dimensional space and hence is bounded in norm. Consequently, there exists a constant $c > 0$ such that : $|\lambda_{in}| \leq c \quad \forall (n = 1, 2, \ldots), (i = 1, 2, \ldots)$. Therefore,

$$| < x, f_n > | = | \sum_{i=1}^{m} < x, \lambda_{in} e_i > | \leq c \sum_{i=1}^{m} | < x, e_i > | \quad (n = 1, 2, \ldots)$$

Using Hölder's inequality with $p > 1$ and $p' = \dfrac{p}{p-1}$. we get

$$| < x, f_n > |^p \leq c^p m^{\frac{p}{p'}} \sum_{i=1}^{m} | < x, e_i > |^p \quad (n = 1, 2, \ldots).$$

Therefore we get from property i)

$$\begin{aligned} \sum_{n=1}^{\infty} | < x_n, f_n > |^p &\leq c^p m^{\frac{p}{p'}} \sum_{n=1}^{\infty} \sum_{i=1}^{m} | < x_n, e_i > |^p \\ &= c^p m^{\frac{p}{p'}} \sum_{i=1}^{m} \sum_{n=1}^{\infty} | < x_n, e_i > |^p < +\infty. \end{aligned}$$

This contradicts iii) and hence dim $X = \infty$.

2) If X is a complemented subspace of a Banach space Z, then there exists a continuous projection P from Z into X. Then, $P^* : X^* \longrightarrow Z^*$. We define in Z^* a sequence $\{h_n\}_{n=1}^{\infty}$ by the equality $h_n = P^* f_n \quad (n = 1, 2, \dots)$ and we show that the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ satisfy the conditions i), ii), iii).

   i) If $f \in Z^*$, then considering f as a functional on X, we get:

$$\sum_{n=1}^{\infty} | < x_n, f > |^p < +\infty.$$

   ii) For any element $x \in Z$ we have $Px \in X$ and hence,

$$\sum_{n=1}^{\infty} | < x, h_n > |^2 = \sum_{n=1}^{\infty} | < x, P^* f_n > |^2 = \sum_{n=1}^{\infty} | < Px, f_n > |^2 < +\infty.$$

   iii)

$$\begin{aligned} \sum_{n=1}^{\infty} | < x_n, h_n > |^p &= \sum_{n=1}^{\infty} | < x_n, P^* f_n > |^p \\ &= \sum_{n=1}^{\infty} | < Px_n, f_n > |^p = \sum_{n=1}^{\infty} | < x_n, f_n > |^p = +\infty. \end{aligned}$$

3) We define a sequence of positive numbers $\{\beta_n\}_{n=1}^{\infty}$ as follows:

$$\beta_n = | < x_n, f_n > |^{\frac{p-q}{q}} (\frac{1}{S_n})^{\frac{1}{q}}.$$

where $S_n = \sum_{i=1}^{n} | < x_i, f_i > |^p$. Putting $y_n = \beta_n x_n$, we prove that the sequences $\{y_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$, satisfy conditions i), iii) with q (condition (ii) does not depend on p). We use Abel Dini Theorem.

Condition iii)
$$\begin{aligned} \sum_{n=1}^{\infty} | < y_n, f_n > |^q &= \sum_{n=1}^{\infty} | < x_n, f_n > |^{p-q} \frac{1}{S_n} | < x_n, f_n > |^q \\ &= \sum_{n=1}^{\infty} | < x_n, f_n > |^p \frac{1}{S_n} = +\infty \end{aligned}$$

Condition i) Let $s = \dfrac{p}{p-q} > 1$ *i.e.* $\dfrac{q}{p} + \dfrac{1}{s} = 1$ Using Hölder's inequality with power $\dfrac{p}{q}$ we get:

$$\sum_{n=1}^{\infty} | < y_n, f > |^q = \sum_{n=1}^{\infty} | < x_n, f_n > |^{p-q} \frac{1}{S_n} | < x_n, f > |^q \leq$$
$$\leq \left\{ \sum_{n=1}^{\infty} | < x_n, f > |^p \right\}^{\frac{q}{p}} \times \left\{ \sum_{n=1}^{\infty} | < x_n, f_n > |^p (\frac{1}{S_n})^{\frac{p}{p-q}} \right\}^{\frac{p-q}{p}} < +\infty.$$

■

Proposition 4.3.

1. For $r \in (1, 2]$, the space ℓ^r is of type λ_p for any $p \in [1, +\infty)$;
2. For any $r > 2$ the space ℓ^r is of type λ_p for $p < \dfrac{2r}{r-2}$.

Proof. Let $x^n = (0, 0, \dots, 1, 0, 0, \dots)$ (unity is in the n^{th} place) and let f^n be defined by the equality: $f^n(\xi) = \xi_n$ for any $\xi = (\xi_k)_{k=1}^{\infty}$ $(n = 1, 2, 3, \dots)$.

1) Let $1 < r \leq 2$. We verify conditions i)-iii) under $p \geq r' = \dfrac{r}{r-1}$. In fact, iii) is clear; ii) $\sum_{n=1}^{\infty} | < \xi, f^n > |^2 = \sum_{n=1}^{\infty} |\xi_n|^2 \leq \left(\sum_{n=1}^{\infty} |\xi_n|^r \right)^{\frac{2}{r}} < +\infty$ $\forall \xi = (\xi_k)_{k=1}^{\infty} \in \ell^r$.
i) For any $\varphi = (\varphi_k)_{k=1}^{\infty} \in \ell^{r'}$, where $\dfrac{1}{r} + \dfrac{1}{r'} = 1$, we have

$$\sum_{n=1}^{\infty} | < x^n, \varphi > |^p = \sum_{n=1}^{\infty} |\varphi_n|^p < +\infty \text{ for any } p \geq r'.$$

From 3) of proposition (4.2) we get the proof of (1).

2) Let $r > 2$. Then $r' = \dfrac{r}{r-1} < \dfrac{2r}{r-2}$. Take a number p such that:

$$r' < p < \frac{2r}{r-2} \text{ and let } h_n = \left(\frac{1}{n}\right)^{\frac{1}{p}} f^n \ (n = 1, 2, \dots).$$

We will prove that the sequences $\{x^n\}_{n=1}^{\infty}$ and $\{h^n\}_{n=1}^{\infty}$ satisfy conditions i)-iii).

$$\begin{aligned}
iii)\sum_{n=1}^{\infty}|<x^n,h^n>|^p &= \sum_{n=1}^{\infty}\frac{1}{n}|<x^n,f^n>|^p=\sum_{n=1}^{\infty}\frac{1}{n}=+\infty;\\
ii)\sum_{n=1}^{\infty}|<x,h^n>|^2 &= \sum_{n=1}^{\infty}|<x,f^n>|^2(\frac{1}{n})^{\frac{2}{p}}\leq\\
&\leq \Big(\sum_{n=1}^{\infty}|<x,f^n>|^r\Big)^{\frac{2}{r}}\Big(\sum_{n=1}^{\infty}(\frac{1}{n})^{\frac{2r}{p(r-2)}}\Big)^{\frac{r-2}{r}}\\
&\leq \Big(\sum_{n=1}^{\infty}|x_n|^r\Big)^{\frac{2}{r}}\Big(\sum_{n=1}^{\infty}(\frac{1}{n})^{\frac{2r}{p(r-2)}}\Big)^{\frac{r-2}{r}}<+\infty\\
&\qquad \forall x=\{x_n\}_{n=1}^{\infty}\in\ell^r.\\
i)\sum_{n=1}^{\infty}|<x^n,\varphi>|^p &= \sum_{n=1}^{\infty}|\varphi_n|^p\leq\Big(\sum_{n=1}^{\infty}|\varphi_n|^{r'}\Big)^{\frac{p}{r'}}<+\infty\\
&\qquad \forall\varphi=\{\varphi_n\}_{n=1}^{\infty}\in\ell^{r'}
\end{aligned}$$

From (3) of proposition (4.2) we get the proof of (2) ■.

Remarks 4.4.

1. The space ℓ^{∞} is of type λ_1, but as will be shown is not of type λ_2 (see Corollary 5.5).
2. The space ℓ^1 is not of type λ_1 (see Corollary 5.5), although its dual is of type λ_1.
3. For $r\in(1,+\infty)$ the space ℓ^r is of type λ_2.

Proof. 1) Let x^n and f^n be the same as in Proposition (4.3), and let $h_n=\frac{1}{n}f^n \quad (n=1,2,\dots)$. We show that $\{x^n\}_{n=1}^{\infty}$ and $\{h^n\}_{n=1}^{\infty}$ satisfy the conditions of λ_1. Let $f\in(\ell^{\infty})^*$, $\quad \varepsilon_n=\text{sign}<x^n,f>$. Then,

$$\begin{aligned}
i)\quad \sum_{n=1}^{\infty}|<x^n,f>| &= \lim_k\sum_{n=1}^{k}\varepsilon_n<x^n,f>\\
&= \lim_k f\Big(\sum_{n=1}^{k}\varepsilon_n x^n\Big)\leq\overline{\lim_k}\|f\|\,\|\sum_{n=1}^{k}\varepsilon_n x^n\|\leq\|f\|\\
ii)\quad \sum_{n=1}^{\infty}|<x,h^n>|^2 &\leq \sum_{n=1}^{\infty}\frac{\|x\|_{\infty}^2}{n^2}<+\infty;\\
iii)\quad \sum_{n=1}^{\infty}|<x^n,h^n>| &= \sum_{n=1}^{\infty}\frac{1}{n}=+\infty.
\end{aligned}$$

3) If $r \in (1, 2]$ then from Proposition 4.3 (1), ℓ^r is of type λ_p for any $p \in [1, +\infty)$ and hence of type λ_2. For $r > 2$, then from Proposition 4.3 (2), ℓ^r is of type λ_2, since $\dfrac{2r}{r-2} > 2$. ■

5. Main Results

Theorem 5.1. If the space X is a Banach space of type λ_p and Y is an infinite dimensional Banach space, then $\Pi_p(X, Y)$ does not contain $\mathcal{N}_2^{\infty}(X, Y)$.

Proof. Since X is a space of type λ_p, then there exist two sequences $\{x_n\}_{n=1}^{\infty}$ in X, and $\{f_n\}_{n=1}^{\infty}$ in X^* satisfying conditions: i), ii), iii). Let us take

$$\beta_n = (S_n)^{\frac{-1}{2p}}, \quad where \quad S_n = \sum_{i=1}^{n} | < x_i, f_i > |^p.$$

By the corollary of Morell and Retherford Theorem, there exists a constant $c > 0$ and a biorthogonal system $\{y_n, g_n\}_{n=1}^{\infty}$ in Y such that:

$$\sum_{n=1}^{\infty} | < y_n, g > |^2 < +\infty \forall g \in Y^*; \quad \|g_n\| \le \frac{c}{\beta_n}, (n = 1, 2, \dots).$$

The series $\sum_{n=1}^{\infty} < x, f_n > y_n$ is weakly convergent. In fact,

$$|\sum_{n=1}^{\infty} < x, f_n >< y_n, g > | \le \left\{\sum_{n=1}^{\infty} | < x, f_n > |^2\right\}^{\frac{1}{2}} \left\{\sum_{n=1}^{\infty} | < y_n, g > |^2\right\}^{\frac{1}{2}} < +\infty$$

According to Lemma (3.6), we define an operator A from X into Y as follows:

$$A(x) = \sum_{n=1}^{\infty} \beta_n < x, f_n > y_n$$

Clearly we have $A \in \mathcal{N}_2^{\infty}(X, Y)$. We prove that the operator A is not p-absolutely summing. In fact, we have:

$$\|Ax_k\| \ge \left|\frac{g_k}{\|g_k\|}(Ax_k)\right| = \frac{1}{\|g_k\|}\beta_k | < x_k, f_k > | \ge \frac{1}{c}\beta_k^2 | < x_k, f_k > |$$

Consequently,

$$\sum_{k=1}^{\infty} \|Ax_k\|^p \ \ge \ \frac{1}{c^p}\sum_{k=1}^{\infty}(\beta_k)^{2p} | < x_k, f_k > |^p$$

$$= \ \frac{1}{c^p}\sum_{k=1}^{\infty} | < x_k, f_k > |^p \frac{1}{S_k} = +\infty$$

Since the space X is of type λ_p then from the condition i) we get

$$\sum_{k=1}^{\infty} | < x_k, f > |^p < +\infty \quad \forall f \in X^*,$$

and the operator T is not p-absolutely summing. ■

Corollary 5.2. If X is of type λ_p and Y is infinite dimensional Banach space, then $\Pi_p(X, Y)$ does not contain $L_c(X, Y)$. This follows from the inclusion:

$$\mathcal{N}_2^{\infty}(X, Y) \subseteq L_c(X, Y).$$

Corollary 5.3. Under the same conditions of Corollary (5.2), we get,

$$\Pi_p(X, Y) \neq L(X, Y).$$

Corollary 5.4. If X is a reflexive space of type λ_p and Y is an infinite dimensional Banach space, then

$$\Pi_p(X, Y) \subsetneqq L_c(X, Y).$$

The statement of this corollary follows from the relation

$$\Pi_p(X, Y) \sqsubseteq L_c(X, Y),$$

satisfied for any reflexive Banach space X. See [18] and Corollary (5.2).

Corollary 5.5. The spaces ℓ^1, L^1, are not of type λ_1; and the spaces ℓ^∞, L^∞, and $C(\Omega)$, where Ω is an arbitrary compact set, are not of type λ_2.

Proof. It is well known that [1, 3, 19]

$$\Pi_1(\ell^1, \ell^2) = Ł(\ell^1, \ell^2), \quad \Pi_1(L^1, \ell^2) = Ł(L^1, \ell^2),$$

$$\Pi_2(\ell^\infty, \ell^2) = Ł(\ell^\infty, \ell^2), \quad \Pi_2(L^\infty, \ell^2) = Ł(L^\infty, \ell^2), \quad \Pi_2(C(\Omega), \ell^2) = Ł(C(\Omega), \ell^2)$$

It remains only to apply Corollary (5.3). ■

Theorem 5.6. Let X be a Banach space in which there exist sequences of operators $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$ such that:

1. $A_n : \ell_n^2 \longrightarrow X, \quad B_n : X \longrightarrow \ell_n^2$ and $B_n A_n = I_{\ell_n^2}$.

2. $inf \dfrac{\|A_n\| \|B_n\|}{\sqrt{n}} = 0,$

then it is of type λ_2.

Proof. Let $\{m_k\}_{k=1}^{\infty}$ be a sequence of natural numbers such that

$$\frac{\|A_{m_k}\| \|B_{m_k}\|}{\sqrt{m_k}} < \frac{1}{k^2}$$

Without loss of generality, one can consider that

$$\|A_{m_k}\| = \|B_{m_k}\| < \frac{\sqrt[4]{m_k}}{k}$$

Taking $m_0 = 0$, for $j = 1, 2, \ldots, m_{k+1}$, we define

$$\begin{aligned} \widetilde{x}_{m_0+m_1+\cdots+m_k+j} &= A_{m_{k+1}} e_j, \\ \widetilde{f}_{m_0+m_1+\cdots+m_k+j} &= B^*_{m_{k+1}} e_j, \end{aligned}$$

where $\{e_j\}_{j=1}^{m_{k+1}}$ is the canonical orthonormal basis in $\ell^2_{m_k+1}$. Let us define

$$x_{m_0+m_1+\cdots+m_k+j} = \frac{1}{\sqrt[4]{m_{k+1}}} \widetilde{x}_{m_0+m_1+\cdots+m_k+j},$$

$$f_{m_0+m_1+\cdots+m_k+j} = \frac{1}{\sqrt[4]{m_{k+1}}} \widetilde{f}_{m_0+m_1+\cdots+m_k+j}$$

We show that the sequences $\{x_n\}_{n=1}^{\infty}$, $\{f_n\}_{n=1}^{\infty}$, satisfy the conditions of spaces of type λ_2.

$$\begin{aligned} i) \quad \sum_{n=1}^{\infty} |< x_n, f >|^2 &= \sum_{k=0}^{\infty} \frac{1}{\sqrt{m_{k+1}}} \sum_{j=1}^{m_{k+1}} |< A_{m_{k+1}} e_j, f >|^2 \\ &= \sum_{k=0}^{\infty} \frac{1}{\sqrt{m_{k+1}}} \sum_{j=1}^{m_{k+1}} |< e_j, A^*_{m_{k+1}} f >|^2 \\ &= \sum_{k=0}^{\infty} \frac{1}{\sqrt{m_{k+1}}} \|A^*_{m_{k+1}} f\|^2 \\ &\leq \sum_{k=0}^{\infty} \frac{1}{\sqrt{m_{k+1}}} \Big(\frac{1}{k+1} \sqrt[4]{m_{k+1}} \Big)^2 \|f\|^2 \\ &= \|f\|^2 \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} < +\infty \end{aligned}$$

ii) Similar to i) we get $\sum_{n=1}^{\infty} | < x, f_n > |^2 \leq \|x\|^2 \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} < +\infty$

$$
\begin{aligned}
iii) \quad \sum_{n=1}^{\infty} | < x_n, f_n > |^2 &= \sum_{k=0}^{\infty} \frac{1}{m_{k+1}} \sum_{j=1}^{m_{k+1}} | < A_{m_{k+1}} e_j, B^*_{m_{k+1}} e_j > |^2 \\
&= \sum_{k=0}^{\infty} \frac{1}{m_{k+1}} \sum_{j=1}^{m_{k+1}} | < B_{m_{k+1}} A_{m_{k+1}} e_j, e_j > |^2 \\
&= \sum_{k=0}^{\infty} \frac{1}{m_{k+1}} \sum_{j=1}^{m_{k+1}} | < e_j, e_j > |^2 = \sum_{k=0}^{\infty} 1 = +\infty.
\end{aligned}
$$

■

Corollary 5.7. Any Banach space of type S_p is of type λ_2. This follows from Proposition (3.8).

Theorem 5.8. Let X be an infinite dimensional Banach space, and Y a Banach space of type type λ_2. Then $D\Pi_2(X, Y)$ does not contain $\mathcal{N}_2^{\infty}(X, Y)$.

Proof. Since Y is of type λ_2, then there exist a sequence of elements $\{y_n\}_{n=1}^{\infty}$ and a sequence of functionals $\{g_n\}_{n=1}^{\infty}$ such that:

1. $\sum_{n=1}^{\infty} | < y_n, g > |^2 < +\infty \quad \forall g \in Y^*$
2. $\sum_{n=1}^{\infty} | < y, g_n > |^2 < +\infty \quad \forall y \in Y$
3. $\sum_{n=1}^{\infty} | < y_n, g_n > |^2 = +\infty.$

Let us take $\beta_n = (\frac{1}{S_n})^{\frac{1}{4}}$, *where* $S_n = \sum_{i=1}^{n} | < y_i, g_i > |^2$. Clearly, $\beta_n \to 0$ as $n \to \infty$. From corollary of Morell and Retherford Theorem, there exist a constant $c > 0$ and a biorthogonal system $\{f_k, F_k\}_{k=1}^{\infty}$, $(f_k \in X^*, F_k \in X^{**})$ such that:

$$\sum_{n=1}^{\infty} | < f_n, F > |^2 < +\infty \quad \forall F \in X^{**}, \quad \|F_n\| \leq \frac{c}{\beta_n} \quad (n = 1, 2, \ldots)$$

Using Lemma (3.6) we construct an operator A from $XintoY$ as follows:

$$Ax = \sum_{n=1}^{\infty} \beta_n < x, f_n > y_n.$$

Clearly, $A \in \mathcal{N}_2^{\infty}(X, Y)$. We show that A does not belong to $D\Pi_2(X, Y)$. We have

$$A^* g = \sum_{n=1}^{\infty} \beta_n < y_n, g > f_n \quad (g \in Y^*).$$

Therefore we have

$$\|A^* g_n\| \geq \left| \frac{F_n}{\|F_n\|}(A^* g_n) \right| \geq \frac{1}{c} \beta_n^2 | < y_n, g_n > |.$$

Consequently,

$$\sum_{n=1}^{\infty} \|A^* g_n\|^2 \geq \frac{1}{c^2} \sum_{n=1}^{\infty} \beta_n^4 \, | < y_n, g_n > |^2 = \sum_{n=1}^{\infty} \frac{| < y_n, g_n > |^2}{S_n} = +\infty.$$

Since from (2) and Lemma (3.5) we get $\sum_{n=1}^{\infty} | < G, g_n > |^2 < +\infty \quad \forall G \in Y^{**}$ then A^* does not belong to $\Pi_2(Y^*, X^*)$. ■

Corollary 5.9. If X is an infinite dimensional Banach space, and Y a Banach space of type λ_2. Then $D\Pi_2(X, Y)$ does not contain $L_c(X, Y)$.

Theorem 5.10. Let X, Y be infinite dimensional Banach spaces. If X is of type λ_1 or Y is of type λ_2 then

$$\mathcal{N}(X, Y) \subsetneqq L_c(X, Y)$$

Proof.

1) Let X be of type λ_1. Then by corollary (5.2) of Theorem (5.1), $\Pi_1(X, Y)$ does not contain $L_c(X, Y)$. Since

$$\mathcal{N}(X, Y) \subseteq \Pi_1(X, Y) \quad [18]$$

and $\mathcal{N}(X, Y) \subseteq L_c(X, Y)$, then $\mathcal{N}(X, Y) \subsetneqq L_c(X, Y)$.

2) Let Y be of type λ_2. Then by corollary (5.9) of Theorem (5.8), $D\Pi_2(X, Y)$ does not contain $L_c(X, Y)$. Moreover, if $T \in \mathcal{N}(X, Y)$ then

$$T^* \in \mathcal{N}(Y^*, X^*) \subseteq \Pi_1(Y^*, X^*) \subseteq \Pi_2(Y^*, X^*) \;\; [18].$$

Consequently, $\mathcal{N}(X, Y) \subseteq D\Pi_2(X, Y)$. From which it follows that $\mathcal{N}(X, Y)$ does not contain $L_c(X, Y)$ and finally

$$\mathcal{N}(X, Y) \subsetneqq L_c(X, Y).$$

■

Corollary 5.11. The ideal $\mathcal{N}$ of nuclear operators is small in the class of type λ_1.

6. Conclusion

If X and Y are Banach spaces of type λ_1 (a property satisfied by any Banach space of type λ_p), then $\mathcal{N}(X, Y) \subsetneqq L_c(X, Y)$. This means that for any Banach spaces X and Y of type λ_1 there always exists a compact non-nuclear operator from X into Y. This also proves that Grothendieck conjecture holds in the class of Banach spaces of type λ_1.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally to the manuscript and read and approved the final manuscript.

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