Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 6 (2016), pp. 4991–4997
© Research India Publications
http://www.ripublication.com/gjpam.htm

Wagner Connection on Averaged Riemannian Metrics

Jenizon, Haripamyu and I Made Arnawa
Department of Mathematics,
Faculty of Mathematics and Natural Sciences,
Universitas Andalas, Kampus Unand Limau Manis,
Padang, Indonesia 25163.

Abstract

In this paper we shall investigate conformal theory in Finsler geometry. The main topic in this paper is the averaged connection of the Wagner connection which is called Lyra connection. Therefore we also discuss deformed connection of Lyra connection which is called Weyl connection. The main tool in this paper is the Wagner connection which is a natural extension of Finsler-Weyl connection in Riemannian.

AMS subject classification: 53B40.
Keywords: Riemannian metrics, Finsler-Weyl connection, Lyra connection.

1. Preliminaries

Let M be an n-dimensional smooth connected manifold with a Riemannian metric h. A linear connection ∇ is said to be *conformal* if the parallel transport with respect to ∇ preserves angles but not the metric h. Thus ∇ is conformal if and only if there exists a one-form $w(h)$ such that

$$\nabla h = 2w(h) \otimes h. \tag{1.1}$$

We denote $c = \{e^{\sigma} h | \sigma \in C^{\infty}(M)\}$ is the conformal class of h. A *Weyl structure* on (M, c) is a map $w : c \to \Lambda^1(M)$ satisfying (1.1). The triplet (M, c, w) is called a *Weyl manifold*.

Let $\pi : TM \longrightarrow M$ be the tangent bundle over M . We denote by $V := \ker\{d\pi : TTM \to TM\}$ the *vertical sub-bundle* over TM. Since V is also isomorphic to π^*TM, we can lift any vector field X in M to a vector field X^V which tangent to the fiber T_xM at each point $(x, y) \in T_xM$. We call X^V the *vertical lift* of X.

We used the coordinate system $\{\pi^{-1}(U), (x^1, y^i)_{1\leq i\leq n}\}$ in TM induced from a coordinate system $\{U, x^i\}_{1\leq i\leq n}$ in M, where $(y^i)_{1\leq i\leq n}$ are the fibre coordinates in each T_xM. The action of $\mathbb{R}^+$ induces the so called *Liouville vector field*

$$\mathcal{E} = \sum y^i \frac{\partial}{\partial y^i}. \tag{1.2}$$

Since any nonlinear connection H is also isomorphic to π^*TM, there exists a unique vector field X^H in TM such that $d\pi(X^H) = X$ for any vector field X in M. Such a vector field X^H is called the *horizontal lift* of X with respect to H.

We shall begin to define a Finsler metric on a smooth manifold.

Definition 1.1. Let M be a smooth manifold of dim $M = n$, and $\pi : TM \to M$ its tangent bundle. A function $L : TM \to \mathbb{R}$ is called a *Finsler metric* if it satisfies

(F-1) L is continuous on the total space TM, and is smooth on the slit tangent bundle $TM\backslash\{0_M\}$.

(F-2) $L(x, y) \geq 0$ for every $(x, y) \in TM$, and the equality holds if and only if $y = 0$.

(F-3) $L(x, \lambda y) = \lambda L(x, y)$ for every $(x, y) \in TM$ and $\lambda \in \mathbb{R}^+$.

(F-4) L^2 is strongly convex along the fibers of TM.

The pair (M, L) is called a *Finsler manifold.*

By the assumption (F-4), the matrix (g_{ij}) defined by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j} \tag{1.3}$$

is positive-definite. Then g defines an inner product in the tangent space $T_y(T_xM)$ at $y \in T_xM$. Therefore $g = (g_{ij})$ causes a Riemannian metric $g_x = (g_{ij})$ in T_xM for every $x \in M$.

On the other hand, since the tangent space $T_y(T_xM)$ is naturally identified with the fiber $V_{(x,y)}$ of V over (x, y), the matrix (g_{ij}) defines a metric g in the vertical sub-bundle V by

$$g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g_{ij}. \tag{1.4}$$

Then, from the homogeneity condition (F-3) on L, we have

$$L^2 = g(\mathcal{E}, \mathcal{E}). \tag{1.5}$$

Since the vertical sub-bundle V is relatively flat, we can define a covariant derivative D on V such that it is flat in vertical direction. In this paper, the pair (H, D) of such a covariant derivative D and a nonlinear connection H is called a *Finsler connection* of M.

2. Finsler-Weyl connections and Wagner connections

In this section, we shall extend the notion of Weyl structures to the category of Finsler geometry. Suppose that the Berwald connection (H, D^B) of Finsler manifold (M, L) is conformal, namely, we suppose that (H, D^B) satisfies

$$D^B_{X^H} g = 2\alpha(X) g,$$

in any Finsler manifold (M, L), there exists a conformal Finsler connection $(\mathcal{H}, \mathcal{D})$.

Proposition 2.1. Let (M, L) be a Finsler manifold. For any $\alpha \in \Lambda^1(M)$, there exists a unique nonlinear connection $\mathcal{H}$ and a Finsler connection $(\mathcal{H}, \mathcal{D})$ satisfying the following conditions for any vector fields X, Y in M :

(1) $(\mathcal{H}, \mathcal{D})$ is conformal, i.e.,

$$\mathcal{D}_{X^{\mathcal{H}}} g = 2\alpha(X) g. \tag{2.6}$$

(2) $(\mathcal{H}, \mathcal{D})$ is symmetric, i.e.,

$$\mathcal{D}_{X^{\mathcal{H}}} Y^V - \mathcal{D}_{Y^{\mathcal{H}}} X^V - [X, Y]^V = 0. \tag{2.7}$$

(3) The deflection $(\mathcal{H}, \mathcal{D})$ vanishes, i.e.,

$$\mathcal{D}_{X^{\mathcal{H}}} \mathcal{E} = 0. \tag{2.8}$$

In the case of $\alpha = 0$, the Finsler connection $(\mathcal{H}, \mathcal{D})$ in Proposition 2.1 is just the Rund connection (H, D^R) of (M, L). If α is closed, the assumption (2.6) is written as $\mathcal{D}_{X^{\mathcal{H}}} \left(e^{2\sigma_U} g\right) = 0$ for some $\sigma_U \in C^\infty(U)$. Hence $(\mathcal{H}, \mathcal{D})$ is the Rund connection of a local Finsler metric $e^{\sigma_U} L$.

We denote by $\mathcal{C}$ the conformal equivalence class of Finsler metric on M. We call the triplet $(M, \mathcal{C}, \alpha)$ a *Finsler-Weyl manifold.*

Definition 2.2. **[8]** The connection $(\mathcal{H}, \mathcal{D})$ is called the *Finsler-Weyl connection* of $(M, \mathcal{C}, \alpha)$.

Since g_{ij} are homogeneous of degree zero with respect to the variables $y^1, \cdots, y^n$, (2.6) implies

$$\frac{\partial g_{ij}}{\partial x^k} - \sum \overline{\mathcal{N}}^l_k \frac{\partial g_{ij}}{\partial y^l} - \sum g_{lj} \overline{\mathcal{K}}^l_{ik} - \sum g_{il} \overline{\mathcal{K}}^l_{jk} = 0, \tag{2.9}$$

where we put $\overline{\mathcal{N}}^l_k = \mathcal{N}^l_k + \alpha_k y^l$ and $\overline{\mathcal{K}}^i_{jk} = \mathcal{K}^i_{jk} + \alpha_k \delta^i_j$. These functions defines a nonlinear connection $\overline{\mathcal{H}}$ and a Finsler connection $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$ which is *semi-symmetric*, i.e.,

$$\overline{\mathcal{D}}_{X^{\overline{\mathcal{H}}}} Y^V - \overline{\mathcal{D}}_{Y^{\overline{\mathcal{H}}}} X^V - [X, Y]^V = \alpha_L(X) Y^V - \alpha_L(Y) X^V. \tag{2.10}$$

Further, $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$ satisfies

$$\overline{\mathcal{D}}_{X^{\overline{\mathcal{H}}}}\, g = 0 \tag{2.11}$$

$$\overline{\mathcal{D}}_{X^{\overline{\mathcal{H}}}}\, \mathcal{E} = 0. \tag{2.12}$$

Definition 2.3. ([7]) The Finsler connection $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$ is called the *Wagner connection* of $(M, \mathcal{C}, \alpha)$.

Therefore the Finsler-Weyl connection $(\mathcal{H}, \mathcal{D})$ of $(M, \mathcal{C}, \alpha)$ determines the Wagner connection $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$, and vise-versa.

3. Averaged Riemannian metrics and connections

Let I_x be the indicatrix at $x \in M$, namely, I_x is a compact hypersurface in the tangent space T_xM defined by $I_x = \{y \in T_xM | L(x, y) = 1\}$. We defines a volume form of each indicatrix I_x,

$$d\mu_I = \iota(\mathcal{E})d\mu = \sum(-1)^{j-1} y^j \sqrt{\det g}\, dy^1 \wedge \cdots \wedge \check{dy}^j \wedge \cdots \wedge dy^n,$$

and the volume $v_L(x)$ of I_x is defined by

$$v_L(x) := \int_{I_x} d\mu_I$$

Definition 3.1. ([11]) Let (M, L) be a Finsler manifold. The averaged Riemannian metric of L is a Riemannian metric h in M defined by

$$h(X, Y) = \frac{1}{v_L(x)} \int_{I_x} g(X^V, Y^V) d\mu_I \tag{3.13}$$

for any vector fields X, Y in M.

Let $\widetilde{L} = e^{\sigma} L$ be a conformal deformation of a Finsler metric L. The indicatrix $\widetilde{I}_x$ at $x \in M$ with respect to $\widetilde{L}$ is given by $\widetilde{I}_x = e^{-\sigma} I_x$, and the volume form $d\mu_{\widetilde{I}}$ on $\widetilde{I}_x$ is given by

$$d\mu_{\widetilde{I}} = \sum(-1)^{i-1} \sqrt{\det \widetilde{g}}\, w^i dw^1 \wedge \cdots \wedge \check{dw}^i \wedge \cdots \wedge dw^n$$

at $w = (w^1, \cdots, w^n) \in \widetilde{I}_x$, where $\widetilde{g} = e^{2\sigma} g$ is the metric in V defined by $\widetilde{L}$. For the diffeomorphism $\psi : I_x \ni (x, y) \longmapsto \psi(x, y) = (x, e^{-\sigma} y) \in \widetilde{I}_x$, we obtain

$$\psi^*(d\mu_{\widetilde{I}}) = d\mu_I,$$

which implies

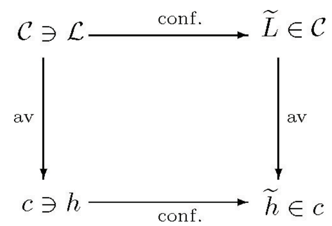
$$v_{\widetilde{L}}(x) = \int_{\widetilde{I}_x} d\mu_{\widetilde{I}} = \int_{I_x} \psi^*(d\mu_{\widetilde{I}}) = \int_{I_x} d\mu_I = v_L(x).$$

Thus, in the sequel, we use the notation $v(x)$ instead of $v_L(x)$ for the volume of the indicatrix I_x of any $L \in \mathcal{C}$. The averaged Riemannian metric $\widetilde{h}$ of $\widetilde{L}$ is given by

$$\widetilde{h} = e^{2\sigma} h \tag{3.14}$$

of the averaged Riemannian metric h of L.

Theorem 3.2. Let $\mathcal{C}$ be a conformal class of Finsler metrics on M, and let g be the metric in V determined by any $L \in \mathcal{C}$. Then, by averaging each metric g by (3.13), the class $\mathcal{C}$ determines a conformal class c of Riemannian metrics on M.



Let $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$ be the Wagner connection of a Finsler-Weyl manifold $(M, \mathcal{C}, \alpha)$. Then

Definition 3.3. ([12]) The **averaged connection** of $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$ is a linear connection $\overline{\nabla}$ on TM defined by

$$h(\overline{\nabla}_X Y, Z) = \frac{1}{v(x)} \int_{I_x} g(\overline{\mathcal{D}}_{X^{\overline{\mathcal{H}}}} Y^V, Z^V) d\mu_I \tag{3.15}$$

for any vector fields X, Y and Z in M, where h is the averaged Riemannian metric of L.

The properties (2.11) and (2.12) of $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$ lead to

$$\mathcal{L}_{X^{\overline{\mathcal{H}}}} L = 0. \tag{3.16}$$

Hence the parallel displacement with respect to Wagner's non-linear connection $\overline{\mathcal{H}}$ preserves every indicatrix, i.e.,

$$I_{\varphi_t(x)} = \varphi_t^{\overline{\mathcal{H}}}(I_x),$$

where φ_t and $\varphi_t^{\overline{\mathcal{H}}}$ denote the flows generated by X and its horizontal lift $X^{\overline{\mathcal{H}}}$ respectively. Therefore we have

$$X\left(\int_{I_x} f d\mu_I\right) = \int_{I_x} \left\{X^{\overline{\mathcal{H}}}(f) d\mu_I + f \mathcal{L}_{X^{\overline{\mathcal{H}}}} d\mu_I\right\} \tag{3.17}$$

for any $f \in C^{\infty}(M)$.

Now we suppose that Wagner's non-linear connection $\overline{\mathcal{H}}$ preserves the density $d\mu$:

$$\mathcal{L}_{X^{\overline{\mathcal{H}}}} d\mu = 0. \tag{3.18}$$

This assumption and $\mathcal{L}_{X^{\overline{\mathcal{H}}}}\mathcal{E} = 0$ lead to $\mathcal{L}_{X^{\overline{\mathcal{H}}}} d\mu_I = 0$, and thus (3.17) implies that the volume function $v(x)$ is constant. If we normalize $\mathcal{C}$ so that $v(x) = 1$, then (2.11) and (3.18) lead to

$$(\overline{\nabla}_X h)(Y, Z) = Xh(Y, Z) - h(\overline{\nabla}_X Y, Z) - h(Y, \overline{\nabla}_X Z) = 0$$

which shows that $\overline{\nabla}$ is compatible with h. Further, from (2.10),

$$\begin{aligned} h(\overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y], Z) &= \int_{I_x} g(\alpha_L(X)Y^V - \alpha_L(Y)X^V - [X, Y]^V, Z^V) d\mu_I \\ &= h(\alpha_L(X)Y - \alpha_L(Y)X, Z), \end{aligned}$$

leads to

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y] = \alpha_L(X)Y - \alpha_L(Y)X. \tag{3.19}$$

Hence $\overline{\nabla}$ is semi-symmetric, that is, $\overline{\nabla}$ is the so-called *Lyra connection* of (M, c) Further the connection ∇ defined by

$$\nabla_X Y = \overline{\nabla}_X Y - \alpha_L(X)Y \tag{3.20}$$

is symmetric, and ∇ satisfies

$$\nabla_X h = 2\alpha_L(X)h. \tag{3.21}$$

Consequently the Finsler-Weyl structure α of $(M, \mathcal{C})$ is a Weyl structure of (M, c), and ∇ is the Weyl connection of (M, c, α).

Theorem 3.4. Let $(M, \mathcal{C}, \alpha)$ be a Finsler-Weyl manifold, and let (M, c, α) be the Weyl manifold determined by $(M, \mathcal{C}, \alpha)$. Suppose that the Wagner's non-linear connection $\overline{\mathcal{H}}$ of $(M, \mathcal{C}, \alpha)$ preserves the density $d\mu$. Then the averaged connection $\overline{\nabla}$ of Wagner connection $(\overline{\mathcal{H}}, \overline{\mathcal{D}})$ is the Lyra connection of (M, c, α), and the deformed connection ∇ defined by (3.20) is the Weyl connection of (M, c, α).

Acknowledgements

The authors was supported by Grant of BOPTN Andalas University,
Grant No. 524/XIV/A/UNAND-2016.

References

[1] T. Aikou, Locally conformal Berwald spaces and Weyl structures, Publ. Math. Debrecen. **49**(1996), 113–120.

[2] T. Aikou, Some remarks on Berwald manifolds and Landsberg manifold, Acta Math. Acad. Paedagog. NYháZi. **26**(2010), 139–148.

[3] T. Aikou, Average Riemannian metrics and connections with application to locally conformal Berwald manifolds, Publ. Math. Debrecen. **80**(2012), 179–198.

[4] T. Aikou and L. Kozma, Global aspects of Finsler geometry, *in Handbook of Global Analysis* (edited by D. Krupka and D. Saunders), pp. 1–39 (2008), Elsevier.

[5] D. Bao, S.S. Chern and Z.Shen, *An Introduction to Riemann-Finsler Geometry*, GTM 200, Springer, 2000.

[6] G. B. Folland, Weyl manifolds, J. Differential Geometry. **4**(1970), 147–153.

[7] M. Hashiguchi and Y. Ichijyō, On conformal transformation in Wagner spaces, Rep. Fac. Sci. Kagoshima Univ. (Math., Phys. & Chem), No **10**(1977), 19–25.

[8] Jenizon and T. Aikou, Finsler-Weyl connections and Wagner connections of conformal Finsler manifolds, preprint, 2014.

[9] L. Kozma, On Finsler-Weyl manifolds and connections, Suppl. Rend. Palermo, Ser. II, No. **43**(1996), 173–179.

[10] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaiseisha-Press, Japan, 1986.

[11] V. S. Matveev, H.-B. Radmacher, M. Troyanov and A. Zeghib, Finsler conformal Lichnerowicz-Obata conjecture, Ann. Inst. Fourier, Grenoble **59**, 3(2009), 937–949.

[12] R. G. Torrome and F. Etayo, On a rigidity condition for Berwald spaces, RACSAM 104(1), 2010, 69–80.