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Global existence of solution for reaction diffusion system with full matrix via the compactness

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Abstract

The purpose of this paper is to prove the global existence in time of solutions for the strongly coupled reaction diffusion system:

$$\begin{cases} \frac{\partial u}{\partial t} - d_1\Delta u - d_2\Delta v = f(u, v) & in\ \mathbb{R}^+ \times \Omega \\ \frac{\partial u}{\partial t} - d_3\Delta u - d_4\Delta v = g(u, v) & in\ \mathbb{R}^+ \times \Omega \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 & in\ \mathbb{R}^+ \times \partial\Omega \\ u(.,0) = u_0(.),\ v(.,0) = v_0(.) & in\ \Omega \end{cases} \qquad \text{(SRD)}$$

with full matrix of diffusion coefficients .Our techniques of proof is based on compact semigroup methods and some L^1 estimates. we show that global solutions exist. Our investigation applied for a wide class of the nonlinear terms f and g.

AMS subject classification: 35K57, 35K40, 35K55.
Keywords: global solution, semi-group, local solution, reaction-diffusion system.

1. Introduction

In this paper we study the following semilinear parabolic system

$$\begin{cases} \frac{\partial u}{\partial t} - d_1\Delta u - d_2\Delta v = f(u, v) & in\mathbb{R}^+ \times \Omega \ \ 1.1 \\ \frac{\partial v}{\partial t} - d_3\Delta u - d_4\Delta v = g(u, v) & in\mathbb{R}^+ \times \Omega \ \ 1.2 \end{cases} \qquad (1)$$

Where Ω is a regular and bounded domain of R^n, $(n \geq 1)$, $u = u(t, x)$.

$v = v(t, x)$, $x \in \Omega$, $t > 0$ are real valued functions, η is the normal vector to $\partial\Omega$ outside, Δ denotes the Laplacian operator, d_1, d_2, d_3, d_4 are positive constants satisfying the condition $(d_2 + d_3)^2 \prec 4d_1d_4$ which reflects the parabolicity of the system and implies at the same time that the matrix of diffusion is positive definite. The eigenvalues λ_1 and λ_2 $(\lambda_1 \prec \lambda_2)$ of the matrix of diffusion are positive. If we assume that $d_1 \prec d_4$, then we have $\lambda_1 \prec d_1 \prec d_4 \prec \lambda_2$.

System (1) is subjected to the following boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \qquad in \mathbb{R}^+ \times \partial\Omega \tag{2}$$

and the initial data

$$u(.,0) = u_0(.), v(.,0) = v_0(.) \qquad in\Omega \tag{3}$$

which are assumed to be nonnegative.

The above system (1)–(3) arises in physics, chemistry and various biological processes including population dynamics. (See [6], [24] and references therein). Condition (2) means that there is no species of immegration.

It is considered the problem (1)–(3) assumes the following hypotheses:

(H1) $u_0, v_0 \in L^1(\Omega) \Rightarrow w_0, z_0 \in L^1(\Omega)$
(H2) $f(0, v) \succeq 0 \forall u \succeq 0, \quad g(u, 0) \succeq 0 \forall v \succeq 0$
(H3) $F(0, z) \succeq 0 \forall w \succeq 0, G(w, 0) \succeq 0 \forall z \succeq 0$
(H4) $\exists c \succeq 0 \; f(u, v) + g(u, v) \preceq c(u + v + 1); \forall u, v \succeq 0$
(H5) $\exists C \succeq 0 \; F(w, z) + G(w, z) \preceq C(w + z + 1); \forall w, z \succeq 0.$

In diagonal case where $d_2 = d_3 = 0$ N. Alikakos [1] established global existence and L^∞-bounds of solutions for positive initial data $f(u, v) = -g(u, v) = -uv^\sigma$ and $1 < \sigma < 1 + \frac{2}{n}$.

In [15] Masuda obtained a global existence result for a large class of the parameter σ. In fact, by using some L^p estimates, he showed that the solution of problem (1) – (3) exists globally in time if $\sigma > 1$.The same result in [17] was obtained by Hollis et al [10] by exploiting the duality arguments on L^p techniques, allowing to derive the uniform boundeness of the solution.

Following Masuda's approach, Haraux and Youkana [8] established a global existence result of system (1)–(3) for a large class of the function f and g.

More precisely they showed that for $f(u, v) = -g(u, v) = -u\Phi(v)$ where

$$\lim_{(v \to +\infty)} \frac{\log(1 + \Phi(v))}{v} = 0.$$

In the general case, that is to say for $f(u, v) = -g(u, v)$.

Kouachi [12] proved that the solution of problem (1)–(3) exists globally in time if

$$\lim_{(v\to+\infty)} \frac{\log(1+f(u,v))}{v} < \frac{8\alpha\beta}{n\,(\alpha-\beta)^2\,\|u_0\|_\infty}$$

Moumeni and Salah Derradji [22] have established the existence of global solution using an approach that involves the Lyapunov's functional for the system (1)–(3) where the functions f and g are assumed to satisfy the condition $f(r,s)+g(r,s) \preceq C(r+s+1)$. Moumeni and barrouk [20] have established the existence of global solutions for systems of reaction diffusion with compact result.

In tridiagonal case where $d_3 > 0$ and $d_2 = 0$ Moumeni and Salah Derradji [23] have established the existence of global solution of the problem (1)–(3), using the Lyapunov method combined with some L^p estimates. In the same direction Moumeni and barrouk [21] have studied the existence of global solution of the problem (1)–(3), using the compact semigroup methods and some L^1 estimates.

For $d_3 > 0$ and $d_2 > 0$ In [11] Kanel and Kirane proved the global existence of solutions for a strongly coupled reaction-diffusion system with homogeneous Neumann boundary conditions and

$$f(u,v) = -g(u,v) = uv^m, \quad m > 0$$

m is an odd integer, Later they improved their results in [17] where they obtained the global existence with

$$f(u,v) = -g(u,v) = u\,\mathrm{F}(v)$$

On the same direction, Kouachi [13] has proved the global existence of solutions for two-component reaction-diffusion systems with a general full matrix of diffusion coefficients, nonhomogeneous boundary conditions and polynomial growth conditions on the nonlinear terms and he obtained in [14] the global existence of solutions for the same system with homogeneous Neumann boundary conditions and

$$g(u,v) = \rho F(u,v), \quad f(u,v) = -\sigma F(u,v) \qquad \rho > 0, \sigma > 0$$

Rebiai and Benachour [27] treat the case of a general full matrix of diffusion coefficients with the homogeneous boundary conditions with nonlinearities of exponentiel growth.

In [4] Boukerrioua generalize a result obtained in [23]. Our techniques are based on invariant regions and Lyapunov functional methods. Finally, Mebarki and Moumeni [19] have established the existence of global solution adopting the Lyapunov method combined with some L^p estimates with $d_2 > 0$ and $d_3 > 0$, where the function f and g are assumed to satisfy the condition $f(r,s) + g(r,s) \preceq C(r+s+1)$.

In this work paper we generalize a result obtained in [20] concerning uniform boundedness and so the global existence of solutions for reaction-diffusion systems with a general full matrix of diffusion coefficients. Our techniques are based on compact semigroup methods and some L^1 estimates. The content of this paper is as follows. Our main results are stated in section 2.

2. Main results

Our goal is to reduce the system as follows:

Multiplying equation (2) one time through by $\frac{d_1 - \lambda_1}{d_3}$ and substracting (1) and another time by $-\frac{d_1 - \lambda_2}{d_3}$ and adding (1) we get:

$$\begin{cases} \frac{\partial w}{\partial t} - \lambda_1 \Delta w = F(w, z) & in\ [0, T[\times\Omega \\ \frac{\partial z}{\partial t} - \lambda_2 \Delta z = G(w, z) & in\ [0, T[\times\Omega \\ \frac{\partial w}{\partial \eta} = \frac{\partial z}{\partial \eta} = 0 & in\ [0, T[\times\partial\Omega \\ w(., 0) = w_0(.), z(., 0) = z_0(.) & in\ \Omega \end{cases}$$

where

$$\begin{aligned} w(t, x) &= -u(t, x) + \frac{d_1 - \lambda_1}{d_3} v(t, x) \\ z(t, x) &= u(t, x) - \frac{d_1 - \lambda_2}{d_3} v(t, x) \end{aligned}$$

for any (t, x) in $[0, T[\times\Omega$ and

$$\begin{aligned} F(w, z) &= -f(u, v) + \frac{d_1 - \lambda_1}{d_3} g(u, v) \\ G(w, z) &= f(u, v) - \frac{d_1 - \lambda_2}{d_3} g(u, v) \end{aligned}$$

Theorem 2.1. We assume that the hypotheses (H_i) are satisfied then there are (w, z)solution of:

$$\begin{cases} w, z \in C([0, +\infty[\,, L^1(\Omega)) \\ F(w, z), G(w, z) \in L^1(\Phi) \text{ where } \Phi = \Omega \times (0, T)\ \forall T \succ 0 \\ w(t, x) = S_{\lambda_1}(t) w_0 + \int_0^t S_{\lambda_1}(t - s) F(w(s), z(s)) ds\ \forall t \in [0, T[\\ z(t, x) = S_{\lambda_2}(t) z_0 + \int_0^t S_{\lambda_2}(t - s) G(w(s), z(s)) ds\ \forall t \in [0, T[\end{cases} \quad (4)$$

where $S_{\lambda_1}(t), S_{\lambda_2}(t)$ are semi contractions groups in $L^1(\Omega)$ generated by $\lambda_1 \Delta w, \lambda_2 \Delta z$ with homogeneous boundary conditions Newmann.

To prove this theorem we will rely on studying a single system through which it is more convenient to derive the evidence.

Theorem 2.2. Let A m-dissipative operator of the dense domain in the Banach space X and $S(t)$ a semi group generated by A, F a function locally Lipschitz, so $\forall u_0 \in X$ it exists $T(u_0) = T_{\max}$ such that the problem

$$\begin{cases} u \in C([0,T], D(A)) \cap C^1([0,T], X) \\ \dfrac{du}{dt} - Au = F(u(s)) \\ u(0) = u_0 \end{cases} \tag{5}$$

admits a unique solution u verifying

$$u = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds \ \ \forall t \in [0, T_{\max}] \tag{6}$$

Remark 2.3. The formula (6) defined a solution $u \in C([0, T_{\max}], X)$.

3. Compactness of the solution

In this section we will give a compactness result of operator L defining the solution of the problem (5) in the case where the initial value equals zero $[u(0) = 0]$

$$L(F)(t) = u(t) = \int_0^t S(t-s)F(u(s))ds \ \ \forall t \in [0, T]$$

Theorem 3.1. If for all $t \succ 0$, the operators $S(t)$ are compact, then L are compact of $L^1([0,T], X)$ in $L^1([0,T], X)$.

Proof.
Step 1: We show that $S(\lambda)L : F \to S(\lambda)L(F)$ is compact in $L^1([0,T], X)$ therefore show that:

The set

$$\{S(\lambda)L(F)(t); \|F\|_1 \preceq 1\}$$

is relatively compact in $L^1([0,T], X)$, $\forall t \in [0,T]$

Since $S(t)$ is compact then, the application : $t \to S(t)$ is continuous of $]0, +\infty[$ in $£(X)$. Therefore

$$\forall \varepsilon \succ 0, \forall \delta \succ 0, \exists \eta \succ 0. \forall 0 \preceq h \preceq \eta, \forall t \succeq \delta, \|S(t+h) - S(t)\|_{£(X)} \preceq \varepsilon$$

we choose $\lambda = \delta$, we have for $\forall 0 \preceq t \preceq T - h$

$$\begin{aligned} & S(\lambda)u(t+h) - S(\lambda)u(t) \\ = & \int_0^{t+h} S(\lambda + t + h - s)F(u(s))ds - \int_0^t S(\lambda + t - s)F(u(s))ds \\ = & \int_t^{t+h} S(\lambda + t + h - s)F(u(s))ds + \int_0^t (S(\lambda + t + h - s) - S(\lambda + t - s))F(u(s))ds \end{aligned}$$

as

$$\|S(\lambda)u(t+h) - S(\lambda)u(t)\|_X \preceq \int_t^{t+h} \|F(u(s)\|_X\, ds + \varepsilon \int_0^t \|F(u(s)\|_X\, ds.$$

We definite $v(t)$ by

$$v(t) = \begin{cases} u(t)\ \forall 0 \preceq t \preceq T \\ 0 \quad \text{if not} \end{cases}$$

Therefore

$$\|S(\lambda)v(t+h) - S(\lambda)v(t)\|_1 \preceq (h + \varepsilon T)\, \|F(u(s)\|_1$$

implying that all $\{S(\lambda)v;\ \|F\|_1 \preceq 1\}$ is equi-integrable , then it is conventional that all $\{S(\lambda)L(F)(t);\ \|F\|_1 \preceq 1\}$is relatively compact in $L^1([0,T], X)$, this way $S(\lambda)L$ is compact.

Step 2: We show that $S(\lambda)L$ converges towards L when λ goes towards 0 in $L^1([0,T], X)$.
We have:

$$S(\lambda)u(t) - u(t) = \int_0^t S(\lambda + t - s)F(u(s))ds - \int_0^t S(t-s)F(u(s))ds$$

So for $t \succeq \delta$ we have:

$$\|S(\lambda)u(t) - u(t)\| \preceq \int_\delta^t \|S(\lambda + s) - S(s)\|_{\pounds(X)}\, \|F(u(s)\|\, ds + 2\int_{t-\delta}^t \|F(u(s)\|\, ds$$

choose $0 \prec \lambda \prec \eta$ then

$$\|S(\lambda)u(t) - u(t)\| \preceq \varepsilon \int_\delta^t \|F(u(s)\|\, ds + 2\int_{t-\delta}^t \|F(u(s)\|\, ds$$

and for $0 \preceq t \preceq \delta$ we have

$$\|S(\lambda)u(t) - u(t)\| \preceq 2\int_0^t \|F(u(s)\|\, ds$$

as $F \in L^1(0, T, X)$ we find

$$\|S(\lambda)u(t) - u(t)\| \preceq (\varepsilon T + 2\delta)\, \|F(u(s)\|_1$$

so if $\lambda \to 0$ then $S(\lambda)u \to u$ in $L^1([0,T], X)$ where the operator L is uniformly bounded with the compact linear operator between two Banach spaces then L is compact in $L^1([0,T], X)$. ■

Remark 3.2. The semi group $S(t)$ generated by the operator Δ is compact in $L^1(\Omega)$.

4. Study of a particular system

For all $n \succ 0$; we define the functions w_{n_0} and z_{n_0} by $w_{n_0} = \min(w_0, n) \succeq 0$ and $z_{n_0} = \min(z_0, n) \succeq 0$ it is clear that w_{n_0} and z_{n_0}verify (H1), so

$$\begin{aligned} w_{n_0} &\in L^1(\Omega), w_{n_0} \succeq 0 \\ z_{n_0} &\in L^1(\Omega), z_{n_0} \succeq 0 \end{aligned}$$

Let us consider the following system:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - d_1 \Delta u_n - d_2 \Delta v_n = f(u_n, v_n) & in\ [0, T[\times\Omega \\ \dfrac{\partial v_n}{\partial t} - d_3 \Delta u_n - d_4 \Delta v_n = g(u_n, v_n) & in\ [0, T[\times\Omega \\ \dfrac{\partial u_n}{\partial \eta} = \dfrac{\partial v_n}{\partial \eta} = 0 & in\ [0, T[\times\partial\Omega \\ u_n(.,0) = u_{n0}(.), v_n(.,0) = v_{n0}(.) & in\ \Omega \end{cases} \qquad (\mathrm{p}_n)$$

4.1. Local Existence and positivity of the solution of the system (p_n)

Converting the system (p_n) in the following system

$$\begin{cases} w_{nt} - \lambda_1 \Delta w_n = F(w_n, z_n) & in\ [0, T[\times\Omega \\ z_{nt} - \lambda_2 \Delta z_n = G(w_n, z_n) & in\ [0, T[\times\Omega \\ \dfrac{\partial w_n}{\partial \eta} = \dfrac{\partial z_n}{\partial \eta} = 0 & in\ [0, T[\times\partial\Omega \\ w_n(0,.) = w_{n0}(x), z_n(0,.) = z_{n0}(x) & in\ \Omega \end{cases} \qquad (s_n)$$

in the Banach space $X = L^1(\Omega) \times L^1(\Omega)$.

So the system (s_n) can be returned to the shape of the system (5), thus if (w_n, z_n) is a solution of (s_n) so it verifies the integral equations:

$$\begin{cases} w_n(t,x) = S_{\lambda_1}(t) w_{n_0} + \displaystyle\int_0^t S_{\lambda_1}(t-s) F(w_n(s), z_n(s)) ds\ \forall t \in [0, T[\\ z_n(t,x) = S_{\lambda_2}(t) z_{n_0} + \displaystyle\int_0^t S_{\lambda_2}(t-s) G(w_n(s), z_n(s)) ds\ \forall t \in [0, T[\end{cases} \qquad (7)$$

where $S_{\lambda_1}(t)$, $S_{\lambda_2}(t)$ are semi contractions groups in $L^1(\Omega)$ generated by $\lambda_1 \Delta w$, $\lambda_2 \Delta z$.

Theorem 4.1. It exists $T_M \succ 0$ and (w_n, z_n) a local solution of (s_n) for all $t \in [0, T_M]$.

Proof. We know that $S_{\lambda_1}(t), S_{\lambda_2}(t)$ are semi groups of contraction and since $\{F(w_n(s), z_n(s)), G(w_n(s), z_n(s))\}$is locally Lipschitz in the space X , so we have $\exists T_M \succ 0$ and (w_n, z_n) is a local solution of (s_n) on $[0, T_M]$. ∎

Lemma 4.2. Let (w_n, z_n) be the solution of the problem (s_n) such that

$$\left\{w_{n_0}(x) \succeq 0, \forall x \in \Omega, z_{n_0}(x) \succeq 0, \forall x \in \Omega\right\}$$

then $w_n(x,t) \succeq 0$ and $z_n(x,t) \succeq 0$, $\forall (x,t) \in \Omega \times (0,T)$.

Proof. 1. Positivity of w_n
Let

$$\overline{w_n}(x,t) = 0 \text{ in } \Omega \times (0,T) \Rightarrow \frac{\partial \overline{w_n}}{\partial t} = 0 \text{ and } \triangle \overline{w_n} = 0$$

then

$$w_{nt} - \lambda_1 \Delta w_n - F(w_n, z_n) = 0 \succeq \overline{w_{n_t}} - \lambda_1 \Delta \overline{w_n} - F(\overline{w_n}, z_n)$$

and

$$w_n(0,x) = w_0(x) \succeq 0 = \overline{w_n}(0,x)$$

Hence, by the comparison theorem we obtain

$$w_n(x,t) \succeq \overline{w_n}(x,t)$$

therefore

$$w_n(x,t) \succeq 0$$

2. Positivity of z_n
Let

$$\overline{z_n}(x,t) = 0 \text{ in } \Omega \times (0,T) \Rightarrow \frac{\partial \overline{z_n}}{\partial t} = 0 \text{ and } \triangle \overline{z_n} = 0$$

then

$$z_{nt} - \lambda_2 \Delta z_n - G(w_n, z_n) = 0 \succeq \overline{z_{n_t}} - \lambda_2 \Delta \overline{z_n} - G(w_n, \overline{z_n})$$

and

$$z_n(0,x) = z_0(x) \succeq 0 = \overline{z_n}(0,x)$$

Hence, by the comparison theorem we obtain

$$z_n(x,t) \succeq \overline{z_n}(x,t)$$

therefore

$$z_n(x,t) \succeq 0$$

■

4.2. Global existence of the solution of the system (p_n)

To prove the global existence of the solution of the system (p_n) for all non-negative $t, \forall t \succ 0$, simply find an estimate of the solution for everthing $t \succeq 0$ according to A. Haraux-M. Kirane [7], D. Henry [9], F. Routh [22].

For this we give the following lemmas according to us shows the existence of an estimate of the solution of (p_n) in $L^1(\Omega)$.

Lemma 4.3. Let (w_n, z_n) the solution of the system (s_n), so it exists $M(t)$ which depends only of t, such that $\forall 0 \preceq t \preceq T_M$, we have:

$$\|w_n(t) + z_n(t)\|_{L^1(\Omega)} \preceq M(t).$$

Proof. We add the first and second equation of (s_n) is obtained

$$(w_n + z_n) - \Delta(\lambda_1 w_n + \lambda_2 z_n) = F(w_n, z_n) + G(w_n, z_n)$$

By taking into account of hypothesis (5) we have:

$$(w_{nt} + z_{nt}) - \Delta(\lambda_1 w_n + \lambda_2 z_n) \preceq c\,(w_n + z_n + 1)\,.$$

Let us integrate over Ω and apply the formula of Green we find

$$\frac{\partial}{\partial t}\int_\Omega (w_n, z_n)dx \preceq c\int_\Omega (w_n + z_n + 1)\,dx$$

which implies

$$\frac{\frac{\partial}{\partial t}\int_\Omega (w_n, z_n)dx}{\int_\Omega (w_n + z_n + 1)\,dx} \preceq c$$

Integrate on $[0, T]$ we obtain

$$\ln\left[\int_\Omega (w_n + z_n + 1)\,dx\right]_0^t \preceq ct$$

thus

$$\ln\frac{\int_\Omega (w_n + z_n + 1)\,dx}{\int_\Omega \left(w_{n_0} + z_{n_0} + 1\right)dx} \preceq ct$$

which implies

$$\frac{\int_\Omega (w_n + z_n + 1)\,dx}{\int_\Omega \left(w_{n_0} + z_{n_0} + 1\right)dx} \preceq \exp(ct)$$

$$\Rightarrow \int_\Omega (w_n + z_n + 1)\,dx \preceq \exp(ct)\int_\Omega \left(w_{n_0} + z_{n_0} + 1\right)dx$$

$$\Rightarrow \int_\Omega (w_n + z_n)\,dx \preceq \int_\Omega (w_n + z_n + 1)\,dx \preceq \exp(ct)\int_\Omega \left(w_{n_0} + z_{n_0} + 1\right)dx$$

$$\Rightarrow \int_{\Omega}(w_n+z_n)\,dx \preceq \exp(ct)\int_{\Omega}(w_0+z_0+1)\,dx \text{ because} w_{n0} \preceq w_0 \text{ and } z_{n0} \preceq z_0$$

Let us put

$$M(t)=\exp(ct)\int_{\Omega}(w_0+z_0+1)\,dx=\exp(ct)\,\|(w_0+z_0+1)\|_{L^1(\Omega)}$$

so w_n, z_n are positives then :

$$\|w_n+z_n\|_{L^1(\Omega)} \preceq M(t), \forall 0 \preceq t \preceq T_M$$

we can conclud from this estimate that the solution (w_n, z_n) given by Theorem (5) is global. ■

4.2.1 Global existence of the solution of the system

On the following lemmas after the estimate is obtained for the global existence of the solution (w_n, z_n) for the system (s_n) in $L^1(\Phi)$.

Lemma 4.4. For any solution (w_n, z_n) of (s_n) there is a constant $K(t)$ which depends only of t such that

$$\|w_n(t)+z_n(t)\|_{L^1(\Phi)} \preceq K(t)(\|w_0+z_0\|_{L^1(\Omega)}+1).$$

Proof. To prove of this lemma; we use the following results: (see S.L. Hollis, H. Martin, M. Pierre [10] and S. Bonafede, D. Schmitt [3]).

So, we introduce $\theta \in C_0^{\infty}(\Phi), \theta \succeq 0$ and $\varphi \in C^{2,1}(\Phi)$ a non-negative solution of the following system:

$$\begin{cases} -\dfrac{\partial\varphi}{\partial t}-d_1\Delta\varphi=\theta \quad \text{in } \Phi \\ \dfrac{\partial\varphi}{\partial\eta}=0 \quad \text{in } [0,T]\times\partial\Omega \\ \varphi(.,T)=0 \text{ in } \Omega \end{cases} \tag{s}$$

According O.A. Ladyzenskaya, V.A. Solonnikov [18] (s) possesses a unique non-negative solution.

Moreover, for $\forall q \in]1,+\infty[$, there exists a positive constant C independant of θ such that:

$$\|\varphi\|_{L^q(\Phi)} \preceq C\,\|\theta\|_{L^q(\Phi)}$$

we have according to S. Bonafede, D. Schmitt [3]:

$$\int_{\Phi}S_{\lambda_1}(t)w_{n0}\left(-\frac{\partial\varphi}{\partial t}-d_1\Delta\varphi\right)dxdt=\int_{\Omega}w_{n0}(x)\varphi(x,0)dx \tag{8}$$

$$\int_{\Phi}S_{\lambda_2}(t)z_{n0}\left(-\frac{\partial\varphi}{\partial t}-d_1\Delta\varphi\right)dxdt=\int_{\Omega}z_{n0}(x)\varphi(x,0)dx \tag{9}$$

$$\int_{\Phi} S_{\lambda_1}(t) z_{n_0} \left(-\frac{\partial \varphi}{\partial t} - d_1 \triangle \varphi\right) dx dt = \int_{\Omega} z_{n_0}(x) \varphi(x,0) dx \tag{10}$$

$$\int_{\Phi} S_{\lambda_2}(t) w_{n_0} \left(-\frac{\partial \varphi}{\partial t} - d_1 \triangle \varphi\right) dx dt = \int_{\Omega} w_{n_0}(x) \varphi(x,0) dx \tag{11}$$

and that

$$\int_{\Phi} (\int_0^t S_{\lambda_1}(t-s) F(w_n, z_n) ds) \left(-\frac{\partial \varphi}{\partial t} - d_1 \triangle \varphi\right) dx dt = \int_{\Phi} F(w_n, z_n) \varphi(x,s) dx ds \tag{12}$$

$$\int_{\Phi} (\int_0^t S_{\lambda_2}(t-s) F(w_n, z_n) ds) \left(-\frac{\partial \varphi}{\partial t} - d_1 \triangle \varphi\right) dx dt = \int_{\Phi} F(w_n, z_n) \varphi(x,s) dx ds \tag{13}$$

$$\int_{\Phi} (\int_0^t S_{\lambda_1}(t-s) G(w_n, z_n) ds) \left(-\frac{\partial \varphi}{\partial t} - d_1 \triangle \varphi\right) dx dt = \int_{\Phi} G(w_n, z_n) \varphi(x,s) dx ds \tag{14}$$

$$\int_{\Phi} (\int_0^t S_{\lambda_2}(t-s) G(w_n, z_n) ds) \left(-\frac{\partial \varphi}{\partial t} - d_1 \triangle \varphi\right) dx dt = \int_{\Phi} G(w_n, z_n) \varphi(x,s) dx ds \tag{15}$$

$$\int_{\Phi} (S_{\lambda_1}(t) w_{n_0}) \theta dx dt = \int_{\Omega} w_{n_0}(x) \varphi(x,0) dx \tag{16}$$

$$\int_{\Phi} (\int_0^t S_{\lambda_1}(t-s) F(w_n, z_n) ds) \theta dx dt = \int_{\Phi} F(w_n, z_n) \varphi(x,s) dx ds \tag{17}$$

$$\int_{\Phi} (S_{\lambda_2}(t) z_{n_0}) \theta dx dt = \int_{\Omega} z_{n_0}(x) \varphi(x,0) dx \tag{18}$$

$$\int_{\Phi} (\int_0^t S_{\lambda_2}(t-s) G(w_n, z_n) ds) \theta dx dt = \int_{\Phi} G(w_n, z_n) \varphi(x,s) dx ds \tag{19}$$

Multiplying the first equation of (7) by θ, and let us integrat on Φ, by using (16) and (17) we get

$$\begin{aligned} \int_{\Phi} w_n \theta dx dt &= \int_{\Phi} (S_{\lambda_1}(t) w_{n_0}) \theta dx dt + \int_{\Phi} (\int_0^t S_{\lambda_1}(t-s) F(w_n, z_n) ds) \theta dx dt \\ &= \int_{\Omega} w_{n_0}(x) \varphi(x,0) dx + \int_{\Phi} F(w_n, z_n) \varphi(x,s) dx ds \end{aligned}$$

Multiplying the second equation of (7) by θ, and let us integrate on Φ, by using (18) and (19) we get

$$\begin{aligned} \int_{\Phi} z_n \theta dx dt &= \int_{\Phi} (S_{\lambda_2}(t) z_{n_0}) \theta dx dt + \int_{\Phi} (\int_0^t S_{\lambda_2}(t-s) G(w_n, z_n) ds) \theta dx dt \\ &= \int_{\Omega} z_{n_0}(x) \varphi(x,0) dx + \int_{\Phi} G(w_n, z_n) \varphi(x,s) dx ds \end{aligned}$$

Therefore

$$\begin{aligned}\int_{\Phi}(w_n+z_n)\theta dxdt &= \int_{\Omega}(w_{n0}(x)+z_{n0}(x))\varphi(x,0)dx \\ &\quad+\int_{\Phi}(F(w_n,z_n)+G(w_n,z_n))\varphi(x,s)dxds \\ &\preceq \int_{\Omega}(w_{n0}(x)+z_{n0}(x))\varphi(x,0)dx \\ &\quad+\int_{\Phi}c\,(w_n+z_n+1)\,\varphi(x,s)dxds\end{aligned}$$

Using Hôlder's inequality we deduce:

$$\int_{\Phi}(w_n+z_n)\theta dxdt \preceq \|w_0+z_0\|_{L^1(\Omega)}\bullet\|\varphi(x,0)\|_{L^\infty(\Omega)}+c\,\|w_n+z_n+1\|_{L^1(\Phi)}\bullet\|\varphi\|_{L^\infty(\Phi)}$$

$$\preceq K_1(\|w_0+z_0\|_{L^1(\Omega)}+\|w_n+z_n\|_{L^1(\Phi)}+1)\bullet\|\theta\|_{L^\infty(\Phi)}$$

as θ is arbitrary in $C_0^\infty(\Phi)$ this implies

$$\|w_n+z_n\|_{L^1(\Phi)}\preceq K_1(\|w_0+z_0\|_{L^1(\Omega)}+\|w_n+z_n\|_{L^1(\Phi)}+1)$$

we take $K=\dfrac{K_1(t)}{1-K_1(t)}$ we find:

$$\|w_n+z_n\|_{L^1(\Phi)}\preceq K(t)(\|w_0+z_0\|_{L^1(\Omega)}+1)$$

Proof of Theorem 1:

Let us define the application L as follows:

$$L:(m_0,h)\to S_d(t)m_0+\int_0^t S_d(t-s)h(s)ds$$

where $S_d(t)$ the semi group of contraction generated by the operator $d\Delta$ according of the previous result in the Theorem 3.1 and $S_d(t)$ is compact, then the application L, it is adding two compact applications on $L^1(\Phi)$ then the application L is compact $L^1(\Phi)\times L^1(\Phi)$ in $L^1(\Phi)$.

Therefore, there exist a subsequence (w_{n_j},z_{n_j}) of (w_n,z_n) in $L^1(\Phi)\times L^1(\Phi)$ such that (w_{n_j},z_{n_j}) converges towards (w,z).

Let us show that (w_{n_j},z_{n_j}) is a solution of (7) we have:

$$\begin{cases} w_{n_j}(t,x)=S_{\lambda_1}(t)w_{n_0}+\displaystyle\int_0^t S_{\lambda_1}(t-s)F(w_{n_j},z_{nj})ds\ \forall t\in[0,T[\\ z_{n_j}(t,x)=S_{\lambda_2}(t)z_{n_0}+\displaystyle\int_0^t S_{\lambda_2}(t-s)G(w_{nj},z_{nj})ds\ \forall t\in[0,T[\end{cases}\qquad (\mathrm{p}_j)$$

so it is enough to show that (w, z) satisfies (4).

Clearly if $j \to \infty$, we have the following limits:

$$\begin{array}{rcl} F(w_{n_j}, z_{nj}) & \to & F(w, z) \quad a.e \\ G(w_{nj}, z_{nj}) & \to & G(w, z) \quad a.e. \end{array} \tag{20}$$

and

$$\begin{array}{rcl} w_{n_0} & \to & w_0 \\ z_{n_0} & \to & z_0 \end{array}$$

thus to show that if (w, z) verifies (4), it remains to show that:

$$\begin{array}{rcl} F(w_{n_j}, z_{nj}) & \to & F(w, z) \\ G(w_{nj}, z_{nj}) & \to & G(w, z) \end{array}$$

in $L^1(\Phi)$ when $j \to \infty$.

We integrate the first and second equations of (s_n)on Φ by taking into account that

$$-\lambda_1 \int_\Phi \Delta w_{n_j} dxdt = 0, -\lambda_2 \int_\Phi \Delta z_{n_j} dxdt = 0$$

we have

$$\begin{array}{rcl} \int_\Omega w_{n_j} dx - \int_\Omega w_{n0} dx & = & \int_\Phi F(w_{n_j}, z_{nj}) dxdt \\ \int_\Omega z_{n_j} dx - \int_\Omega z_{n0} dx & = & \int_\Phi G(w_{n_j}, z_{nj}) dxdt \end{array}$$

where

$$-\int_\Phi F(w_{n_j}, z_{nj}) dxdt \preceq \int_\Omega w_0 dx \tag{21}$$

$$-\int_\Phi G(w_{n_j}, z_{nj}) dxdt \preceq \int_\Omega z_0 dx \tag{22}$$

Let us put

$$\begin{aligned} N_n &= C\left(w_{n_j} + z_{nj} + 1\right) - F(w_{n_j}, z_{nj}) \\ M_n &= C\left(w_{n_j} + z_{nj} + 1\right) - F(w_{n_j}, z_{nj}) - G(w_{n_j}, z_{nj}) \\ &= N_n - G(w_{n_j}, z_{nj}) \end{aligned}$$

It is clear that M_n and N_n are positives, according to (H3) and (H4) of (21) and (22) we obtain:

$$\int_\Phi N_n dxdt \preceq C \int_\Phi (w_{n_j} + z_{nj} + 1) dxdt + \int_\Omega w_0 dx$$

$$\int_{\Phi} M_n dxdt \preceq C \int_{\Phi} (w_{n_j} + z_{nj} + 1) dxdt + \int_{\Omega} (w_0 + z_0) dx$$

the lemma 4.4 gives us:

$$\begin{cases} \int_{\Phi} N_n dxdt \prec +\infty \\ \int_{\Phi} M_n dxdt \prec +\infty \end{cases} \tag{23}$$

which implies

$$\begin{aligned} \int_{\Phi} \left| F(w_{n_j}, z_{nj}) \right| dxdt &\preceq C \int_{\Phi} (w_{n_j} + z_{nj} + 1) dxdt + \int_{\Phi} N_n dxdt \prec +\infty \\ \int_{\Phi} \left| G(w_{n_j}, z_{nj}) \right| dxdt &\preceq \int_{\Phi} M_n dxdt + \int_{\Phi} N_n dxdt \prec +\infty \end{aligned}$$

Let

$$H_n = N_n + C(w_{n_j} + z_{nj} + 1), \quad \Psi_n = N_n + M_n$$

$$H_n, \Psi_n \in L^1(\Phi)$$

H_n and Ψ_n are positives such as

$$\begin{aligned} \left| F(w_{n_j}, z_{nj}) \right| &\preceq H_n \\ \left| G(w_{n_j}, z_{nj}) \right| &\preceq \Psi_n \end{aligned}$$

this result is combined with (20) and applying the convergence theorem dominated Lebsegue:

We get:

$$\begin{aligned} F(w_{n_j}, z_{nj}) &\rightarrow F(w, z) \quad \text{in } L^1(\Phi) \\ G(w_{nj}, z_{nj}) &\rightarrow G(w, z) \quad \text{in } L^1(\Phi) \end{aligned}$$

by passage in the limit $j \rightarrow \infty$ of (p_j) in $L^1(\Phi)$.

We get:

$$\begin{cases} w(t, x) = S_{\lambda_1}(t) w_0 + \int_0^t S_{\lambda_1}(t - s) F(w(s), z(s)) ds \ \forall t \in [0, T[\\ z(t, x) = S_{\lambda_2}(t) z_0 + \int_0^t S_{\lambda_2}(t - s) G(w(s), z(s)) ds \ \forall t \in [0, T[\end{cases} \tag{1}$$

then (w, z) satisfies (4) therefore (w, z) the solution of (1)–(3). ■

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