Differential equations associated with Genocchi polynomials

Seonhee Kim
Department of Mathematics,
Kyungpook National University,
Daegu, 41566, Republic of Korea.

Byung-Moon Kim
Department of Mech. System Engineering,
Dongguk University,
Gyungju-si, 38066, Republic of Korea.

Jongkyum Kwon
Department of Mathematics
Education and RINS,
Gyeongsang National University,
Jinju, Gyeongsangnam-do, 52828,
Republic of Korea.

Abstract

Recently, several authors have studied nonlinear differential equations arising from the generating functions of various special functions (see [3, 8–15]). We derive a family of non-linear differential equations from the generating functions of the Genocchi polynomials and study the solutions of these differential equations.

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1Corresponding author.

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1. Introduction

The Euler polynomials are defined by the generating function to be

\[ \frac{2}{e^t + 1} e^{tx} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad (\text{see } [4 - 7]). \]  

When \( x = 0, E_n = E_n(0), (n \geq 0) \), are called Euler numbers.

The Genocchi polynomials \( G_n(x) \) are defined by generating functions as follows:

\[ \frac{2t}{e^t + 1} e^{tx} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad (\text{see } [1, 2, 5, 7]). \]

In the special case \( x = 0 \), \( G_n(0) = G_n \) for \( n = 0, 1, \ldots \) are called the \( n \)-th Genocchi numbers.

For \( r \in \mathbb{N} \), we consider that the higher order Genocchi polynomials \( G_n^{(r)}(x) \) are defined by generating functions as follows:

\[ \left( \frac{2t}{e^t + 1} \right)^r e^{tx} = \sum_{n=0}^{\infty} G_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see } [1, 2]). \]

In the special case \( x = 0 \), \( G_n^{(r)}(0) = G_n^{(r)} \) for \( n = 0, 1, \ldots \) are called the higher order \( n \)-th Genocchi numbers.

Recently, T. Kim et al. have studied nonlinear differential equations arising from Bernoulli, Euler, Frobenius-Euler, Changhee, Mittag-Leffler, degenerate Bell, \( \lambda \)-Changhee, degenerate Bernoulli polynomials. (see [3, 8-15]). In this paper, we use the idea recently developed by Kim et al. (see [14]). We derive a family of non-linear differential equations from the generating function of Genocchi polynomials and study the solutions of these differential equations.

2. The non-linear differential equations for Genocchi polynomials

Let

\[ F = F(t) = \frac{2}{e^t + 1}. \]

From (4), we get

\[ F' = F'(t) = \frac{-2e^t}{(e^t + 1)^2} = -F + \frac{1}{2} F^2. \]

It is arranged as following.

\[ \frac{1}{2} F^2 = F + F'. \]
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Let us take derivative in (6) with respect to \( t \). Then we obtain

\[
\frac{1}{2} 2! F' = F' + F''.
\]  

(7)

From (5) and (7), we can derive the following equation.

\[
\frac{1}{2} 2! F^3 = 2F + 3F' + F''.
\]  

(8)

and furthermore

\[
\frac{1}{2} 3! F^4 = 6F + 11F' + 6F'' + F^{(3)}.
\]  

(9)

\[
\frac{1}{2} 4! F^5 = 24F + 50F' + 35F'' + 10F^{(3)} + F^{(4)}.
\]  

(10)

Continuing this process, we set

\[
\frac{1}{2N} N! F^{N+1} = \sum_{k=0}^{N} \alpha_k^{(N)} F^{(k)}, \quad N = 1, 2, \ldots,
\]  

(11)

where \( \left( \frac{d}{dt} \right)^k F(t) = F^{(k)} \).

Let us take derivative in (11) with respect to \( t \). From (5) and (11), we obtain

\[
\frac{1}{2N} (N + 1)! F^N \left( -F + \frac{1}{2} F^2 \right) = \sum_{k=0}^{N} \alpha_k^{(N)} F^{(k+1)}
\]  

so that

\[
\frac{1}{2(N+1)} (N + 1)! F^{N+2} = \sum_{k=0}^{N} \alpha_k^{(N)} F^{(k+1)} + (N + 1) \sum_{k=0}^{N} \alpha_k^{(N)} F^{(k)}.
\]  

(12)

On the other hand, by replacing \( N \) by \( N + 1 \) in (11), we have

\[
\frac{1}{2(N+1)} (N + 1)! F^{N+2} = \sum_{k=0}^{N+1} \alpha_k^{(N+1)} F^{(k)}
\]  

(13)

so that, by (12) and (13), we get the followings.

\[
\alpha_0^{(N+1)} = (N + 1) \alpha_0^{(N)},
\]

\[
\alpha_{N+1}^{(N+1)} = \alpha_N^{(N)}
\]  

(14)

and for \( k = 1, 2, \ldots, N \),

\[
\alpha_k^{(N+1)} = (N + 1) \alpha_k^{(N)} + \alpha_{k-1}^{(N)}.
\]

(15)
We see $\alpha^{(1)}_0 = \alpha^{(1)}_1 = 1$ from (6), so that, from (14) we have the followings.

$$
\alpha^{(N+1)}_0 = (N + 1)\alpha^{(N)}_0 = (N + 1)N\alpha^{(N-1)}_0 \\
= \cdots = (N + 1) \cdots 2\alpha^{(2)}_0 = (N + 1)!, \\
\alpha^{(N+1)}_{N+1} = \alpha^{(N)}_{N} = \cdots = \alpha^{(1)}_1 = 1
$$

(16)

For $k = 1, 2, \ldots, N$, we get from (15),

$$
\begin{align*}
\alpha^{(N+1)}_k &= (N + 1)\alpha^{(N)}_k + \alpha^{(N)}_{k-1} \\
&= (N + 1)(N\alpha^{(N-1)}_k + \alpha^{(N-1)}_{k-1}) + \alpha^{(N)}_{k-1} \\
&= (N + 1) <2^>\alpha^{(N-1)}_k + (N + 1)\alpha^{(N-1)}_{k-1} + \alpha^{(N)}_{k-1} \\
&= (N + 1) <3^>\alpha^{(N-2)}_k + (N + 1) <2^>\alpha^{(N-2)}_{k-1} + (N + 1)\alpha^{(N-1)}_{k-1} \\
&\quad + \alpha^{(N)}_{k-1} \\
&= \cdots \\
&= (N + 1) <N+1-k^>\alpha^{(k)}_k + \sum_{n_1=k}^{N} (N + 1) <N-n_1^>\alpha^{(n_1)}_{k-1} \\
&= \sum_{n_1=k-1}^{N} (N + 1) <N-n_1^>\alpha^{(n_1)}_{k-1}
\end{align*}
$$

(17)

where $N^<_i^> = N(N-1) \cdots (N-i+1)$.

Continuing the work,

$$
\begin{align*}
\alpha^{(N+1)}_k &= \sum_{n_1=k-1}^{N} (N + 1) <N-n_1^> \sum_{n_2=k-2}^{n_1-1} n_1 <n_1-1-n_2^>\alpha^{(n_2)}_{k-2} \\
&= \cdots \\
&= \sum_{n_1=k-1}^{N} \sum_{n_2=k-2}^{n_1-1} \cdots \sum_{n_m=k-m}^{n_{m-1}-1} (N + 1) <N-n_1^> \sum_{n_{m-1}-m}^{n_{m-1}-1} <m_{m-1}-m^>\alpha^{(n_m)}_{k-m} \\
&= \cdots \\
&= \sum_{n_1=k-1}^{N} \sum_{n_2=k-2}^{n_1-1} \cdots \sum_{n_{k-1}-1}^{n_{k-1}-1} (N + 1) <N-n_1^> \sum_{n_{k-1}-k}^{n_{k-1}-1} <n_{k-1}-k^>\alpha^{(n_k)}_{0} \\
&= \cdots
\end{align*}
$$
so that it gives

\[
\alpha_k^{(N+1)} = \sum_{n_1=k-1}^{N} (N + 1)^{<N-n_1>} \sum_{n_2=k-2}^{n_1-1} n_1^{<n_1-1-n_2>} \times \\
\cdots \times \sum_{n_k=0}^{n_k-1} n_k^{<n_k-1-n_1>} n_k^{<n_k>} \\
= \left( \prod_{m=1}^{k} \sum_{n_m=k-m}^{n_m-1} n_m^{<n_m-1-n_m>} \right) (n_k)!,
\]

(18)

where \(n_0 = N + 1\).

Therefore, we obtain the following theorem.

**Theorem 2.1.** Let \(F^{(k)}(t) = \frac{d^k F(t)}{dt^k}\) and \(F^N(t) = \frac{F(t) \times \cdots \times F(t)}{N\text{-times}}\). For \(N = 2, 3, \ldots\), the following non-linear differential equation with respect to \(t\):

\[
\frac{1}{2N!} F^{N+1} = \sum_{k=0}^{N} \alpha_k^{(N)} F^{(k)}
\]

(19)

has a solution \(F = F(t) = \frac{2t}{e^t + 1}\). Where,

\[
\alpha_k^{(N+1)} = \sum_{n_1=k-1}^{N} (N + 1)^{<N-n_1>} \sum_{n_2=k-2}^{n_1-1} n_1^{<n_1-1-n_2>} \times \\
\cdots \times \sum_{n_k=0}^{n_k-1} n_k^{<n_k-1-n_1>} n_k^{<n_k>} \\
= \left( \prod_{m=1}^{k} \sum_{n_m=k-m}^{n_m-1} n_m^{<n_m-1-n_m>} \right) (n_k)!,
\]

(20)

where \(n_0 = N + 1\).
It is not difficult to show that
\[
F^{(k)}(t) = \left( \frac{d}{dt} \right)^k F(t) = \left( \frac{d}{dt} \right)^k \left( \frac{1}{t} \cdot \frac{2t}{e^t + 1} \right)
\]
\[
= \left( \frac{d}{dt} \right)^k \left( \frac{1}{t} \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} \right) = \left( \frac{d}{dt} \right)^k \left( \frac{1}{t} + \sum_{n=1}^{\infty} G_n \frac{t^{n-1}}{n!} \right)
\]
\[
= \left( \frac{d}{dt} \right)^k \left( \frac{1}{t} + \sum_{n=0}^{\infty} \frac{G_{n+1}}{n+1} \cdot \frac{t^n}{n!} \right)
\]
\[
= (-1)^k k! t^{k-1} + \sum_{n=0}^{\infty} \frac{G_{n+k+1}}{n+k+1} \cdot \frac{t^n}{n!}.
\]

From (21), we get
\[
N+1 \sum_{k=0}^{N} \alpha_k^{(N)} F^{(k)} = \sum_{k=0}^{N} \alpha_k^{(N)} (-1)^k k! t^{N-k}
\]
\[
+ \sum_{k=0}^{N} \sum_{n=0}^{\infty} \alpha_k^{(N)} \frac{G_{n+k+1}}{n+k+1} \cdot \frac{t^{N+1+n}}{n!}
\]
\[
= \sum_{n=0}^{N} \left( \alpha_n^{(N)} (-1)^{N-n} (N-n)! n! \right) t^n
\]
\[
+ \sum_{n=N+1}^{N} \left( \sum_{k=0}^{N} \alpha_k^{(N)} n^{<N+1>} \frac{G_{n-N+k}}{n-N+k} \cdot \frac{t^n}{n!} \right).
\]

On the other hand,
\[
\frac{1}{2N} N! F^{N+1} = \frac{1}{2N} N! \left( \frac{2t}{e^t + 1} \right)^N
\]
\[
= \frac{1}{2N} N! \sum_{n=0}^{\infty} G_n^{(N+1)} \frac{t^n}{n!}.
\]

Therefore, by Theorem 2.1, (22) and (23), we obtain the following.

**Theorem 2.2.** For \( n \in \mathbb{N} \cup \{0\} \) and \( N \in \mathbb{N} \), we have the followings.

For \( n \geq N+1 \),
\[
G_n^{(N+1)} = \sum_{k=0}^{N} 2^N \alpha_k^{(N)} (N+1) \left( \binom{n}{N+1} \frac{G_{n-N+k}}{n-N+k} \right).
\]

For \( 0 \leq n \leq N \),
\[
G_n^{(N+1)} = 2^N \alpha_{n-n}^{(N)} (-1)^{N-n} \frac{1}{(\binom{n}{\binom{n}{N+1})}}.
\]
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References


