Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 6 (2016), pp. 4661–4675

http://www.ripublication.com/gjpam.htm

Note on fractional edge covering games[1]

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Abstract

In this paper, we study cooperative games arising from fractional edge covering problems on graphs. We introduce two games, a rigid fractional edge covering game and its relaxed game, as generalizations of a rigid edge covering game and its relaxed game studied by Liu and Fang [4]. We show that main results in [4] are well extended to fractional edge covering games. That is, we give a characterization of the cores of both games, find relationships between them.

AMS subject classification:
Keywords: Cooperative game, core, balancedness, fractional edge covering game.

1. Introduction

A *transferable utility game* (or game for short) is an ordered pair (V,c) of a player set V and a characteristic function $c : 2^V \to \mathbb{R} \cup \{\infty\}$ with $c(\emptyset) = 0$. For each $S \subseteq V$, $c(S)$ represents the cost which the players in S achieve together. Given a game (V,c), the main issue is how to distribute fairly the total cost $c(V)$ among all the players. By imposing different requirements for fairness of distributions, there are many concepts of fair distributions. The core is one of these distribution concepts, which is considered an important distribution for $c(V)$. Given a game (V,c) with $V = \{v_1, v_2, \ldots, v_n\}$, the *core* of (V,c) is a set of vectors in $\mathbb{R}^n$ defined by

$$\mathscr{C}(V,c) = \left\{ (z_1,\ldots,z_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} z_i = c(V) \text{ and } \forall S \subseteq V, \sum_{i:v_i \in S} z_i \le c(S) \right\},$$

[1]This work was supported by research grants from the Catholic University of Daegu in 2015.

where z_i denotes the value of the entry of z corresponding to v_i. The constraint imposed on the definition of $\mathscr{C}(V,c)$, which is called group rationality, ensures that no coalition would have an incentive to split from the grand coalition V, and do better on its own. We say a game is *balanced* if and only if the core of a game is nonempty. Since the core of a game is one of important distribution method, characterizing the cores of games and determining the balancedness for games have been considered as important research questions. In this paper, we introduce cooperative cost games that arises from fractional edge covering problems on graphs and investigate the cores of those games.

Games arising from problems on graph structures have been studied many researchers (see [1, 2, 3, 4, 5, 6, 7, 8, 9]). Especially, Velzen [9] studied cooperative games that arise from domination problem on graphs to model the cost allocation problem arising from domination problems on graphs. He introduced three kinds of cooperative games, rigid dominating set games, intermediate dominating set games and relaxed dominating set games, and presented a common necessary and sufficient condition for balancedness of all the games. Kim and Fang [5] extended his work into integer dominating set games. Then Liu and Fang [4] studied a rigid edge covering game and its relaxed game that arise from the edge covering problems on graphs. In this paper, we extends most of the results obtained by Liu and Fang [4] into fractional edge covering games.

This paper is organized as follows. Section 2 introduces the rigid fractional edge covering game and the relaxed fractional edge covering game associated with a weighted graph. Section 3 investigates properties of fractional edge covering functions. Section 4 gives results on the cores of fractional edge covering games which are extension of results in [4], that is, characterizing the cores of those games, finding relationships between of them. Section 5 gives some remarks on balancedness of those games.

2. Definitions of fractional edge covering games

Throughout this paper, we assume that a graph means a simple graph with no isolated vertices. An edge of G with endpoints u and v is denoted by uv. A weighted graph is a graph G with an edge weight function $\omega : E(G) \to \mathbb{R}_+ \setminus \{0\}$ ($\mathbb{R}_+$ is the set of nonnegative real numbers). For a graph G, and $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and by $E(G[S])$ the set of edges in $G[S]$, and by $\widetilde{E}(G[S])$ the set of edges which have an end in S. Given a vertex v of G, the set of vertices that are adjacent to v is denoted by $N_G(v)$.

Let $[a,b]$ be the closed real interval for real numbers a,b with $a \leq b$. For a graph G, a function $g : E(G) \to [0,1]$ is a fractional edge covering function (or FEC function for short) of G if for every vertex $v \in V(G)$, $\sum_{x \in N_G(v)} g(xv) \geq 1$. Note that since we consider only graphs with no isolated vertices, each graph has a fractional edge covering function. For a weighted graph G with an edge weight function ω, if a FEC function g of G minimizes the total weight $\sum_{e \in E(G)} g(e)\omega(e)$, then g is called a *minimum FEC function* of G. The fractional edge covering problem (or FEC problem for short) of a weighted graph G with an edge weight function ω is to find the total weight $\sum_{e\in E(G)} g(e)\omega(e)$ of a minimum FEC function g of G.

The FEC problem of a weighted graph is one of edge-domination problems on graphs which are widely studied in graph theory. The FEC problem has some practical applications: we assume that each vertex is a player, who wants to obtain some kind of service through a FEC function. From the authority's point of view, one wishes to minimize the total cost for providing the service to every player. This is just the fractional edge covering problem on a graph.

We introduce a rigid FEC game and its relaxed game by considering the cost allocation problem arising from FEC problems on graphs. Let G be a weighted graph with an edge weight function ω. For $S \subseteq V(G)$ such that $G[S]$ has no isolated vertex, a function $g_S : E(G[S]) \to [0,1]$ is called a *rigid fractional covering function for S* (or rigid FEC function for S for short), if $\sum_{x\in N_{G[S]}(v)} g_S(xv) \geq 1$ for each $v \in S$. In addition, for any $S \subseteq V(G)$, a function $\widetilde{g}_S : \widetilde{E}(G[S]) \to [0,1]$ is called a *relaxed fractional covering function for S* (or relaxed FEC function for S for short), if $\sum_{x\in N_G(v)} \widetilde{g}_S(xv) \geq 1$ for each $v \in S$. By the assumption that G is a graph with no isolated vertex, it is true that G has a relaxed FEC function for any $S \subset V(G)$.

Now we introduce two games associated with a weighted graph in the following.

Definition 2.1. Given a weighted graph G with an edge weight function ω, (V,c) is called the *rigid fractional covering game* (or rigid FEC game for short) *associated with* G if $V = V(G)$ and a function $c : 2^{V(G)} \to \mathbb{R}\cup\{\infty\}$ is defined as follows:

i) $c(\emptyset) = 0$;

ii) for $\emptyset \subsetneq S \subseteq V(G)$, if $G[S]$ has an isolated vertex then $c(S) = \infty$, and if $G[S]$ has no isolated vertex then

$$c(S) = \min\{ \sum_{e\in E(G[S])} g_S(e)\omega(e) \mid g_S \text{ is a rigid FEC function for } S.\}.$$

Definition 2.2. Given a weighted graph G with an edge weight function ω, $(V,\widetilde{c})$ is called the *relaxed fractional edge covering game* (or relaxed FEC game for short) *associated with* G if $V = V(G)$ and a function $\widetilde{c} : 2^{V(G)} \to \mathbb{R}\cup\{\infty\}$ is defined as follows:

i) $\widetilde{c}(\emptyset) = 0$;

ii) for $\emptyset \subsetneq S \subseteq V(G)$,

$$\widetilde{c}(S) = \min\{ \sum_{e\in \widetilde{E}(G[S])} \widetilde{g}_S(e)\omega(e) \mid \widetilde{g}_S \text{ is a relaxed FEC function for } S.\}.$$

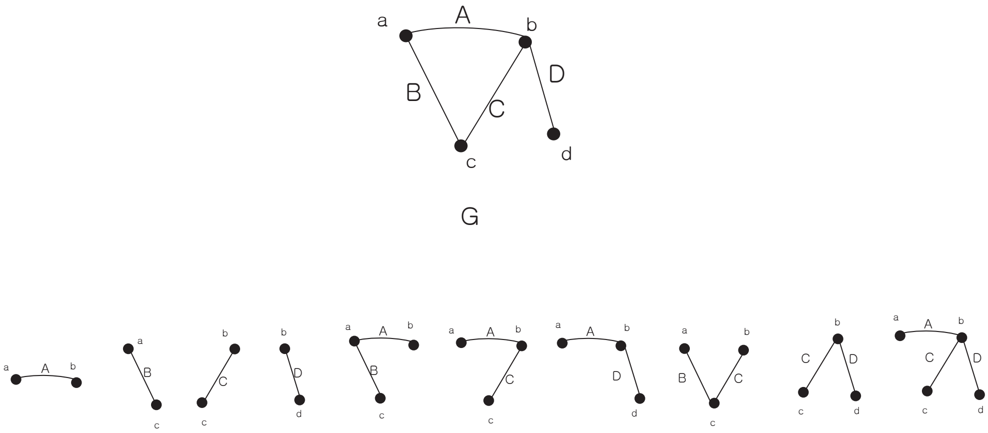

Figure 1: A graph G and the elements of $\mathscr{T}(G)$

If we restrict the codomain of rigid (resp. the relaxed) FEC functions to $\{0,1\}$, then the rigid (resp. the relaxed) FEC game associated with G is equal to the rigid (resp. the relaxed) EC game associated with G, which was introduced by Liu and Fang [4]. In this paper, we show that the results obtained by Liu and Fang can be well extended into fractional edge covering games.

3. A property of FEC function of a graph

In this section, we investigate a property of FEC function of a graph which will play an important role in characterizing the core of a rigid FEC game in Section 4.

Given a graph G, the subgraph induced by edges incident to a vertex v of G is called a *star* or *v-star* and we call v a *center* of the star. We denote by $\mathscr{T}(G)$ the set of all stars of G (see Figure 1 for an illustration). For a vertex v of G, the set of all stars containing the vertex v is denoted by $\mathscr{T}_v(G)$. For an edge e of G, the set of all stars containing the edge e is denoted by $\mathscr{T}_e(G)$.

The following theorem states a property of FEC functions of a graph.

Theorem 3.1. Let g be a minimum FEC function of a weighted graph G. Then there exists a function $\alpha : \mathscr{T}(G) \to [0,1]$ such that

(i) for each edge e of G, $\displaystyle\sum_{T\in\mathscr{T}_e(G)} \alpha(T) = g(e)$,

(ii) for each vertex v of G, $\displaystyle\sum_{T\in\mathscr{T}_v(G)} \alpha(T) = 1$.

We prove Theorem 3.1 in the rest of this section. The matrices considered in this section are square matrices having nonnegative integer elements and its rows and columns are labeled by the vertices of a graph in a same order. If rows and columns of a matrix A are labeled by the vertices of a graph G, then we say A is a matrix of G.

Given a graph G and a FEC function g of G, the *g-adjacency matrix* of G is a symmetric matrix of G defined by

$$A = \sum_{xy \in E(G)} g(xy) E_{xy},$$

where E_{xy} is the (0,1)-matrix of G such that the (x,y)-entry and the (y,x)-entry are the only entry having nonzero elements. Since g is a FEC function, for each $v \in V(G)$,

$$\sum_{x \in N_G(v)} [A]_{vx} \geq 1. \tag{3.1}$$

A (0,1)-matrix R of a graph G is called a *v-star matrix* of G if there exists a nonempty set $S \subseteq N_G(v)$ such that

$$R = \sum_{u \in S} E_{vu}.$$

A *star matrix* of a graph G is a v-star matrix of G for some $v \in V(G)$. It is easy to see that

$$\begin{gathered}\text{if a star matrix } R \text{ has at least two nonzero elements in row } v,\\ \text{then } R \text{ cannot be } w\text{-star matrix for any } w \in V(G) \setminus \{v\}.\end{gathered} \tag{$*$}$$

Note that given a $(0,1)$-matrix R of G, R is a star matrix if and only if $\{uv \mid [R]_{uv} \neq 0\} \subset E(G)$ and the subgraph of G induced by the edges $\{uv \mid [R]_{uv} \neq 0\}$ is a star of G.

The following lemma is a simple observation but plays a key role in proving Theorem 3.1.

Lemma 3.2. Let G be a graph, S be a subset of $N_G(v)$ with $|S| \geq 2$ for some $v \in V(G)$. Let A be a matrix of G defined by

$$A = \sum_{u \in S} n_u E_{vu},$$

where n_u be a positive real number with $n_u \leq 1$. For a real number ℓ satisfying

$$\max_{u \in S} n_u \leq \ell < \sum_{u \in S} n_u, \tag{3.2}$$

A can be represented by a linear combination of all v-star matrices of G such that each coefficient is a real number in $[0,1]$ and the sum of coefficients is exactly ℓ.

Proof. Let u^* be a vertex in S such that $n_{u^*} = \max_{u \in S} n_u$. Since $|S| \geq 2$, $S \setminus \{u^*\} \neq \emptyset$ and there exists a real number ℓ satisfies (3.2). By subtracting n_{u^*} from each side of (3.2), we have

$$0 \leq \ell - n_{u^*} < \sum_{u \in S \setminus \{u^*\}} n_u.$$

Then there exist real numbers x_u with $0 \le x_u < n_u$ for all $u \in S \setminus \{u^*\}$ satisfying $\ell - n_{u^*} = \sum_{u \in S\setminus\{u^*\}} x_u$.

Now let

$$B = A - \sum_{u \in S\setminus\{u^*\}} x_u E_{vu}. \tag{3.3}$$

Then

$$\begin{aligned} B &= A - \sum_{u \in S\setminus\{u^*\}} x_u E_{vu} \\ &= \sum_{u \in S} n_u E_{vu} - \sum_{u \in S\setminus\{u^*\}} x_u E_{vu} \\ &= n_{u^*} E_{vu^*} + \sum_{u \in S\setminus\{u^*\}} (n_u - x_u) E_{vu}. \end{aligned} \tag{3.4}$$

Note that all coefficients of terms in (3.4) are positive real numbers, and let N be the set of coefficients of terms in (3.4), that is, $N = \{n_u - x_u \mid u \in S \setminus \{u^*\}\} \cup \{n_{u^*}\} = \{a_1, a_2, \ldots, a_k\}$ with $0 = a_0 < a_1 < a_2 < \cdots < a_k$. Note that $\sum_{j=1}^{k} (a_j - a_{j-1}) = a_k = n_{u^*}$. For each $1 \le j \le k$, we define a matrix R_j of G by

$$R_j = \sum_{u:\, [B]_{vu} \ge a_j} E_{vu}.$$

From (3.4), we know that the maximum value among the elements of the matrix B is n_{u^*} and so $u^* \in \{u \mid [B]_{vu} \ge a_j\}$ for each $1 \le j \le k$. Therefore, for each $1 \le j \le k$, $\{u \mid [B]_{vu} \ge a_j\} \ne \emptyset$ and so R_j is well-defined. It is easy to see that R_j is a v-star matrix for each $1 \le j \le k$. Now we will show that

$$B = \sum_{j=1}^{k} (a_j - a_{j-1}) R_j. \tag{3.5}$$

Take two vertices x, y of G. If $\{x,y\} \ne \{v,u\}$ for each $u \in S$, then $[B]_{xy} = 0$ by (3.4) and so $\left[\sum_{j=1}^{k} R_j\right]_{xy} = 0$ by the definition of R_j.

Suppose that $\{x,y\} = \{v,u\}$ for some $u \in S$. Note that $[B]_{vu} \in N$ and so $[B]_{vu} = a_p$ for some $a_p \in N$. Then by the definition of R_j, it holds that $[(a_j - a_{j-1})R_j]_{xy} = (a_j - a_{j-1})$ for all $j \le p$, and $[R_j]_{vu} = 0$ for all $j > p$. Then $\left[\sum_{j=1}^{k} (a_j - a_{j-1}) R_j\right]_{vu} = \sum_{j \le p} (a_j - a_{j-1}) = a_p - a_0 = a_p$ and so $\left[\sum_{j=1}^{k} R_j\right]_{vu} = [B]_{vu}$. Therefore, (3.5) holds.

By (3.3) and (3.5),

$$A = \sum_{j=1}^{k}(a_j - a_{j-1})R_j + \sum_{u\in S\setminus\{u^*\}} x_u E_{vu}.$$

Thus, each matrix on the right hand side of the above equality is v-star matrix of G and the sum of coefficient is

$$\sum_{j=1}^{k}(a_j - a_{j-1}) + \sum_{u\in S\setminus\{u^*\}} x_u = n_{u^*} + (\ell - n_{u^*}) = \ell.$$

Moreover, since $0 \le n_u \le 1$ for each $u \in S$, it follows that $a_j - a_{j-1} \in [0,1]$ for each j. Hence, by adding terms $0 \cdot R$ (the zero matrix) for all v-star matrices R, we can conclude that A is a linear combination of all v-star matrices of G such that each coefficient is a real number in $[0,1]$ the sum of coefficients is exactly ℓ. We complete the proof. ■

Lemma 3.3. Let g be a FEC function of a graph G, A be the g-adjacency matrix of G, and Ω be a set of vertices defined by

$$\Omega = \left\{ v \in V(G) \mid \sum_{u\in N_G(v)} [A]_{uv} > 1 \right\}.$$

If Ω is an empty set or the submatrix of A induced by Ω is a zero matrix, then there exists a function $\alpha : \mathscr{T}(G) \to [0,1]$ such that

(i) for each edge e of G, $\sum_{T\in\mathscr{T}_e(G)} \alpha(T) = g(e)$,

(ii) for each vertex v of G, $\sum_{T\in\mathscr{T}_v(G)} \alpha(T) = 1$.

Proof. We first consider the case where $\Omega = \emptyset$. Then by (3.1), $\sum_{x\in N_G(u)} [A]_{xu} = 1$ for each vertex $u \in V(G)$. We define a function $\alpha : \mathscr{T}(G) \to [0,1]$ by

$$\alpha(T) = \begin{cases} g(e) & \text{if } |E(T)| = 1 \text{ and } E(T) = \{e\} \\ 0 & \text{otherwise} \end{cases}.$$

For an edge $e \in E(G)$, $\sum_{T\in\mathscr{T}_e(G)} \alpha(T) = g(e)$ by the definition of α. For a vertex $v \in V(G)$,

$$\sum_{T\in\mathscr{T}_v(G)} \alpha(T) = \sum_{x\in N_G(u)} g(xu) = \sum_{x\in N_G(u)} [A]_{xu} = 1.$$

Therefore, α satisfies (i) and (ii) when $\Omega = \emptyset$.

Now we consider the case where $\Omega \neq \emptyset$. By assumption, the submatrix of A induced by Ω is a zero matrix. Thus, for each edge xy with $g(xy) > 0$, we have $\{x,y\} \not\subseteq \Omega$ and so either $|\{x,y\} \cap \Omega| = 1$ or $\{x,y\} \cap \Omega = \emptyset$. This implies $\{xy \in E(G) \mid g(xy) > 0\}$ can be partitioned into the two subsets $P_1 = \{xy \in E(G) \mid g(xy) > 0, |\{x,y\} \cap \Omega| = 1\}$ and $P_2 = \{xy \in E(G) \mid g(xy) > 0, \{x,y\} \cap \Omega = \emptyset\}$. In fact,

$$P_1 = \bigcup_{x \in \Omega} \{xy \in E(G) \mid g(xy) > 0\}$$

where each union is a disjoint union. Therefore, A can be represented as follows:

$$A = \sum_{x \in \Omega} \sum_{\substack{xy:\ xy \in E(G) \\ g(xy) > 0}} g(xy) E_{xy} + \sum_{\substack{xy:\ xy \in E(G) \\ g(xy) > 0 \\ \{x,y\} \cap \Omega = \emptyset}} g(xy) E_{xy}.$$

For the sake of convenience, let

$$A(x) = \sum_{\substack{xy:\ xy \in E(G) \\ g(xy) > 0}} g(xy) E_{xy}$$

for each $x \in \Omega$. Then

$$A = \sum_{x \in \Omega} A(x) + \sum_{\substack{xy:\ xy \in E(G) \\ g(xy) > 0 \\ \{x,y\} \cap \Omega = \emptyset}} g(xy) E_{xy}. \tag{3.6}$$

For each $x \in \Omega, A(x)$ can be represented by a linear combination of all x-star matrices of G such that the sum of coefficients is exactly 1, and each coefficient is a real number in $[0,1]$. To see why, take a vertex $x \in \Omega$ and let $S = \{y \mid xy \in E(G), g(xy) > 0\}$. Then $S \subseteq N_G(x)$ and

$$A(x) = \sum_{u \in S} g(xu) E_{xu}.$$

In addition, $\sum_{u \in S} g(xu) = \sum_{u \in N_G(v)} g(xu)$. Since $x \in \Omega$, $\sum_{u \in S} g(xu) > 1$. Since $g(xu) \leq 1$ for each $u \in S$, it holds that $|S| \geq 2$. Moreover,

$$\max_{u \in S} g(xu) \leq 1 < \sum_{u \in S} g(xu).$$

Then, by Lemma 3.2, $A(x)$ can be represented by a linear combination of all x-star matrices such that the sum of coefficients is exactly 1, and each coefficient is a real number in $[0,1]$. Therefore, for each $x \in \Omega$, there exist real numbers $a_{(R,x)} \in [0,1]$ for all x-star matrices R such that

$$A(x) = \sum_{R:\ x\text{-star matrix}} a_{(R,x)} R \quad \text{and} \quad \sum_{R:\ x\text{-star matrix}} a_{(R,x)} = 1.$$

Therefore, by (3.6),

$$A = \sum_{x\in\Omega} \sum_{R:\ x\text{-star matrix}} a_{(R,x)} R + \sum_{\substack{xy:\ xy\in E(G) \\ g(xy)>0 \\ \{x,y\}\cap\Omega=\emptyset}} g(xy) E_{xy}. \tag{3.7}$$

Note that each matrix in (3.7) is a star matrix of G. For each star matrix R of G, we denote by $T(R)$ a star of G which is the subgraph induced by $\{xy \mid [R]_{xy} \neq 0\}$. We define a function $\alpha : \mathscr{T}(G) \to [0,1]$ by, for each star $T \in \mathscr{T}(G)$,

$$\alpha(T) = \begin{cases} a_{(R,x)} & \text{if } T = T(R) \text{ for some } x\text{-star matrix } R \text{ where } x \in \Omega \\ g(xy) & \text{if } T = E_{xy} \text{ for some vertices } x, y \text{ with } \{x,y\} \cap \Omega = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Note that α is well-defined. By the definition of α, it is true that

$$\sum_{T\in\mathscr{T}_e(G)} \alpha(T) = (\text{the sum of coefficients of matrices } R \text{ in (3.7) s.t. } T(R) \in \mathscr{T}_e(G)),$$

$$\sum_{T\in\mathscr{T}_v(G)} \alpha(T) = (\text{the sum of coefficients of matrices } R \text{ in (3.7) s.t. } T(R) \in \mathscr{T}_v(G)).$$

We will show that α satisfies (i) and (ii). It is easy to see that for each edge uv, the sum of coefficients of matrices R in (3.7) such that $T(R) \in \mathscr{T}_e(G)$ is equal to $[A]_{uv}$. Since $[A]_{uv} = g(uv)$ by the definition of g-adjacency matrix, we have $\sum_{T\in\mathscr{T}_e(G)} \alpha(T) = g(uv)$, and so (i) holds. Now it remains to show that for each vertex $v \in V(G)$, $\sum_{T\in\mathscr{T}_v(G)} \alpha(T) = 1$. To do so, take a vertex $v \in V(G)$. If $v \in \Omega$, then by the definition of α, it is easy to check that

$$\sum_{T\in\mathscr{T}_v(G)} \alpha(T) = \sum_{R:\ v\text{-star matrix}} a_{(R,v)} = 1.$$

Suppose that $v \notin \Omega$. Then $\sum_{u\in V(G)} [A]_{vu} = 1$. First, we will show that $\sum_{u\in V(G)} [R]_{vu} \in \{0,1\}$ for each star matrix R in (3.7). To reach a contradiction, suppose that $\sum_{u\in V(G)} [R]_{vu} > 1$ for some star matrix R in (3.7). Then by the property $(*)$ for a star matrix which we previously observed right before stating Theorem 3.1, R is a v-star matrix of G and $R \neq E_{vw}$ for any $w \in V(G)$. Therefore R must be one of star matrices in the first summation in (3.7), which implies that R is a x-star matrix for some $x \in \Omega$. Since $v \notin \Omega$, we reach a contradiction. Therefore, $\sum_{u\in V(G)} [R]_{vu} \leq 1$. As R is a (0,1)-matrix, $\sum_{u\in V(G)} [R]_{vu} \in \{0,1\}$ for each star matrix R in (3.7). Therefore for a star matrix R in (3.7), $T(R) \in \mathscr{T}_v(G)$ if $\sum_{u\in V(G)} [R]_{vu} = 1$, and $T(R) \notin \mathscr{T}_v(G)$ if $\sum_{u\in V(G)} [R]_{vu} = 0$. Then

$$\sum_{u\in V(G)} [A]_{uv} = \sum_{M:\ \text{a summand in (3.7)}} \Big(\sum_{u\in V(G)} [M]_{vu} \Big) = \sum_{\substack{R:\ \text{a matrix in (3.7)} \\ \text{with } T(R)\in\mathscr{T}_v(G)}} \alpha(T(R)).$$

Therefore the sum of coefficients of matrices R in (3.7) such that $T(R) \in \mathscr{T}_v(G)$ is equal to $\sum_{u\in V(G)} [A]_{vu}$. Since $\sum_{u\in V(G)} [A]_{vu} = 1$, we complete the proof. ■

Lemma 3.4. Let g be a minimum FEC function of a weighted graph G, A be the g-adjacency matrix of G, and Ω be a set of vertices defined by

$$\Omega = \{v \in V(G) \mid \sum_{u\in N_G(v)} [A]_{uv} > 1\}.$$

Then Ω is an empty set or the submatrix of A induced by Ω is a zero matrix.

Proof. By contradiction. Suppose that Ω is not an empty set and that the submatrix of A induced by Ω is not a zero matrix. Then there exist vertices $u, v \in \Omega$ such that $[A]_{uv} \neq 0$, and so $[A]_{uv} = g(uv) > 0$. Since $\sum_{x\in N_G(v)} [A]_{xv} > 1$ and $\sum_{x\in N_G(u)} [A]_{xu} > 1$, we can find a positive real number r such that

$$r = \min\left\{ \sum_{x\in N_G(v)} [A]_{xv} - 1, \sum_{x\in N_G(u)} [A]_{xu} - 1, g(uv) \right\}.$$

We define a function $g' : E(G) \to [0,1]$ by

$$g'(xy) = \begin{cases} g(xy) - r & \text{if } xy = uv; \\ g(xy) & \text{otherwise.} \end{cases}$$

As $0 < r \leq g(uv)$, g' is well-defined. Furthermore, g' is a FEC function of G. To see why, take a vertex $x \in V(G)$. If $x \in V(G) \setminus \{u, v\}$, we have $\sum_{y\in N_G(x)} g(xy) = \sum_{y\in V(G)} g'(xy)$, and so

$$\sum_{y\in N_G(x)} g'(xy) \geq 1.$$

If $x \in \{u, v\} \subseteq \Omega$, we have $\sum_{y\in N_G(x)} g(xy) \geq 1 + r$ and so

$$\sum_{y\in N_G(x)} g'(xy) = \sum_{y\in N_G(x)\setminus\{u,v\}} g'(xy) + (g(uv) - r) = \sum_{y\in N_G(x)} g(xy) - r \geq 1,$$

by the definition of r. Thus we conclude that $\sum_{y\in N_G(x)} g'(xy) \geq 1$ for every $x \in V(G)$, and so g' is a FEC function of G.

Let ω be an edge weight function of G. Note that ω is a positive real valued function by our assumption. By the definition of g', it is easy to check that $\sum_{e\in E(G)} g'(e)\omega(e) =$

$\sum_{e\in E(G)} g(e)\omega(e) - \omega(uv)r$. Since $\omega(uv)r > 0$, it holds that $\sum_{e\in E(G)} g'(e)\omega(e) < \sum_{e\in E(G)} g(e)\omega(e)$, which contradicts the minimality of g. Hence we complete the proof. ■

Proof of Theorem 3.1. It immediately follows from Lemma 3.3 and Lemma 3.4. ■

4. Characterization of the cores

Throughout this section, we assume that G is a weighted graph with an edge weight function ω where the vertex set of G is $V = \{v_1, v_2, \dots, v_n\}$. Given $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ and $S \subseteq V$, we let

$$z(S) = \sum_{i:v_i \in S} z_i.$$

Recall that given a game (V, c), the core of (V, c) is defined by

$$\mathscr{C}(V,c) = \{z \in \mathbb{R}^n \mid z(V) = c(V) \text{ and } \forall S \subseteq V, z(S) \leq c(S)\},$$

and a game is balanced if the core is not empty. This section investigates the cores of rigid FEC games and relaxed FEC games.

We will present characterization results of the cores for both rigid and relaxed FEC games. Theorem 4.1 and Theorem 4.3 are characterizations for the cores of the rigid and the relaxed FEC games, respectively. First, we see a characterization result of the cores for rigid FEC games. For $E' \subseteq E$, we let $\omega(E') = \sum_{e\in E'} \omega(e)$.

Theorem 4.1. Let (V, c) be the rigid k-EC game associated with a weighted graph G whose edge weight function is ω. Then $z \in \mathscr{C}(V, c)$ if and only if it holds that

(a) $z(V) = c(V)$

(b) for any $T \in \mathscr{T}(G)$, $z(V(T)) \leq \omega(E(T))$.

The following is a characterization of a core element of a rigid EC game given by Liu and Fang [4]. It is easy to see that Theorem 4.2 is well extended to Theorem 4.1.

Theorem 4.2. [4] Let (V, c) be the rigid EC game associated with a weighted graph G whose edge weight function is ω. Then $z \in \mathscr{C}(V, c)$ if and only if it holds that

(a)′ $z(V) = c(V)$

(b)′ for any $T \in \mathscr{T}(G)$, $z(V(T)) \leq \omega(E(T))$.

Now we prove Theorem 4.1.

Proof of Theorem 4.1. The 'only if' part is easy. Let z be an element of $\mathscr{C}(V,c)$. Then by the definition of the core, (a) follows immediately. To show (b), take any $T \in \mathscr{T}(G)$. Then for $V(T) = S$, $z(S) \leq c(S)$ by definition of the core. We define a $g_S : E(G[S]) \to [0,1]$ as $g_S(e) = \begin{cases} 1 & \text{if } e \in E(T) \\ 0 & \text{otherwise} \end{cases}$. Then it is easy to check that g_S is a rigid FEC function for S. Therefore $c(S) \leq \sum_{e \in E(G[S])} g_S(e)\omega(e) = \sum_{e \in E(T)} \omega(e) = \omega(E(T))$, and so (b) holds.

Now we will show the 'if' part. Suppose that z satisfies (a) and (b). It is sufficient to show that $z(S) \leq c(S)$ for any $S \subset V$. Take $S \subset V$. If $G[S]$ has an isolated vertex, $c(S) = \infty$ and so $z(S) \leq \infty = c(S)$ holds. Now we consider the case that $G[S]$ has no isolated vertex. Then there is a minimum rigid FEC function for S $g_S : E_G(S) \to [0,1]$. Let $\mathscr{T}^* = \{T \in \mathscr{T}(G) \mid V(T) \subset S\}$, $\mathscr{T}_v^* := \mathscr{T}_v(G) \cap \mathscr{T}^*$, and $\mathscr{T}_e^* := \mathscr{T}_e(G) \cap \mathscr{T}^*$. Note that $\mathscr{T}^* = \mathscr{T}(G[S])$, $\mathscr{T}_v^* = \mathscr{T}_v(G[S])$, and $\mathscr{T}_e^* = \mathscr{T}_e(G[S])$. By Theorem 3.1, there is a function $\alpha : \mathscr{T}^* \to [0,1]$ satisfying:

i) for each edge $e \in E(G[S])$, $\sum_{T \in \mathscr{T}_e^*} \alpha(T) = g_S(e)$,

ii) for each vertex $v \in S$, $\sum_{T \in \mathscr{T}_v^*} \alpha(T) = 1$.

Therefore

$$z(S) = \sum_{v_i \in S} z_i = \sum_{v_i \in S} z_i \sum_{T \in \mathscr{T}_{v_i}^*} \alpha(T) = \sum_{T \in \mathscr{T}^*} z(V(T))\alpha(T) \tag{4.8}$$

and

$$c(S) = \sum_{e \in E(G[S])} \omega(e) g_S(e) = \sum_{e \in E(G[S])} \omega(e) \sum_{T \in \mathscr{T}_e^*} \alpha(T) = \sum_{T \in \mathscr{T}^*} \omega(E(T))\alpha(T). \tag{4.9}$$

On the other hand, by the hypothesis (b), note that for all $T \in \mathscr{T}^*$,

$$z(V(T)) \leq \omega(E(T)). \tag{4.10}$$

Then

$$z(S) = \sum_{T \in \mathscr{T}^*} z(V(T))\alpha(T) \leq \sum_{T \in \mathscr{T}^*} \omega(E(T))\alpha(T) = c(S),$$

where first equality holds by (4.8), second inequalities hold by (4.10) and the last holds by (4.9). Hence $z(S) \leq c(S)$. ■

Now we give a characterization of a core element of a relaxed FEC game.

Theorem 4.3. Let $(V, \widetilde{c})$ be the relaxed FEC game associated with a weighted graph G whose edge weight function is ω. Then $z \in \mathscr{C}(V, \widetilde{c})$ if and only if it holds that

(a) $z(V) = \widetilde{c}(V)$

(b) $z \in \mathbb{R}^n_+$

(c) for any edge v_iv_j, $z_i + z_j \leq \omega(v_iv_j)$.

Proof. We will show the 'only if' part first. Let z be an element of $\mathscr{C}(V, \widetilde{c})$. By the definition of $\mathscr{C}(V,c)$ (a) holds. Since $\widetilde{c}(S) \leq \widetilde{c}(T)$ for every $S \subseteq T \subseteq V$, $z_i = z(V) - z(V \setminus \{v_i\}) \geq \widetilde{c}(V) - \widetilde{c}(V \setminus \{v_i\}) \geq 0$ for every $v_i \in V$. From this observation, (b) holds immediately. To show (c), take any edge $e^* = v_iv_j$. Then for $S = \{v_i, v_j\}$, $z_i + z_j \leq \widetilde{c}(S)$. Let $\widetilde{g}_S : \widetilde{E}(G[S]) \to [0,1]$ be a relaxed FEC function that has value 1 at edge e^* and value 0 elsewhere. Then $\widetilde{c}(S) \leq \sum_{e \in \widetilde{E}(G[S])} \widetilde{g}_S(e)\omega(e) = \omega(v_iv_j)$. Thus $z_i + z_j \leq \widetilde{c}(S) \leq \omega(v_iv_j)$.

To show the 'if' part, suppose that z satisfies (a), (b) and (c). It is sufficient to show that $z(S) \leq \widetilde{c}(S)$ for any $S \subset V$. Take a subset $S \subset V$. Then there is a relaxed FEC function $\widetilde{g}_S : \widetilde{E}(G[S]) \to [0,1]$ such that $\widetilde{c}(S) = \sum_{e \in \widetilde{E}(G[S])} \widetilde{g}_S(e)\omega(e)$ and $\sum_{x \in N_G(v)} \widetilde{g}_S(vx) \geq 1$ for all $v \in S$. Note that $z \in \mathbb{R}^n_+$ and so $z_i \geq 0$ for any $v_i \in V$. Therefore $z_i \leq z_i \left(\sum_{x \in N_G(v_i)} \widetilde{g}_S(xv_i) \right)$ for any $v_i \in V$. Then

$$
\begin{aligned}
z(S) = \sum_{v_i \in S} z_i \leq \sum_{v_i \in S} z_i \left(\sum_{x \in N_G(v_i)} \widetilde{g}_S(xv_i) \right) &= \sum_{v_i \in S} \sum_{x \in N_G(v_i)} z_i \widetilde{g}_S(xv_i) \\
&= \sum_{v_iv_j \in \widetilde{E}(G[S])} (z_i + z_j)\widetilde{g}_S(v_iv_j).
\end{aligned}
$$

By the hypothesis (c), $(z_i + z_j) \leq \omega(v_iv_j)$ for each edge $v_iv_j \in \widetilde{E}(G[S])$. It implies that

$$
z(S) \leq \sum_{v_iv_j \in \widetilde{E}(G[S])} (z_i + z_j)\widetilde{g}_S(v_iv_j) \leq \sum_{e \in \widetilde{E}(G[S])} \omega(e)\widetilde{g}_S(e) = \widetilde{c}(S),
$$

hence $z(S) \leq \widetilde{c}(S)$. ■

Let (V,c) be a rigid FEC game and $(V,\widetilde{c})$ be a relaxed FEC game, both of which associate with a weighted graph G whose edge weight function is ω. For $S \subset V$, if g_S is a rigid FEC function for S, then

$$
\widetilde{g}_S(e) = \begin{cases} g_S(e) & e \in E(G[S]) \\ 0 & \text{otherwise} \end{cases}
$$

is a relaxed FEC function for S. By definition,

$$
\sum_{e \in \widetilde{E}(G[S])} \widetilde{g}_S(e)\omega(e) = \sum_{e \in E(G[S])} g_S(e)\omega(e)
$$

and so $\widetilde{c}(S) \leq c(S)$. A rigid FEC function for the grand coalition V is a relaxed FEC function for V, and vise versa. Therefore $\widetilde{c}(V) = c(V)$. Now take $z \in \mathscr{C}(V,\widetilde{c})$. Then $z(V) = \widetilde{c}(V) = c(V)$ and, for $S \subset V$, $z(S) \leq \widetilde{c}(S) \leq c(S)$. Thus $z \in \mathscr{C}(V,c)$. Moreover, by Theorem 4.3, $\mathscr{C}(V,\widetilde{c}) \subset \mathbb{R}^n_+$ and therefore $z \in \mathscr{C}(V,c) \cap \mathbb{R}^n_+$. We have shown that $\mathscr{C}(V,\widetilde{c}) \subset \mathscr{C}(V,c) \cap \mathbb{R}^n_+$. On the other hand, if $z \in \mathscr{C}(V,c) \cap \mathbb{R}^n_+$, then z satisfies (a), (b), and (c) of Theorem 4.3 and so $z \in \mathscr{C}(V,\widetilde{c})$. Thus $\mathscr{C}(V,c) \cap \mathbb{R}^n_+ \subset \mathscr{C}(V,\widetilde{c})$. Hence, the core of a rigid FEC game is the nonnegative part of the core of its relaxed game.

Theorem 4.4. Let (V,c) be a rigid FEC game and $(V,\widetilde{c})$ be a relaxed FEC game, both of which associate with a weighted graph G. Then $\mathscr{C}(V,\widetilde{c}) = \mathscr{C}(V,c) \cap \mathbb{R}^n_+$.

5. Concluding Remark

In this paper, we introduced two games, a rigid fractional edge covering game and its relaxed game, as generalizations of a rigid edge covering game and its relaxed game studied by Liu and Fang [4]. Also, we give characterizations of the cores of both games, which are extensions of results in [4].

Let G be a weighted graph with an edge weight function ω. Let M be the incidence matrix of G, that is, a (0,1)-matrix whose rows are labeled as the vertices of G and columns are labeled as the edges of G such that the entry in row v and column e is 1 if and only if the vertex v is an end of e. Note that each of the edge weight function ω and a FEC function for G can be considered as a vector of size $1 \times |E(G)|$. Let $\mathbf{1}$ be the vector whose all entries are 1. Then a vector x of size $1 \times |E(G)|$ is a minimum FEC function for G if and only if x is an optimal solution of the following linear programming:

$$\text{(P)} \qquad \begin{array}{ll} \text{minimize} & \omega x \\ \text{subject to} & Mx \geq \mathbf{1}, \quad x \in [0,1]^{|E(G)|} \end{array}$$

The (LP) in the following is a relaxation of (P), and (DLP) is its LP-dual problem:

$$\text{(LP)} \qquad \begin{array}{ll} \text{minimize} & \omega x \\ \text{subject to} & Mx \geq \mathbf{1}, \quad x \in \mathbb{R}_+^{|E(G)|} \end{array}$$

$$\text{(DLP)} \qquad \begin{array}{ll} \text{maximize} & y(k\mathbf{1}) \\ \text{subject to} & yM \leq \omega, \quad y \in \mathbb{R}_+^{|V(G)|} \end{array}$$

The optimal value of (P) (resp. (LP) and (DLP)) is denoted by opt(P) (resp. opt(LP) and opt(DLP)). It can be easily checked that z is the core of the relaxed FEC game associate with G if and only if z is an optimal solution to (DLP). To see why, suppose that z is in the core of the relaxed FEC game, Theorem 4.3 implies that z is a feasible solution to (DLP) and $z(V) \leq$ opt(LP). Then $z(V) \geq$ opt(LP) $=$ opt(DLP) $\geq z(V)$. Therefore (DLP) has an optimal solution z. Conversely, suppose that z is an optimal solution to (DLP), then $z(V) =$ opt(DLP) $=$ opt(LP) $= \widetilde{c}(V)$. Since z is a feasible solution to (DLP), $z \geq 0$ and $z_i + z_j \leq \omega(v_i v_j)$ for any $v_i v_j \in E$. Thus z is in the core of the relaxed FEC game by Theorem 4.3.

Since (DLP) always has an optimal solution, the core of the relaxed FEC game associated with a weighted graph is always non empty. Thus the relaxed FEC game associated with a weighted graph is always balanced and finding an element of the core can be carried out in polynomial time.

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