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Tutte Polynomials with Applications

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Abstract

We give the general form of the Tutte polynomial of a family of positive-signed connected planar graphs, and specialize it to the Jones polynomial of the alternating links that correspond to these graphs. Moreover, we give combinatorial interpretation of some of evaluations of the Tutte polynomial, and also recover the chromatic polynomial from it as a special case.

AMS subject classification: Primary 05C31; Secondary 57M27.
Keywords: Tutte polynomial, Jones polynomial, chromatic polynomial.

1. Introduction

The Tutte polynomial was introduced by Tutte [16] in 1954 as a generalization of chromatic polynomials studied by Birkhoff [1] and Whitney [19]. This graph invariant became popular because of its universal property that any multiplicative graph invariant with a deletion/contraction reduction must be an evaluation of it, and because of its applications in computer science, engineering, optimization, physics, biology, and knot theory.

In 1985, Jones [8] revolutionized knot theory by defining the Jones polynomial as a knot invariant via Von Neumann algebras. However, in 1987 Kauffman [10] introduced a state-sum model construction of the Jones polynomial that was purely combinatorial and remarkarkably simple, we follow this construction. T-equivalence of some families of ladder type graphs has been studied in [13].

Our primary motivation to study the Tutte polynomial came from the remarkable connection between the Tutte and the Jones polynomials that *up to a sign and multiplication by a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_G(-t, -t^{-1})$* [7, 12, 15].

This paper is organized as follows: In Section 2 we give some basic notions about graphs and knots along with definitions of the Tutte and the Jones polynomials. Moreover, in this section we give the relation between graphs and knots, and the relation between the Tutte and the Jones polynomials. Then the main theorem is given in Section 3. Finally, we present the results about the Jones, chromatic, flow, and reliability polynomials in Section 3. Moreover, we give here interpretations of some evaluations of the Tutte polynomial.

2. Preliminary Notions

2.1. Basic concepts of graphs

A *graph* G is an ordered pair of disjoint sets (V, E) such that E is a subset of the set V^2 of unordered pairs of V. The set V is the set of vertices and E is the set of edges. If G is a graph, then $V = V(G)$ is the vertex set of G, and $E = E(G)$ is the edge set. An edge x, y is said to join the vertices x and y, and is denoted by xy; the vertices x and y are the end vertices of this edge. If $xy \in E(G)$, then x and y are *adjacent*, or neighboring, vertices of G, and the vertices x and y are *incident* with the edge xy. Two edges are adjacent if they have exactly one common end vertex.

We say that $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. In this case we write $G' \subset G$. If G' contains all edges of G that join two vertices in V' then G' is said to be the subgraph induced or spanned by V', and is denoted by $G[V']$. Thus, a subgraph G' of G is an *induced subgraph* if $G' = G[V(G')]$. If $V = V'$ then G' is said to be a *spanning subgraph* of G.

Two graphs are *isomorphic* if there is a correspondence between their vertex sets that preserves adjacency. Thus, $G = (V, E)$ is isomorphic to $G' = (V', E')$, denoted $G \simeq G'$, if there is a bijection $\varphi : V \to V'$ such that $xy \in E$ if and only if $\varphi(xy) \in E'$.

The dual notion of a cycle is that of cut or cocycle. If $\{V1, V2\}$ is a partition of the vertex set, and the set C, consisting of those edges with one end in V_1 and one end in V_2, is not empty, then C is called a *cut*. A cycle with one edge is called a *loop* and a cocycle with one edge is called a bridge. We refer to an edge that is neither a loop nor a bridge as *ordinary*.

A graph is *connected* if there is a path from one vertex to any other vertex of the graph. A connected subgraph of a graph G is called the *component* of G. We denote by $k(G)$ the number of connected components of a graph G, and by $c(G)$ the number of non-trivial connected components, that is the number of connected components not counting isolated vertices. A graph is k-connected if at least k vertices must be remarkoved to disconnect the graph.

A *tree* is a connected graph without cycles. A *forest* is a graph whose connected components are all trees. Spanning trees in connected graphs play a fundamental role in the theory of the Tutte polynomial. Observe that a loop in a connected graph can be characterized as an edge that is in no spanning tree, while a bridge is an edge that is in every spanning tree.

A graph is *planar* if it can be drawn in the plane without edges crossings. A drawing of a graph in the plane separates the plane into regions called faces. Every plane graph G has a *dual graph*, G^*, formed by assigning a vertex of G^* to each face of G and joining two vertices of G^* by k edges if and only if the corresponding faces of G share k edges in their boundaries. Note that G^* is always connected. If G is connected, then $(G^*)^* = G$. If G is planar, it may have many dual graphs.

A *graph invariant* is a function f on the collection of all graphs such that $f(G_1) = f(G_2)$, whenever $G_1 \cong G_2$. A graph polynomial is a graph invariant where the image lies in some polynomial ring.

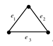
2.2. The Tutte polynomial

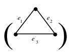
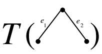
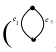
The following two operations are essential to understand the Tutte polynomial definition for a graph G. These are: *edge deletion* denoted by $G' = G - e$, and *edge contraction* $G'' = G/e$.

The deletion and contraction operations

Definition 2.1. ([16, 17, 18]) The *Tutte polynomial* of a graph G is a two-variable polynomial $T_G(x, y)$ defined as follows:

$$T_G(x, y) = \begin{cases} 1 & \text{if } E \text{ is empty,} \\ xT(G/e) & \text{if } e \text{ is a bridge,} \\ yT(G-e) & \text{if } e \text{ is a loop,} \\ T(G-e) + T(G/e) & \text{if } e \text{ is neither a bridge nor a loop.} \end{cases}$$

Example 2.2. Here is the Tutte polynomial of the graph $G=$ .

$T($  $) = T($  $) + T($  $)$

$= xT($  $) + T($  $) + T($  $)$

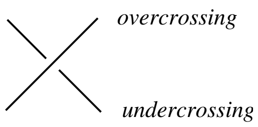
$= x^2T(\bullet) + xT(\bullet) + y$

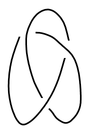
$= x^2 + x + y.$

Remark 2.3. The definition of the Tutte polynomial outlines a simple recursive procedure to compute it, but the order of the rules applied is not fixed.

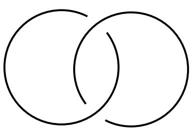
2.3. Basic concepts of Knots

A *knot* is a circle embedded in $\mathbb{R}^3$, and a *link* is an embedding of a union of such circles. Since knots are special cases of links, we shall often use the term link for both knots and links. Links are usually studied via projecting them on a plan; a projection with exampletra information of *overcrossing* and *undercrossing* is called the *link diagram*.





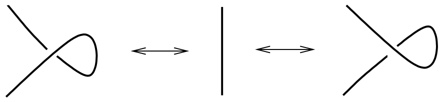
Trefoil knot



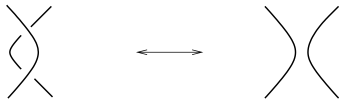
Hopf link

Two links are called *isotopic* if one of them can be transformed to the other by a diffeomorphism of the ambient space onto itself. A fundamental result about the isotopic link diagrams is:

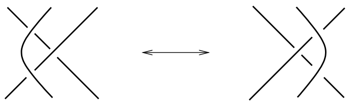
Two unoriented links L_1 and L_2 are equivalent if and only if a diagram of L_1 can be transformed into a diagram of L_2 by a finite sequence of ambient isotopies of the plane and local (Reidemeister) moves of the following three types:



R_1



R_2



R_3

The set of all links that are equivalent to a link L is called a *class* of L. By a link L we shall always mean a class of the link L.

2.4. The Jones polynomial

The main question of knot theory is *Which two links are equivalent and which are not*? To address this question one needs a *knot invariant*, a function that gives one value on all links in a single class and gives different values (but not always) on links that belong to different classes. In 1985, Jones [8] revolutionized knot theory by defining the Jones polynomial as a knot invariant via Von Neumann algebras. However, in 1987 Kauffman [10] introduced a state-sum model construction of the Jones polynomial that was purely combinatorial and remarkarkably simple.

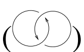
Definition 2.4. ([8, 9, 10]) The *Jones polynomial* $V_K(t)$ of an oriented link L is a Laurent polynomial in the variable $\sqrt{t}$ satisfying the skein relation

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{1/2} - t^{-1/2})V_{L_0}(t),$$

and that the value of the unknot is 1. Here L_+, L_-, and L_0 are three oriented links having diagrams that are isotopic everywhere examplecept at one crossing where they differ as in the figure below:

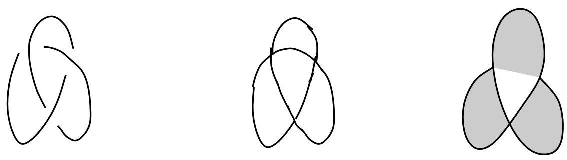
L_+ L_- L_0

Example 2.5. The Jones polynomials of the Hopf link and the trefoil knot are respectively

$V($$) = -t^{-5/2} - t^{-1/2}$ and $V($$) = -t^{-4} + t^{-3} + t.$

2.5. A Connection between Knots and graphs

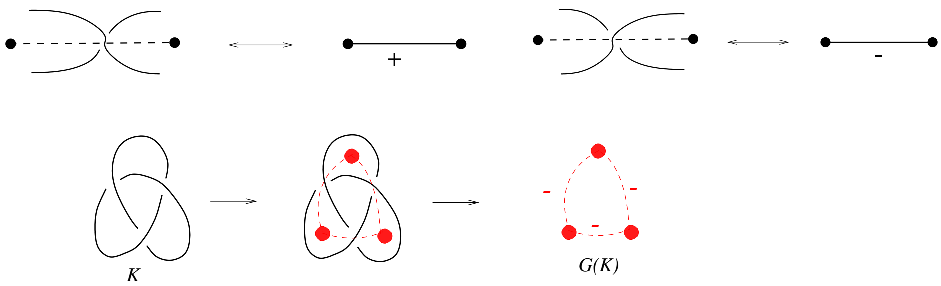
Corresponding to every connected link diagram we can find a connected signed planar graph and vice versa. The process is as follows: Suppose K is a knot and K' its projection. The projection K' divides the plane into several regions. Starting with the outermost region, we can color the regions either white or black. By our convention, we color the outermost region white. Now, we color the regions so that on either side of an edge the colors never agree.



The graph G corresponding to the knot projection K'

Next, choose a vertex in each black region. If two black regions R and R' have common crossing points $c_1, c_2, \ldots, c_n$, then we connect the selected vertices of R and R' by simple edges that pass through $c_1, c_2, \ldots, c_n$ and lie in these two black regions. In this way, we obtain from K' a plane graph G [14].

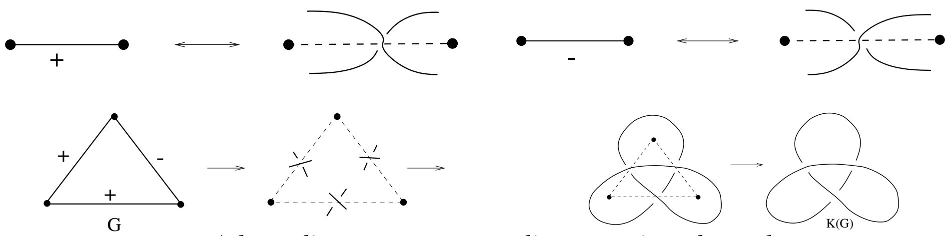
However, in order for the plane graph to embody some of the characteristics of the knot, we need to use the regular diagram rather than the projection. So, we need to consider the *under*- and *over*-crossings. To this end, we assign to each edge of G either the sign $+$ or $-$ as you can see in the following figure.



A signed graph corresponding to a knot diagram

A signed plane graph that has been formed by means of the above process is said to be the graph of the knot K [14].

Conversely, corresponding to a connected signed planar graph, we can find a connected planar link diagram. The construction is clear from the following figure.



A knot diagram corresponding to a signed graph

The fundamental combinatorial result connecting knots and graphs is:

Theorem 2.6. ([11, 12]) The collection of connected planar link diagrams is in one-to-one correspondence with the collection of connected signed planar graphs.

2.6. Connection between the Tutte and the Jones polynomials

The primary motivation to study the Tutte polynomial came from the following remarkable connection between the Tutte and the Jones polynomials.

Theorem 2.7. (Thistlethwaite) ([7, 12, 15]) Up to a sign and multiplication by a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_G(-t, -t^{-1})$.

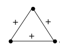
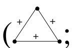
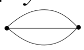
For positive-signed connected graphs, we have the precise connection:

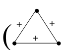
Theorem 2.8. ([2, 5, 6]) Let G be the positive-signed connected planar graph of an alternating oriented link diagram L. Then the Jones polynomial of the link L is

$$V_L(t) = (-1)^{wr(L)} t^{\frac{(b(L)-a(L)+3wr(L))}{4}} T_G(-t, -t^{-1}),$$

where $a(L)$ is the number of vertices in G, $b(L)$ is the number of vertices in the dual of G, and $wr(L)$ is the writhe of L.

Remark 2.9. In this paper, we shall compute Jones polynomials of links that correspond only to positive-signed graphs.

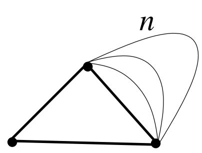
Example 2.10. Corresponding to the positive-signed graph G: , we receive the right-handed trefoil knot L: . It is easy to check, by definitions, that $V($$; t) = -t^4 + t^3 + t$ and $T($$; x, y) = x^2 + x + y$. Further note that the number of vertices in G is 3, number of vertices in the dual  of G is 2, and withe of L is 3. Now notice that

$$V($$$$; t) = (-1)^3 t^{\frac{2-3+3(3)}{4}} T($$$$; -t, -t^{-1}) = -t^2(t^2 - t - t^{-1}),$$

which agrees with the known value.

3. The Main Results

First of all we give the general formula of the Tutte polynomial of the following graph, which we denote by $G(3, n)$.



$G(3, n)$

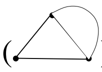
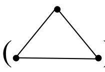
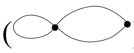
Here n is the number of additional edges to the basic figure of the cycle graph, C_3.

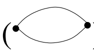
Proposition 3.1. The Tutte polynomial of the graph $G_{3,n}$ is

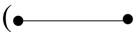
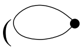
$$G_{3,n}(x, y) = x^2 + (x + y)\sum_{i=0}^{n} y^i.$$

Proof. We proof it by induction on n.

For $n = 1$, we have

$T($  $) = T($  $) + T($  $)$

$= (x^2 + x + y) + yT($  $)$

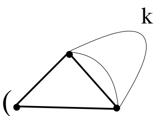
$= (x^2 + x + y) + y[T($  $) + T($  $)]$

$$= x^2 + (x + y) + y(x + y)$$

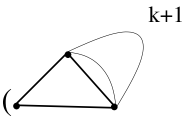
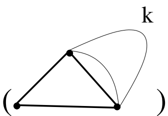
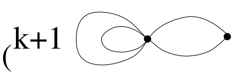
$$= x^2 + (x + y)(1 + y)$$

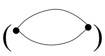
$$= x^2 + (x + y)\sum_{i=0}^{1} y^i.$$

Let the result be true for $n = k$, that is

$T($  $) = x^2 + (x + y)\sum_{i=0}^{k} y^i.$

Now for $n = k + 1$, we have

$T($  $) = T($  $) + T($  $)$

$= x^2 + (x + y)\sum_{i=0}^{k} y^i + y^{k+1}T($  $)$

$$= x^2 + (x + y)\sum_{i=0}^{k} y^i(x + y) + y^{k+1}(x + y)$$

$$= x^2 + (x + y)\left(\sum_{i=0}^{k} y^i + y^{k+1}\right)$$

$$= x^2 + (x + y)\sum_{i=0}^{k+1} y^i.$$

This completes the proof. ■

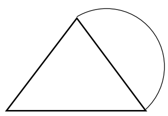
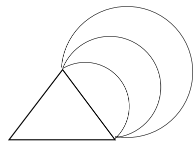
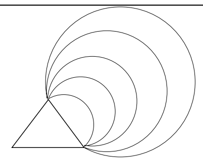
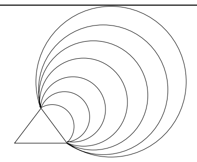
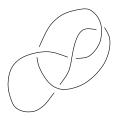
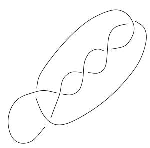
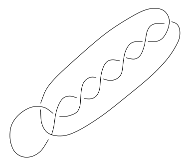
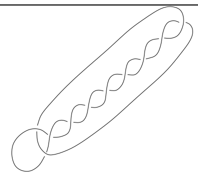
Now we specialize this Tutte polynomial to the Jones polynomial of the the alternating links that corresponds to $G_{3,n}$.

3.1. The Jones polynomial

The alternating links L that correspond to the graphs $G_{3,n}$ fall into two categories, the 1-component links (when n is odd) and 1-component links (when n is even).

The graphs along with the corresponding 1-component links (which are simply knots) are given in the following table.

Case I. When n is odd

n	1	3	5	7	$\cdots$
G					$\cdots$
L					$\cdots$
$a(L)$	3	3	3	3	$\cdots$
$b(L)$	3	5	7	9	$\cdots$
$wr(L)$	0	2	4	6	$\cdots$

Remark 3.2. Observe from the above table that $a(L) = 3$, $b(L) = n+2$, and $wr(L) = n - 1$.

Proposition 3.3. The Jones polynomial of the link L corresponding to the graph $G_{3,n}$ is

$$V_L(t) = \frac{1}{(1+t)}\big(t^{n+2} - t^{n-1} + t^{-2} + 1\big).$$

Proof. We prove it by specializing the Tutte polynomial of graph $G_{3,n}$ by using Theorem 2.6

$$V_L(t) = (-1)^{wr(L)} t^{\frac{(b(L)-a(L)+3wr(L))}{4}} T_{G_{3,n}}(-t, -t^{-1}).$$

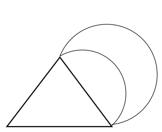
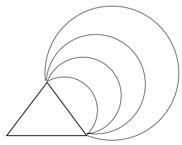
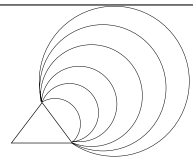
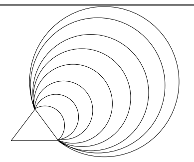
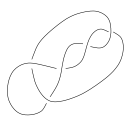
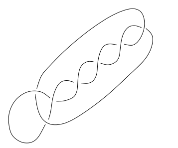
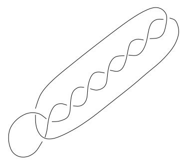
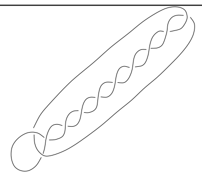
From the above Remark 3.2 the factor $(-1)^{wr(L)} t^{\frac{(b(L)-a(L)+3wr(L))}{4}}$ reduces to t^{n-1}.

$$\begin{aligned}
V_L(t) &= t^{n-1} T(-t, -t^{-1}) \\
&= t^{n-1}\left[t^2 + (-t - t^{-1})\left(\frac{(-t)^{-n-1} - 1}{(-t^{-1} - 1)}\right)\right] \\
&= t^{n-1}\left[t^2 + t(t + t^{-1})\left(\frac{t^{-n-1} - 1}{1 + t}\right)\right] \\
&= \frac{t^{n-1}}{(1+t)}\left[t^2(1+t) + (t^2+1)(t^{-n-1} - 1)\right] \\
&= \frac{t^{n-1}}{(1+t)}\left(t^2 + t^3 + t^{-n+1} - t^2 + t^{-n-1} - 1\right) \\
&= \frac{1}{(1+t)}\left(t^{n+2} - t^{n-1} + t^{-2} + 1\right).
\end{aligned}$$

■

Case II. When n is even

The graphs along with the corresponding 1-component links (which are simply knots) are given in the following table.

n	2	4	6	8	$\cdots$
G					$\cdots$
L					$\cdots$
$a(L)$	3	3	3	3	$\cdots$
$b(L)$	4	6	8	10	$\cdots$
$wr(L)$	5	7	9	11	$\cdots$

Remark 3.4. Observe from the above table that $a(L) = 3$, $b(L) = n+2$, and $wr(L) = n + 3$.

Proposition 3.5. The Jones polynomial of the link L corresponding to the graph $G_{3,n}$ is

$$V_L(t) = \frac{1}{(1+t)}\left(-t^{n+5} + t^{n+2} + t^3 + t\right).$$

Proof. The proof is similar to the proof of Proposition 3.3; the only difference is that the factor $(-1)^{wr(L)} t^{\frac{(b(L)-a(L)+3wr(L))}{4}}$ now reduces to $-t^{n+2}$. ■

3.2. Chromatic plynomial

A common problem in the study of graph theory is the coloring of the vertices of a graph. The coloring of a graph in such a way that no two adjacent vertices have the same color is called proper coloring of graphs.That is If we have a positive integer λ, then a λ-coloring of a graph G is a mapping of $V(G)$ into the set $\{1, 2, 3, \ldots, \lambda\}$ of λ colors. Thus, there are exactly λ^n colorings for a graph on n vertices.

The chromatic polynomial, because of its theoretical and applied importance, has generated a large body of work. Chia [3] provides an extensive bibliography on the chromatic polynomial, and Dong et al. [4] give a comprehensive treatment.

Definition 3.6. The *chromatic polynomial* $P_G(\lambda)$ of a graph G is a one-variable graph invariant and is defined by the following deletion-contraction relation:

$$P_G(\lambda) = P(G - e) - P(G/e).$$

Since the chromatic polynomial counts the number of distinct ways to color a graph with λ colors, we recover it from the Tutte polynomial. The following theorem gives a relation between these polynomials.

Theorem 3.7. ([2, 3, 4]) The chromatic polynomial of a graph $G = (V, E)$ is

$$P_G(\lambda) = (-1)^{|V|-k(G)}\lambda^{k(G)}T_G(1 - \lambda, 0),$$

where $k(G)$ denote the number of connected components of G.

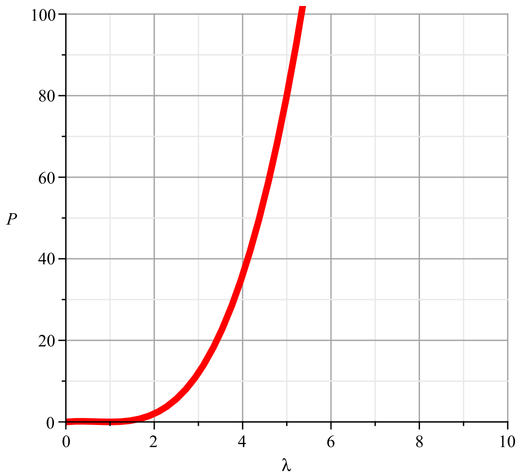
Now we recover the chromatic polynomial of $G_{3,n}$ from the Tutte polynomial.

Proposition 3.8. The chromatic polynomial of the graph $G_{3,n}$ is

$$P_{G_{3,n}}(\lambda) = \lambda(1 - \lambda)^2.$$

Proof. Since $|V| = 3$ and $k(G) = 1$, by using Theorem 3.7 the factor $(-1)^{|V|-k(G)}\lambda^{k(G)}$ reduces to λ and $T_G(1 - \lambda, 0)$ reduces to $(1 - \lambda)^2$.then we get the final result as

$$P_{G_{3,n}}(\lambda) = \lambda(1 - \lambda)^2.$$



The chromatic polynomials verses number of colours
The curve for n=1,2,3,4 appears from left to right

■

3.3. Flow Polynomial

A function which could count the number of flows in a connected graph is known as flow polynomial.

Definition 3.9. Suppose that G be a graph with an arbitrary but fixed orientation, and let K be an Abelian group of order $|K|$ and with 0 as its identity element. A K-flow is a mapping ϕ of the oriented edges $\overrightarrow{E}(G)$ into the elements of the group K such that

$$\sum_{\overrightarrow{e}=u\rightarrow v} \phi(\overrightarrow{e}) + \sum_{\overrightarrow{e}=u\leftarrow v} \phi(\overrightarrow{e}) = 0 \tag{3.1}$$

for every vertex v, and where the first sum is taken over all arcs towards v and the second sum is over all arcs leaving v.

A K-flow is nowhere zero if ϕ never takes the value 0. The relation (3.1) is called the conservation law (that is, the Kirchhoff's law is satisfied at each vertex of G).

It is well known [2] that the number of proper K-flows does not depend on the structure of the group, it depends only on its cardinality, and this number is a polynomial function of $|K|$ that we refer to as the *flow polynomial*.

The following, due to Tutte [16], relates the Tutte polynomial of G with the number of nowhere zero flows of G over a finite Abelian group (which, in our case, is $\mathbb{Z}_k$).

Theorem 3.10. **([16])** Let $G = (V, E)$ be a graph and K a finite Abelian group. If $F_G(|K|)$ denotes the number of nowhere zero K-flows then

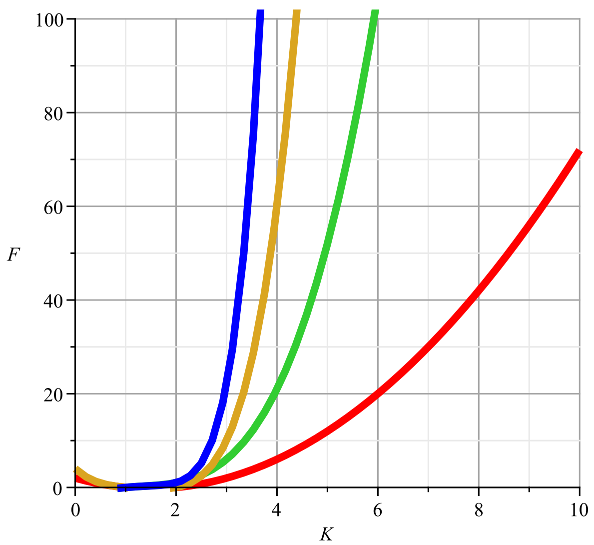
$$F_G(|K|) = (-1)^{|E|-|V|+k(G)}T(0, 1-|K|).$$

Proposition 3.11. The Flow polynomial of the graph $G_{3,n}$ is

$$F_G(|K|) = (-1)^{n+1}\frac{1}{|K|}\big[(1-|K|)^{n+2} + |K| - 1\big].$$

Proof. Since $|V| = 3$, $|E| = n + 3$ and $k(G) = 1$, then by using Theorem 3.10 the factor $(-1)^{|E|-|V|+k(G)}$ reduces to $(-1)^{n+1}$.

$$\begin{aligned} F_G(|K|) &= (-1)^{n+1} \cdot T_G(0, 1 - |K|) \\ &= (-1)^{n+1}\left[0 + (0 + (1 - |K|) \sum_{i=0}^{n} (1 - |K|)^i\right] \\ &= (-1)^{n+1}(1 - |K|)\left[\frac{[(1 - |K|)^{n+1} - 1]}{-|K|}\right] \\ &= (-1)^{n+1}\frac{1}{-|K|}\left[(1 - |K|)^{n+2} + |K| - 1\right] \\ &= (-1)^{n}\frac{1}{|K|}\left[(1 - |K|)^{n+2} + |K| - 1\right]. \end{aligned}$$



The flow polynomials F verses the order |K| of group
The curve for m,n=1,2,3,4 appears from left to right

■

3.4. Reliability polynomial

Definition 3.12. Let G be a connected graph or network with $|V|$ vertices and $|E|$ edges, and supposed that each edge is independently chosen to be active with probability p. Then the (all terminal) reliability polynomial of the network is

$$\begin{aligned} R_G(p) &= \sum_A p^{|A|}(1 - p)^{|E-A|} \\ &= \sum_{i=o}^{|E|-|V|+1} g_i(p^{i+|V|-1})(1 - p)^{|E|-i-|V|+1}, \end{aligned}$$

where A is the connected spanning subgraph of G and g_i is the number of spanning connected subgraphs with $i + |V| - 1$ edges.

Theorem 3.13. ([5]) If G is a connected graph with m edges and n vertices, then

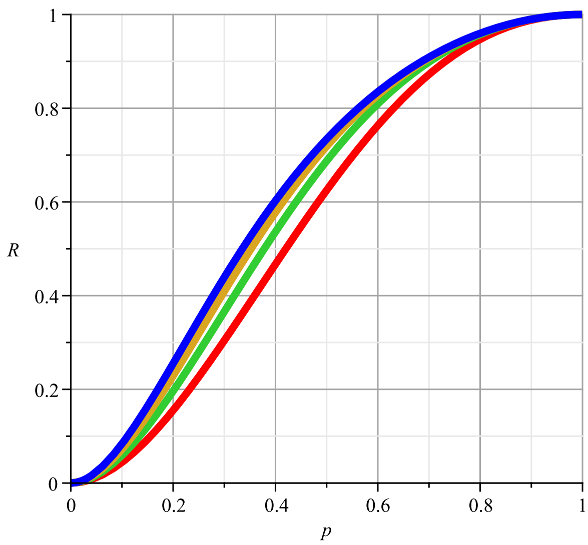
$$R_G(p) = p^{|V|-1}(1-p)^{|E|-|V|+1} T_G\left(1, \frac{1}{1-p}\right).$$

Proposition 3.14. The Reliability polynomial of the graph $G_{3,n}$ is

$$R_G(p) = p^2(1-p)^{n+1} + p(2-p)\big(1-(1-p)^{n+1}\big).$$

Proof. Since $|V| = 3$ and $|E| = n+3$, by using Theorem 3.13 the factor $(p)^{|V|-1}(1-p)^{|E|-|V|+1}$ reduces to $(p)^2(1-p)^{n+1}$.

$$\begin{aligned}
R_G p) &= p^2(1-p)^{n+1} \cdot T_G\left(1, \frac{1}{1-p}\right) \\
&= p^2(1-p)^{n+1}\left[1 + \left(1 + \frac{1}{1-p}\right)\frac{(\frac{1}{1-p})^{n+1} - 1]}{\frac{1}{1-p} - 1}\right] \\
&= p^2(1-p)^{n+1}\left[1 + \left(\frac{1-p+1}{1-p}\right)\frac{(1-(1-p)^{n+1})(1-p)}{(1-p)^{n+1}(1-1+p)}\right] \\
&= p^2(1-p)^{n+1}\left[1 + \frac{(2-p)(1-(1-p)^{n+1})}{(1-p)^{n+1}p}\right] \\
&= p^2(1-p)^{n+1} + p(2-p)\big(1-(1-p)^{n+1}\big).
\end{aligned}$$



The reliability polynomials R verses the probability p
The curve for m,n=1,2,3,4 appears from left to right

■

3.5. Evaluations

In this section we evaluate the Tutte polynomial at some special points to get useful combinatorial information about the graph $G_{3,m,n}$.

Theorem 3.15. ([5]) If $G = (V, E)$ is a connected graph, then

(1) $T_G(1, 1)$ is the number of spanning trees of G.

(2) $T_G(2, 1)$ equals the number of spanning forests of G.

(3) $T_G(1, 2)$ is the number of spanning connected subgraphs of G.

(4) $T_G(2, 2)$ equals $2^{|E|}$, and is the number of subgraphs of G.

Theorem 3.16. ([5]) If $G = (V, E)$ is a connected graph, then

(1) $T_G(2, 0)$ equals the number of cyclic orientations of G, that is, orientation without oriented cycles.

(2) $T_G(1, 0)$ equals the number of acyclic orientations with exactly one predefined source.

(3) $T_G(0, 2)$ equals the number of totally cyclic orientations of G, that is, orientation in which every arc is a directed cyclic.

(4) $T_G(2, 1)$ equals the number of score vectors of orientations of G.

Proposition 3.17. The following statements hold for the connected, planar graph $G_{3,n}$.

(1) $T_{G_{3,n}}(1, 1) = 3n + 5$.

(2) $T_{G_{3,n}}(2, 1) = 12n + 15$.

(3) $T_{G_{3,n}}(2, 2) = 16n + 24$.

(4) $T_{G_{3,n}}(1, 2) = 4n + 9$.

Proof. We prove it using directly Theorems 3.1 and 3.15.

Substituting $x = y = 1$ in proposition 3.1 we have

$$\begin{aligned} T(1, 1) &= 1 + 1 + (1^2 + 1 + 1) \sum_{i=0}^{n} 1^i \\ &= 2 + 3(n + 1) \\ &= 3n + 5, \end{aligned}$$

which is the required result. The proofs of the rest statements are similar. ■

Proposition 3.18. The following statements hold for the connected, planar graph $G_{3,n}$.

(1) $T_{G_{3,n}}(2,0) = 12n + 12.$

(2) $T_{G_{3,n}}(1,0) = 2n + 2.$

(3) $T_{G_{3,n}}(0,2) = 4.$

(4) $T_{G_{3,n}}(2,1) = 4n + 17.$

Proof. We prove it directly by using Theorems 3.1 and 3.16.

Substituting $x = 2$ and $y = 0$ in Theorem 3.1 we have

$$\begin{aligned} T(1,1) &= 0 + 0 + (2^2 + 2 + 0)\sum_{i=0}^{n} 2^i \\ &= (6)(2)(n+1) \\ &= 12n + 12, \end{aligned}$$

which is the required result. The proofs of the rest statements are similar. ■

Acknowledgments

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