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Strong and $\triangle$-Convergence Theorems for Total Asymptotically Nonexpansive Nonself-Mappings in CAT(κ) Spaces

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Abstract

In this pape, we study the demiclosed principle, $\triangle$-convergence and strong convergence of the iterative schemes for total asymptotically nonexpansive nonself-mappings in CAT(κ) spaces for $\kappa > 0$. Our results generalize, unify and extend the several well-known results.

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1. Introduction

The concept of total asymptotically nonexpansive mappings was first introduced in Banach spaces by Alber et al. [1]. It is generalization of the asymptotically nonexpansive mappings introduced by Goebel and Kirk [2] as well as the nearly asymptotically nonexpansive mappings introduced by Sahu [3].

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In 2012, Chang et al. [4] studied the demiclosed principle and convergence theorems for total asymptotically nonexpansive mappings in CAT(0) spaces. Since then the convergence theorems of several iteration procedures for this type of mappings have been rapidly developed (see e.g., [5]-[15]).

We know that any CAT(κ) space is a CAT(κ') space for $\kappa' \geq \kappa$. Thus, all results for CAT(0) spaces immediately apply to any CAT(κ) space with $\kappa \leq 0$.

In this article, we study the demiclosed principle, strong convergence and $\triangle$-convergence of the iterative schemes for total asymptotically nonexpansive nonself-mappings in the general setting of CAT(κ) space for $\kappa > 0$.

2. Preliminaries

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image $c([0, l])$ of c is called a *geodesic (or metric) segment* joining x and y. If it is unique, then this geodesic segment is denoted by $[x, y]$, that is, $z \in [x, y]$ means that there exists $\alpha \in [0, 1]$ such that

$$d(x, z) = (1 - \alpha)d(x, y) \quad \text{and} \quad d(y, z) = \alpha d(x, y).$$

In this case, we write $z = \alpha x \oplus (1 - \alpha)y$. The pair (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset C of X is said to be *convex* if C includes every geodesic segment joining any two of its points. The set C is said to be *bounded* if

$$diam(C) := \sup\{d(x; y) : x, y \in C\} < \infty.$$

The metric $d : X \times X \to R$ is said to be *convex* in a geodesic space (X, d), if for any $x, y, z \in X$ and $\alpha \in [0, 1]$,

$$d(x, \alpha y \oplus (1 - \alpha)z) \leq \alpha d(x, y) + (1 - \alpha)d(x, z).$$

For $D \in (0, \infty]$, (X, d) is called a *D-geodesic space* if any two points of X with their distance smaller than D are joined by a geodesic segment. We know that (X, d) is a geodesic space if and only if it is a D-geodesic space.

Now we introduce the model spaces M^n_κ, for more details on these spaces (see [16]). We denote by $(\cdot|\cdot)$ the Euclidean scalar product in R^n for $n \in N$, that is,

$$(x|y) = x_1y_1 + \cdots + x_ny_n, \quad \text{where} \quad x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n).$$

Let S^n denote the n-dimensional sphere defined by

$$S^n = \{x = (x_1, \cdots, x_{n+1}) \in R^{n+1} : (x|x) = 1\},$$

with metric $d_{S^n}(x, y) = \arccos(x|y)$, $x, y \in S^n$.

Let $H^{n,1}$ denote the vector space R^{n+1} endowed with the symmetric bilinear form which associates to vectors $u = (u_1, \cdots, u_{n+1})$ and $v = (v_1, \cdots, v_{n+1})$. Then the real number $\langle u|v\rangle$ defined by

$$\langle u|v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_i v_i.$$

Let H^n denote the *hyperbolic n-space* defined by

$$H^n = \{u = (u_1, \cdots, u_{n+1}) \in H^{n,1} : \langle u|u\rangle = -1, u_{n+1} > 0\},$$

with metric d_{H^n} such that

$$\cosh d_{H^n}(x, y) = -\langle x|y\rangle, \quad x, y \in H^n.$$

From now on, we assume that $\kappa \geq 0$ and define

$$D_\kappa := \frac{\pi}{\sqrt{\kappa}} \quad \text{if } \kappa > 0 \quad \text{and} \quad D_\kappa := \infty \quad \text{if } \kappa = 0.$$

We denote by M_κ^n the following metric spaces:

(i) if $\kappa = 0$ then M_0^n is the Euclidean space R^n;

(ii) if $\kappa > 0$ then M_κ^n is obtained from the spherical space S^n by multiplying the distance function by the constant $\frac{1}{\sqrt{\kappa}}$.

A *geodesic triangle* $\triangle(x, y, z)$ in a geodesic space X consists of three points x, y, z of X and three geodesic segments joining each pair of vertices. A *comparison triangle* of a geodesic triangle $\triangle(x, y, z)$ is the triangle $\bar{\triangle}(\bar{x}, \bar{y}, \bar{z})$ in M_κ^2 such that

$$d(x, y) = d_{M_\kappa^2}(\bar{x}, \bar{y}), \quad d(y, z) = d_{M_\kappa^2}(\bar{y}, \bar{z}), \quad \text{and} \quad d(z, x) = d_{M_\kappa^2}(\bar{z}, \bar{x}).$$

If $\kappa > 0$, then such a triangle exists whenever $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$. A point $\bar{p} \in [\bar{x}, \bar{y}]$ is called a *comparison point* for $p \in [x, y]$ if $d(x, p) = d_{M_\kappa^2}(\bar{x}, \bar{p})$.

A geodesic triangle $\triangle(x, y, z)$ is said to satisfy the $CAT(\kappa)$ *inequality* if for any $p, q \in \triangle(x, y, z)$ and $\bar{p}, \bar{q} \in \bar{\triangle}(\bar{x}, \bar{y}, \bar{z})$, we have

$$d(p, q) \leq d_{M_\kappa^2}(\bar{p}, \bar{q}).$$

Definition 2.1. X is called a *CAT(κ) space* for $\kappa > 0$, if X is D_κ-geodesic and any geodesic triangle $\triangle(x, y, z)$ in X with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ satisfies the CAT(κ) inequality.

Notice that in a CAT(0) space (X, d), if $x, y, z \in X$, then the CAT(0) inequality implies

$$d^2(x, \frac{1}{2}y \oplus \frac{1}{2}z) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(y, z). \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [17] and it is extended by Dhompongsa and Panyanak [18] as

$$d^2(x, (1-\alpha)y \oplus \alpha z) \leq (1-\alpha)d^2(x, y) + \alpha d^2(x, z) - \alpha(1-\alpha)d^2(y, z), \tag{CN*}$$

for all $\alpha \in [0, 1]$ and $x, y, z \in X$.

Note that, if X is a geodesic space then the following statements are equivalent:

(i) X is a CAT(0) space;

(ii) X satisfies (CN);

(iii) X satisfies (CN*).

Recall that a geodesic space (X, d) is said to be R-convex for $R \in (0, 2]$ (see [18]), if for any $x, y, z \in X$, we have

$$d^2(x, (1-\alpha)y \oplus \alpha z) \leq (1-\alpha)d^2(x, y) + \alpha d^2(x, z) - \frac{R}{2}\alpha(1-\alpha)d^2(y, z). \tag{2.1}$$

It follows from (CN*) that a geodesic space (X, d) is a CAT(0) space if and only if (X, d) is R-convex for $R = 2$.

The following lemma is a consequence of Proposition 3.1 in [19].

Lemma 2.2. (cf. [19]) Let (X, d) be a CAT(κ) space for $\kappa > 0$ and $\operatorname{diam}(X) \leq \frac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Then (X, d) is R-convex for $R = (\pi - 2\varepsilon)\tan\varepsilon$.

Lemma 2.3. [16] Let (X, d) be a complete CAT(κ) space for $\kappa > 0$ and $\operatorname{diam}(X) \leq \frac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Then

$$d((1-\alpha)x \oplus \alpha y, z) \leq (1-\alpha)d(x, z) + \alpha d(y, z), \tag{2.2}$$

for all $x, y, z \in X$ and $\alpha \in [0, 1]$.

Lemma 2.4. Let (X, d) be a CAT(κ) space for $\kappa > 0$ and $\operatorname{diam}(X) \leq \frac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Then X is uniformly convex, i.e., for any given $r > 0, \sigma \in (0, 2]$ and $\alpha \in [0, 1]$, there exists $\eta(r, \sigma) = \frac{R\sigma^2}{8}$ such that for any $x, y, z \in X$,

$$\left.\begin{array}{l} d(x, z) \leq r \\ d(y, z) \leq r \\ d(x, y) \geq \sigma r \end{array}\right\} \Rightarrow d((1-\alpha)x \oplus \alpha y, z) \leq \left(1 - 2\alpha(1-\alpha)\frac{R\sigma^2}{8}\right)r, \tag{2.3}$$

where the function $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ is a modulus of uniform convexity of CAT(κ).

Proof. For $r > 0$, $\sigma \in (0, 2]$, $x, y, z \in X$, let $d(x, z) \leq r$, $d(y, z) \leq r$, $d(x, y) \geq \sigma r$. Then from (2.1), we have

$$\begin{aligned} d^2((1-\alpha)x \oplus \alpha y, z) &\leq (1-\alpha)d^2(x, z) + \alpha d^2(y, z) - \frac{R}{2}\alpha(1-\alpha)d^2(x, y) \\ &\leq (1-\alpha)r^2 + \alpha r^2 - \frac{R}{2}\alpha(1-\alpha)\sigma^2 r^2 \\ &\leq (1 - \frac{R}{4}\alpha(1-\alpha)\sigma^2)^2 r^2. \end{aligned}$$

Hence, we have

$$d((1-\alpha)x \oplus \alpha y, z) \leq \left(1 - 2\alpha(1-\alpha)\frac{R\sigma^2}{8}\right)r.$$

Therefore, X is uniformly convex, and $\eta(r, \sigma) = \frac{R\sigma^2}{8}$ is a modulus of uniform convexity for X. ■

By Lemma 2.4, we can obtain the following lemma (cf. [4], Lemma 3.2).

Lemma 2.5. (cf. [4]) Let (X, d) be a CAT(κ) space for $\kappa > 0$ and $\operatorname{diam}(X) \leq \frac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let $x \in X$ be a given point and $\{t_n\}$ be a sequence in $[b, c]$ with $b, c \in (0, 1)$ and $0 < b(1-c) \leq \frac{1}{2}$. Let $\{x_n\}$ and $\{y_n\}$ be any sequences in X such that

$$\limsup_{n\to\infty} d(x_n, x) \leq r, \quad \limsup_{n\to\infty} d(y_n, x) \leq r,$$

and

$$\lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r,$$

for some $r \geq 0$. Then

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{2.4}$$

We now give some elementary concepts about CAT(κ) ($\kappa > 0$) spaces. Most of them are proved in CAT(0) spaces. For completeness, we state the results in CAT(κ) for $\kappa > 0$.

Let $\{x_n\}$ be a bounded sequence in a CAT(κ) space (X, d). For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is defined by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}.$$

The *asymptotic radius* $r_C(\{x_n\})$ *of* $\{x_n\}$ *with respect to* $C \subset X$ is defined by

$$r_C(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

The *asymptotic center* $A_C(\{x_n\})$ *of* $\{x_n\}$ *with respect to* $C \subset X$ is the set

$$A_C(\{x_n\}) = \{x \in C : r(x, \{x_n\}) = r_C(\{x_n\})\}.$$

We know that the asymptotic center $A(\{x_n\})$ consists of exactly one point in a CAT(κ) space X with $\mathrm{diam}(X) \leq \frac{\pi}{2\sqrt{\kappa}}$, from Proposition 4.1 of [19].

Now, we give the concept of $\triangle$-convergence and collect some of its basic properties.

Definition 2.6. ([20],[21]) A sequence $\{x_n\}$ in X is said to be $\triangle$- *convergent* to $x \in X$, if x is the unique asymptotic center of $\{u_n\}$ for each subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\triangle$-$\lim_{n\to\infty} x_n = x$ and call x the $\triangle$-*limit* of $\{x_n\}$.

Lemma 2.7. [22] Let (X, d) be a complete CAT(κ) space for $\kappa > 0$ and $\mathrm{diam}(X) \leq \frac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Then the following statements hold:

(i) every sequence in X has a $\triangle$-convergent subsequence;

(ii) if $\{x_n\} \subset X$ and $\triangle - \lim_{n\to\infty} x_n = x$, then

$$x \in \bigcap_{m=1}^{\infty} \overline{conv}\{x_m, x_{m+1}, \dots\},$$

where $\overline{conv} A = \bigcap \{B : B \supseteq A\}$ for closed and convex set B.

By the uniqueness of asymptotic center, we can obtain the following lemma (cf. [18], Lemma 2.8).

Lemma 2.8. (cf. [18]) Let (X, d) be a complete CAT(κ) space for $\kappa > 0$ and $\mathrm{diam}(X) \leq \frac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. If $\{x_n\}$ is a sequence in X with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$, and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.

Lemma 2.9. Let (X, d) be a complete CAT(κ) space for $\kappa > 0$ and $\mathrm{diam}(X) \leq \frac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$, and let C be a closed convex subset of X. If $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.

Proof. Let x be an asymptotic center of $\{x_n\}$. It is known that the nearest point projection $P : X \to C$ exists and is nonexpensive [16]. If $x \notin C$, then $r(x, \{x_n\}) \leq r(Px, \{x_n\})$, and we have a contradiction. ■

Remark 2.10. Let (X, d) be a CAT(κ) space for $\kappa > 0$, let C be a closed convex subset of X and $\{x_n\}$ be a bounded sequence in C. In what follows we denote by

$$x_n \rightharpoonup w \iff \Phi(w) = \inf_{x \in C} \Phi(x), \tag{2.5}$$

where $\Phi(x) := \limsup_{n\to\infty} d(x_n, x)$.

Now we give a relation between the $\rightharpoonup$ convergence and $\triangle$-convergence.

Proposition 2.11. Let (X, d) be a complete CAT(κ) space for $\kappa > 0$ and $\operatorname{diam}(X) \leq \frac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a closed convex subset of X and $\{x_n\}$ be a bounded sequence in C. Then $\triangle - \lim_{n\to\infty} x_n = p$ implies that $x_n \rightharpoonup p$.

Proof. If $\triangle - \lim_{n\to\infty} x_n = p$, then it follows from Lemma 2.9 that $p \in C$. Since $A(\{x_n\}) = \{p\}$, we have $r(\{x_n\}) = r(p, \{x_n\})$. This implies that $\Phi(p) = \inf_{y \in C} \Phi(y)$, that is, $x_n \rightharpoonup p$. ■

Let (X, d) be a metric space and C be a nonempty subset of X. Recall that C is said to be a *retract* of X, if there exists a continuous map $P : X \to C$ such that $Px = x$, for all $x \in C$. A map $P : X \to C$ is said to be a *retraction*, if $P^2 = P$. If P is a retraction, then $Py = y$ for all y in the range of P.

Definition 2.12. [10] Let C be a nonempty subset of a CAT(κ) space (X, d). $T : C \to X$ is said to be a $(\{\mu_n\}, \{\nu_n\}, \zeta)$*-total asymptotically nonexpansive nonself-mapping*, if there exist nonnegative sequences $\{\mu_n\}, \{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\zeta : [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq d(x, y) + \nu_n \zeta(d(x, y)) + \mu_n, \tag{2.6}$$

for all $n \geq 1$ and $x, y \in C$, where P is a nonexpansive retraction of X onto C.

Definition 2.13. [10] Let C be a nonempty subset of a CAT(κ) space (X, d). $T : C \to X$ is said to be a *uniformly L-Lipschitzian* nonself-mapping, if there exists a constant $L > 0$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq Ld(x, y), \tag{2.7}$$

for all $n \geq 1$ and $x, y \in C$.

A point $x \in C$ is called a *fixed point* of T if $x = Tx$. We denote by $F(T)$ the set of fixed points of T.

The following lemma is also needed for the proof of main theorems.

Lemma 2.14. [23] Let $\{s_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers satisfying the inequality:

$$s_{n+1} \leq s_n + t_n, \quad \forall n \in N.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, then the $\lim_{n\to\infty} s_n$ exists.

3. Demiclosed principle

First, we introduce and prove the demiclosed principle for the $\rightharpoonup$ convergence (cf. [24]).

Theorem 3.1. Let (X, d) be a complete CAT(κ) space for $\kappa > 0$ and diam$(X) \leq \frac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X and $T : C \to X$ be a uniformly L-Lipschitzian and $(\{\mu_n\}, \{\nu_n\}, \zeta)$-total asymptotically nonexpansive nonself-mapping. Let $\{x_n\}$ be a bounded sequence in C such that $x_n \rightharpoonup q$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $Tq = q$.

Proof. Since $x_n \rightharpoonup q$, $A_C(\{x_n\}) = \{q\}$. By Lemma 2.9, we have $A(\{x_n\}) = \{q\}$. Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, by induction we can prove that

$$\lim_{n\to\infty} d(x_n, T(PT)^m x_n) = 0, \quad \text{for each } \ m \geq 0. \tag{3.1}$$

In fact, it is obvious that the conclusion is true for $m = 0$. Suppose that the conclusion holds for $m \geq 1$, now we have to prove that (3.1) is also true for $m + 1$. Indeed, since $x_n \in C$, we have $x_n = Px_n$. In addition, since T is uniformly L-Lipschitzian, we have

$$\begin{aligned} d(x_n, T(PT)^m x_n) &\leq d(x_n, T(PT)^{m-1} x_n) + d(T(PT)^{m-1} x_n, T(PT)^m x_n) \\ &\leq d(x_n, T(PT)^{m-1} x_n) + Ld(x_n, PTx_n) \\ &= d(x_n, T(PT)^{m-1} x_n) + Ld(Px_n, PTx_n) \\ &\leq d(x_n, T(PT)^{m-1} x_n) + Ld(x_n, Tx_n) \\ &\to 0 \ \text{ as } \ n \to \infty, \end{aligned}$$

so, (3.1) is proved. Hence for each $x \in C$ and $m \geq 1$, from (3.1), we have

$$\Phi(x) := \limsup_{n\to\infty} d(x_n, x) = \limsup_{n\to\infty} d(T(PT)^{m-1} x_n, x). \tag{3.2}$$

Letting $x = T(PT)^{m-1} q$, $m \geq 1$ in (3.2), then we have

$$\begin{aligned} \Phi(T(PT)^{m-1} q) &= \limsup_{n\to\infty} d(T(PT)^{m-1} x_n, T(PT)^{m-1} q) \\ &\leq \limsup_{n\to\infty} \left\{ d(x_n, q) + \nu_m \zeta(d(x_n, q)) + \mu_m \right\}. \end{aligned}$$

Taking the limsup on the both sides of the above inequality, we get

$$\limsup_{m\to\infty} \Phi(T(PT)^{m-1}q) \leq \Phi(q). \tag{3.3}$$

Furthermore, for any $n, m \geq 1$, it follows from inequality (2.1) with $\alpha = \frac{1}{2}$ that

$$\begin{aligned} d^2\left(x_n, \frac{1}{2}q \oplus \frac{1}{2}T(PT)^{m-1}q\right) &\leq \frac{1}{2}d^2(x_n, q) + \frac{1}{2}d^2(x_n, T(PT)^{m-1}q) \\ &\quad - \frac{R}{8}d^2(q, T(PT)^{m-1}q). \end{aligned}$$

where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. Taking the limsup on the both sides of the above inequality, then for any $m \geq 1$, we get

$$\begin{aligned} &\Phi^2\left(\frac{1}{2}q \oplus \frac{1}{2}T(PT)^{m-1}q\right) \\ &\leq \frac{1}{2}\Phi^2(q) + \frac{1}{2}\Phi^2(T(PT)^{m-1}q) - \frac{R}{8}d^2(q, T(PT)^{m-1}q). \end{aligned}$$

Since $A(\{x_n\}) = \{q\}$, for any $m \geq 1$, we have

$$\begin{aligned} \Phi^2(q) &\leq \Phi^2\left(\frac{1}{2}q \oplus \frac{1}{2}T(PT)^{m-1}q\right) \\ &\leq \frac{1}{2}\Phi^2(q) + \frac{1}{2}\Phi^2(T(PT)^{m-1}q) - \frac{R}{8}d^2(q, T(PT)^{m-1}q), \end{aligned}$$

this implies that

$$d^2(q, T(PT)^{m-1}q) \leq \frac{4}{R}\Big[\Phi^2(T(PT)^{m-1}q) - \Phi^2(q)\Big]. \tag{3.4}$$

By (3.3) and (3.4), we have $\lim_{n\to\infty} d(q, T(PT)^{m-1}q) = 0$. Since T is a uniformly L-Lipschitzian, we have

$$\begin{aligned} d(Tq, q) &\leq d(Tq, T(PT)^m q) + d(T(PT)^m q, q) \\ &\leq Ld(q, (PT)^m q) + d(T(PT)^m q, q) \\ &= Ld(Pq, PT(PT)^{m-1}q) + d(T(PT)^m q, q) \\ &\leq Ld(q, T(PT)^{m-1}q) + d(T(PT)^m q, q) \\ &\to 0, \quad \text{as } m \to \infty. \end{aligned}$$

Hence, we have $Tq = q$. ■

Next, the following demiclosed principle can be obtained from Theorem 3.1 immediately for the $\triangle$-convergence.

Theorem 3.2. Let (X, d) be a complete CAT(κ) space for $\kappa > 0$ and $\mathrm{diam}(X) \le \frac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X and T be a mapping satisfying one of the following conditions:

(1) $T : C \to X$ is an asymptotically nonexpansive nonself-mapping.

(2) $T : C \to X$ is a uniformly Lipschitzian and total asymptotically nonexpansive nonself-mapping.

If $\{x_n\}$ is a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\triangle$-$\lim_{n\to\infty} x_n = q$, then $Tq = q$.

4. Convergence theorems

Now we will prove the $\triangle$-convergence theorem for the iterative sequence in a CAT(κ) space.

Theorem 4.1. Let (X, d) be a complete CAT(κ) space for $\kappa > 0$ and $\mathrm{diam}(X) \le \frac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let C be a nonempty closed convex subset of X and P be a nonexpansive retraction of X onto C and $T_i : C \to X$ be a uniformly L_i-Lipschitzian and $(\{\mu_n^{(i)}\}, \{\nu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive nonself-mapping for each $i = 1, 2$. For arbitrarily chosen $x \in C$, $\{x_n\}$ is defined by

$$\begin{cases} y_n = P((1-\beta_n)x_n \oplus \beta_n T_1(PT_1)^{n-1}x_n), \\ x_{n+1} = P((1-\alpha_n)x_n \oplus \alpha_n T_2(PT_2)^{n-1}y_n), \quad n \ge 1, \end{cases} \tag{4.1}$$

where $\{\nu_n^{(i)}\}$, $\{\mu_n^{(i)}\}$, $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

(1) $\sum_{n=1}^{\infty} \nu_n^{(i)} < \infty, \ \sum_{n=1}^{\infty} \mu_n^{(i)} < \infty, \ i = 1, 2,$

(2) there exist constants $a, b \in (0, 1)$ with $0 < b(1-a) \le \frac{1}{2}$ such that $\{\alpha_n\} \subset [a, b]$ and $\{\beta_n\} \subset [a, b]$.

If $S := F(T_1) \cap F(T_2) \ne \emptyset$, then the sequence $\{x_n\}$ defined by (4.1) is $\triangle$-convergent to a common fixed point of T_1 and T_2.

Proof. We would like to prove the theorem by three steps.
Step 1: We first prove that the following limit exists for each $q \in F(T_1) \cap F(T_2)$.

$$\lim_{n\to\infty} d(x_n, q). \tag{4.2}$$

Let $M = diam(C)$. Set $\nu_n = \max\{\nu_n^{(1)}, \nu_n^{(2)}\}$, $\mu_n = \max\{\mu_n^{(1)}, \mu_n^{(2)}\}$, $n = 1, 2, \ldots$, and $\zeta = \max\{\zeta^{(1)}, \zeta^{(2)}\}$. Since $\sum_{n=1}^{\infty} \nu_n^{(i)} < \infty$, $\sum_{n=1}^{\infty} \mu_n^{(i)} < \infty$, $i = 1, 2$, we know that $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$. Since T_i $(i = 1, 2)$ is a total asymptotically nonexpansive nonself-mapping, by Lemma 2.3, for each $q \in F(T_1) \cap F(T_2)$, we have

$$\begin{aligned} d(y_n, q) &\le d(P((1-\beta_n)x_n \oplus \beta_n T_1(PT_1)^{n-1}x_n), q) \\ &\le d((1-\beta_n)x_n \oplus \beta_n T_1(PT_1)^{n-1}x_n, q) \\ &\le (1-\beta_n)d(x_n, q) + \beta_n d(T_1(PT_1)^{n-1}x_n, q) \\ &\le (1-\beta_n)d(x_n, q) + \beta_n d(T_1(PT_1)^{n-1}x_n, T_1(PT_1)^{n-1}q) \\ &\le (1-\beta_n)d(x_n, q) + \beta_n[d(x_n, q) + \nu_n\zeta(d(x_n, q)) + \mu_n] \\ &\le d(x_n, q) + \beta_n\nu_n\zeta(M) + \beta_n\mu_n. \end{aligned} \tag{4.3}$$

This implies that

$$\begin{aligned} d(x_{n+1}, q) &\le d(P((1-\alpha_n)x_n \oplus \alpha_n T_2(PT_2)^{n-1}y_n), q) \\ &\le d((1-\alpha_n)x_n \oplus \alpha_n T_2(PT_2)^{n-1}y_n, q) \\ &\le (1-\alpha_n)d(x_n, q) + \alpha_n d(T_2(PT_2)^{n-1}y_n, q) \\ &= (1-\alpha_n)d(x_n, q) + \alpha_n d(T_2(PT_2)^{n-1}y_n, T_2(PT_2)^{n-1}q) \\ &\le (1-\alpha_n)d(x_n, q) + \alpha_n[d(y_n, q) + \nu_n\zeta(d(y_n, q)) + \mu_n] \\ &\le d(x_n, q) + \alpha_n(1+\beta_n)[\nu_n\zeta(M) + \mu_n]. \end{aligned} \tag{4.4}$$

Since $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, by Lemma 2.14, $\lim_{n\to\infty} d(x_n, q)$ exists for each $q \in F(T_1) \cap F(T_2)$.

Step 2: We show that

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \quad i = 1, 2. \tag{4.5}$$

Let $q \in F(T_1) \cap F(T_2)$. Then from the proof of Step 1, we obtain that $\lim_{n\to\infty} d(x_n, q)$ exists. We may assume that $\lim_{n\to\infty} d(x_n, q) = l$. From (4.3), we have

$$d(y_n, q) \le d(x_n, q) + \beta_n\nu_n\zeta(M) + \beta_n\mu_n. \tag{4.6}$$

Taking the limsup on both sides in (4.6), we have

$$\limsup_{n\to\infty} d(y_n, q) \le l. \tag{4.7}$$

In addition, since

$$\begin{aligned} d(T_2(PT_2)^{n-1}y_n, q) &= d(T_2(PT_2)^{n-1}y_n, T_2(PT_2)^{n-1}q) \\ &\leq d(y_n, q) + \nu_n\zeta(d(y_n, q)) + \mu_n \\ &\leq d(y_n, q) + \nu_n\zeta(M) + \mu_n, \end{aligned}$$

we have

$$\limsup_{n\to\infty} d(T_2(PT_2)^{n-1}y_n, q) \leq l.$$

From (4.4), we have

$$\lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n T_2(PT_2)^{n-1}y_n, q) \leq l.$$

Since $\lim_{n\to\infty} d(x_{n+1}, q) = l$, it is easy to prove that

$$\lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n T_2(PT_2)^{n-1}y_n, q) = l.$$

It follows from Lemma 2.5 that

$$\lim_{n\to\infty} d(x_n, T_2(PT_2)^{n-1}y_n) = 0. \tag{4.8}$$

On the other hand, since

$$\begin{aligned} d(x_n, q) &\leq d(x_n, T_2(PT_2)^{n-1}y_n) + d(T_2(PT_2)^{n-1}y_n, q) \\ &\leq d(x_n, T_2(PT_2)^{n-1}y_n) + d(T_2(PT_2)^{n-1}y_n, T_2(PT_2)^{n-1}q) \\ &\leq d(x_n, T_2(PT_2)^{n-1}y_n) + d(y_n, q) + \nu_n\zeta(M) + \mu_n, \end{aligned}$$

we have $\lim_{n\to\infty} d(y_n, q) \geq l$. Combining this with (4.7), it yields that

$$\lim_{n\to\infty} d(y_n, q) = l. \tag{4.9}$$

Hence we have

$$\lim_{n\to\infty} d((1-\beta_n)x_n \oplus \beta_n T_1(PT_1)^{n-1}x_n, q) = l.$$

It is easy to show that

$$\limsup_{n\to\infty} d(T_1(PT_1)^{n-1}x_n, q) \leq l. \tag{4.10}$$

So, it follows from (4.10) and Lemma 2.5 that

$$\lim_{n\to\infty} d(x_n, T_1(PT_1)^{n-1}x_n) = 0. \tag{4.11}$$

Observe that

$$\begin{aligned} d(x_n, T_2(PT_2)^{n-1}x_n) &\le d(x_n, T_2(PT_2)^{n-1}y_n) \\ &\quad + d(T_2(PT_2)^{n-1}y_n, T_2(PT_2)^{n-1}x_n) \\ &\le d(x_n, T_2(PT_2)^{n-1}y_n) + d(x_n, y_n) \\ &\quad + \nu_n\zeta(d(x_n, y_n)) + \mu_n \\ &\le d(x_n, T_2(PT_2)^{n-1}y_n) \\ &\quad + d(x_n, y_n) + \nu_n\zeta(M) + \mu_n, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} d(x_n, y_n) &= d(x_n, P((1-\beta_n)x_n \oplus \beta_n T_1(PT_1)^{n-1}x_n)) \\ &\le \beta_n d(x_n, T_1(PT_1)^{n-1}x_n). \end{aligned}$$

It follows from (4.11) that

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{4.13}$$

Thus, from (4.8), (4.12) and (4.13), we have

$$\lim_{n\to\infty} d(x_n, T_2(PT_2)^{n-1}x_n) = 0. \tag{4.14}$$

In addition, since

$$\begin{aligned} d(x_{n+1}, x_n) &= d(P((1-\alpha_n)x_n \oplus \alpha_n T_2(PT_2)^{n-1}y_n), x_n) \\ &\le d((1-\alpha_n)x_n \oplus \alpha_n T_2(PT_2)^{n-1}y_n, x_n) \\ &\le \alpha_n d(T_2(PT_2)^{n-1}y_n, x_n), \end{aligned}$$

from (4.8), we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{4.15}$$

Finally, since

$$\begin{aligned} d(x_n, T_1x_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T_1(PT_1)^n x_{n+1}) \\ &\quad + d(T_1(PT_1)^n x_{n+1}, T_1(PT_1)^n x_n) + d(T_1(PT_1)^n x_n, T_1x_n) \\ &\le (1+L)d(x_n, x_{n+1}) + d(x_{n+1}, T_1(PT_1)^n x_{n+1}) \\ &\quad + Ld(T_1(PT_1)^{n-1}x_n, x_n), \end{aligned}$$

it follows from (4.11) and (4.15) that $\lim_{n\to\infty} d(x_n, T_1x_n) = 0$.

Similarly, since

$$\begin{aligned} d(x_n, T_2x_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T_2(PT_2)^n x_{n+1}) \\ &\quad + d(T_2(PT_2)^n x_{n+1}, T_2(PT_2)^n x_n) + d(T_2(PT_2)^n x_n, T_2x_n) \\ &\le (1+L)d(x_n, x_{n+1}) + d(x_{n+1}, T_2(PT_2)^n x_{n+1}) \\ &\quad + Ld(T_2(PT_2)^{n-1}x_n, x_n), \end{aligned}$$

it follows from (4.14) and (4.15) that $\lim_{n\to\infty} d(x_n, T_2x_n) = 0$.

Step 3: We show that $\{x_n\}$ is $\triangle$-convergent to a common fixed point of T_1 and T_2. Let $W_\omega(x_n) := \bigcup_{\{u_n\}\subset\{x_n\}} A(\{u_n\})$. We first prove that $W_\omega(x_n) \subset F(T_1) \cap F(T_2)$. In fact, let $u \in W_\omega(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.7 and Lemma 2.9, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $\triangle$-$\lim_{i\to\infty} u_{n_i} = v \in C$. In view of (4.5),

$$\lim_{n\to\infty} d(x_n, T_1x_n) = \lim_{n\to\infty} d(x_n, T_2x_n) = 0.$$

It follows from Theorem 3.2 that $v \in F(T_1)\cap F(T_2)$, so, by (4.2), the $\lim_{n\to\infty} d(x_n, v)$ exists. By Lemma 2.8, $u = v \in F(T_1) \cap F(T_2)$. This implies that $W_\omega(x_n) \subset F(T_1) \cap F(T_2)$.

Next, let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in W_\omega(x_n) \subset F(T_1) \cap F(T_2)$, from (4.2) the $\lim_{n\to\infty} d(x_n, u)$ exists. In view of Lemma 2.8, $x = u$. This implies that $W_\omega(x_n)$ consists of exactly one point. This means that $\{x_n\}$ is $\triangle$-convergent to a common fixed point of T_1 and T_2. The proof is completed. ■

As a consequence of Theorem 4.1, we obtain the following corollary.

Corollary 4.2. Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d). Let $T_i : C \to X$, $i = 1, 2$ be a uniformly L_i-Lipschitzian and $(\{\mu_n^{(i)}\}, \{\nu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive nonself-mapping. If $F(T_1) \cap F(T_2) \neq \emptyset$ and the following conditions are satisfied:

(1) $\sum_{n=1}^{\infty} \nu_n^{(i)} < \infty$, $\sum_{n=1}^{\infty} \mu_n^{(i)} < \infty$, $i = 1, 2$,

(2) there exist constants $a, b \in (0, 1)$ with $0 < b(1 - a) \leq \frac{1}{2}$ such that $\{\alpha_n\} \subset [a, b]$ and $\{\beta_n\} \subset [a, b]$,

then the sequence $\{x_n\}$ defined by (4.1) is $\triangle$-convergent to a common fixed point of T_1 and T_2.

We can get the following corollary for the self-mapping.

Corollary 4.3. Let C and X be the same as in Theorem 4.1 and let $T_i : C \to C$, $i = 1, 2$ be a uniformly L_i-Lipschitzian and $(\{\mu_n^{(i)}\}, \{\nu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive mapping. If $F(T_1) \cap F(T_2) \neq \emptyset$ and the conditions (1) and (2) in Theorem 4.1 are satisfied, then the sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in C, \\ y_n = (1 - \beta_n)x_n \oplus \beta_n T_1^n x_n, & n \geq 1, \\ x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n T_2^n y_n, \end{cases} \tag{4.16}$$

is $\triangle$-convergent to a common fixed point of T_1 and T_2.

Proof. Since T_i, $i = 1, 2$ is a self-mapping from C into C, take $P = I$ (the identity mapping on C), then $T_i(PT_i)^{n-1} = T_i^n$. The conclusion of Corollary 4.3 is obtained from Theorem 4.1. ■

In order to prove the strong convergence of the sequence defined by (4.1), we need some more condition on the mappings T_1 and T_2. In this stage, we will give the demi-compactness condition.

Recall that a mapping $T : C \to X$ is said to be *demi-compact* if for any sequence $\{x_n\}$ in C such that $d(x_n, Tx_n) \to 0$, as $n \to \infty$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $x^* \in C$.

Theorem 4.4. Under the assumptions of Theorem 4.1, if one of T_1 and T_2 is demi-compact, then the sequence defined by (4.1) converges strongly to a common fixed point of T_1 and T_2.

Proof. By virtue of (4.5), $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$, $i = 1, 2$, and one of T_1 and T_2 is demi-compact, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some point $p \in C$. Moreover, by the uniform continuity of T_1 and T_2, for each $i = 1, 2$, we have

$$d(T_i(p), p) \leq d(T_i(p), T_i(x_{n_j})) + d(T_i(x_{n_j}), x_{n_j}) + d(x_{n_j}, p) \to 0, \quad \text{as} \quad j \to \infty.$$

This implies that $p \in F(T_1) \cap F(T_2)$. Again by (4.2), the $\lim_{n\to\infty} d(x_n, p)$ exists, thus p is the strong limit of the sequence $\{x_n\}$. ■

Corollary 4.5. Under the assumptions of Corollary 4.2, if one of T_1 and T_2 is demi-compact, then the sequence defined by (4.1) converges strongly to a common fixed point of T_1 and T_2.

Corollary 4.6. Under the assumptions of Corollary 4.3, if one of T_1 and T_2 is demi-compact, then the sequence defined by (4.16) converges strongly to a common fixed point of T_1 and T_2.

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