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Symmetric identities of modified q-Euler polynomials under the symmetric group of degree n

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Abstract

In this paper, we introduce the modified q-Euler polynomials. The main objective of this paper is to consider symmetric identities of the modified q-Euler polynomials under the symmetric group of degree n.

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1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$ and the q-analogue of the number x is defined as $[x]_q = \frac{1-q^x}{1-q}$.

The Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \text{ (see } [1-13]). \tag{1.1}$$

When $x = 0$, $E_n = E_n(0)$, $(n \geq 0)$, are called Euler numbers.

Thus, by (1.3), the Witt's formula for Euler polynomials is given by

$$\int_{\mathbb{Z}_p} e^{(x+y)t} du_{-1}(y) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1.2}$$

That is,

$$\int_{\mathbb{Z}_p} (x+y)^n du_{-1}(y) = E_n(x), (n \geq 0), \text{(see [8, 9, 10])}. \tag{1.3}$$

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) du_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x \text{ (see [1-5, 7, 10-12])}, \tag{1.4}$$

where $f(x)$ is a continuous functions on $\mathbb{Z}_p$.

From (1.1), we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \text{ (see [1 - 5, 7, 10 - 12])}, \tag{1.5}$$

where $f_1(x) = f(x+1)$.

It is not difficult to show that

$$\int_{\mathbb{Z}_p} f_n(x) du_{-1}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} f(x) du_{-1}(x)$$

$$= 2\sum_{l=0}^{n-1}(-1)^{n-1-l} f(l), \text{ (see [1 - 5, 7, 10 - 12])}, \tag{1.6}$$

(1.7)

where $f_n(x) = f(x+n)$ and $n \equiv 1 (mod\ 2)$.

In [8], T. Kim introduced Carlitz-type q-Euler numbers as follows:

$$\mathcal{E}_{0,q} = 1, q(q\mathcal{E}_q + 1)^n + \mathcal{E}_{n,q} = [2]_q\delta_{0,n}, (n \geq 0), \text{ (see [8])}. \tag{1.8}$$

The Carlitz-type q-Euler polynomials are introduced by he generating function as follows:

$$\mathcal{E}_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q}(y), (n \geq 0), \text{ (see [8])}. \tag{1.9}$$

Recently, T. Kim studied Carlitz-type q-Euler numbers and polynomials (see [8]) and some identities of these polynomials (see [13]). we introduce modified q-Euler polynomials which are derived from the fermionic integral on $\mathbb{Z}_p$ different from Carlitz-type q-Euler numbers and polynomials. The purpose of this paper is to investigate symmetric identities of modified q-Euler polynomials under the symmetric group of degree n.

2. Symmetric identities for modified q-Euler polynomials under $\mathbb{S}_n$

In this section, we introduce the modified q-Euler polynomials which are given by the generating function to be

$$\int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} \tilde{\mathcal{E}}_{n,q} \frac{t^n}{n!}, \tag{2.10}$$

when $x = 0$, $\tilde{\mathcal{E}}_{n,q} = \tilde{\mathcal{E}}_{n,q}(0)$, $(n \geq 0)$ are called *modified q-Euler numbers*.

Then, by (2.10), we have

$$\begin{aligned}
\tilde{\mathcal{E}}_{n,q}(x) &= \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-1}(y) \\
&= \int_{\mathbb{Z}_p} \left([x]_q + q^x [y]_q\right)^n d\mu_{-1}(y) \\
&= \sum_{l=0}^{n} \binom{n}{l} q^{lx} \int_{\mathbb{Z}_p} [y]_q^l d\mu_{-1}(y) [x]_q^{n-l} \\
&= \sum_{l=0}^{n} \binom{n}{l} q^{lx} \tilde{\mathcal{E}}_{l,q} [x]_q^{n-l}.
\end{aligned} \tag{2.11}$$

Now, we consider the symmetric identities of modified q-Euler polynomials under the symmetric group of degree n. Let $w_1, w_2, \ldots, w_n \in \mathbb{N}$ such that $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2, \ldots, w_n \equiv 1 \pmod 2$. Then, we consider the following identity.

$$\begin{aligned}
&\int_{\mathbb{Z}_p} e^{\left[(\prod_{j=1}^{n-1} w_j)y + (\prod_{j=1}^{n} w_j)x + w_n \sum_{j=1}^{n-1}\left(\prod_{i=1, i\neq j}^{n-1} w_i\right)k_j\right]_q t} d\mu_{-1}(y) \\
&= \lim_{N\to\infty} \sum_{y=0}^{p^N-1} e^{\left[(\prod_{j=1}^{n-1} w_j)y + (\prod_{j=1}^{n} w_j)x + w_n \sum_{j=1}^{n-1}\left(\prod_{i=1, i\neq j}^{n-1} w_i\right)k_j\right]_q t} (-1)^y \\
&= \lim_{N\to\infty} \sum_{y=0}^{w_n p^N-1} e^{\left[(\prod_{j=1}^{n-1} w_j)y + (\prod_{j=1}^{n} w_j)x + w_n \sum_{j=1}^{n-1}\left(\prod_{i=1, i\neq j}^{n-1} w_i\right)k_j\right]_q t} (-1)^y \\
&= \lim_{N\to\infty} \sum_{m=0}^{w_n-1} \sum_{y=0}^{p^N-1} e^{\left[(\prod_{j=1}^{n-1} w_j)(m+w_n y) + (\prod_{j=1}^{n} w_j)x + w_n \sum_{j=1}^{n}\left(\prod_{i=1, i\neq j}^{n} w_i\right)k_j\right]_q t} \\
&\quad \times (-1)^{m+y}.
\end{aligned} \tag{2.12}$$

From (2.12), we have

$$\begin{aligned}&\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\sum_{i=1}^{n-1}k_i}\int_{\mathbb{Z}_p}e^{\left[(\prod_{j=1}^{n-1}w_j)y+(\prod_{j=1}^{n}w_j)x+w_n\sum_{j=1}^{n-1}\left(\prod_{i=1,i\neq j}^{n-1}w_i\right)k_j\right]_q t}d\mu_{-1}(y)\\&=\lim_{N\to\infty}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}\sum_{m=0}^{w_n-1}\sum_{y=0}^{p^N-1}(-1)^{\sum_{i=1}^{n-1}k_i+m+y}e^{\left[++w_n\sum_{j=1}^{n}\left(\prod_{i=1,i\neq j}^{n}w_i\right)k_j\right]_q t}.\end{aligned}\tag{2.13}$$

Therefore, by (2.12) and (2.13), we obtain the following theorem.

Theorem 2.1. Let $w_1, w_2, \ldots, w_n \in \mathbb{N}$ such that $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2, \ldots,$ $w_n \equiv 1 \pmod 2$. Then, the following expressions

$$\begin{aligned}&\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\sigma(l)}-1}(-1)^{\sum_{i=1}^{n-1}k_i}\\&\times\int_{\mathbb{Z}_p}e^{\left[(\prod_{j=1}^{n-1}w_{\sigma(j)})y+(\prod_{j=1}^{n}w_j)x+w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{i=1,i\neq j}^{n-1}w_{\sigma(i)}\right)k_j\right]_q t}d\mu_{-1}(y)\end{aligned}\tag{2.14}$$

are the same for any $\sigma \in \mathbb{S}_n$, $(n \geq 1)$.

Now, we consider that

$$\begin{aligned}&\left[\left(\prod_{j=1}^{n-1}w_j\right)y+\left(\prod_{j=1}^{n}w_j\right)x+w_n\sum_{j=1}^{n-1}\left(\prod_{i=1,i\neq j}^{n-1}w_i\right)k_j\right]_q\\&=\left[\prod_{j=1}^{n-1}w_j\right]_q\left[y+w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right]_{q^{w_1w_2\cdots w_{n-1}}}.\end{aligned}\tag{2.15}$$

By (2.15), we get

$$\begin{aligned}&\int_{\mathbb{Z}_p}e^{\left[(\prod_{j=1}^{n-1}w_j)y+(\prod_{j=1}^{n}w_j)x+w_n\sum_{j=1}^{n-1}\left(\prod_{i=1,i\neq j}^{n-1}w_i\right)k_j\right]_q t}d\mu_{-1}(y)\\&=\sum_{m=0}^{\infty}\left[(\prod_{j=1}^{n-1}w_j)\right]_q^m\int_{\mathbb{Z}_p}\left[y+w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right]_{q^{w_1w_2\cdots w_{n-1}}}^m d\mu_{-1}(y)\frac{t^m}{m!}\\&=\sum_{m=0}^{\infty}\left[\prod_{j=1}^{n-1}w_j\right]_q^m\tilde{\mathcal{E}}_{m,q^{w_1w_2\cdots w_{n-1}}}\left(w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right)\frac{t^m}{m!}.\end{aligned}\tag{2.16}$$

For $m \geq 0$ and $n \in \mathbb{N}$, from (2.16), we have

$$\begin{aligned}&\int_{\mathbb{Z}_p}\left[(\prod_{j=1}^{n-1} w_j)y + (\prod_{j=1}^{n} w_j)x + w_n\sum_{j=1}^{n-1}\left(\prod_{i=1,i\neq j}^{n-1} w_i\right)k_j\right]_q^m d\mu_{-1}(y)\\&= \left[\prod_{j=1}^{n-1} w_j\right]_q^m \tilde{\mathcal{E}}_{m,q^{w_1w_2\cdots w_{n-1}}}\left(w_nx + w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right).\end{aligned} \tag{2.17}$$

Therefore, by (2.1) and (2.17), we obtain the following theorem.

Theorem 2.2. Let $w_1, w_2, \ldots, w_n \in \mathbb{N}$ such that $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2, \ldots,$ $w_n \equiv 1 \pmod 2$. Then, the following expressions

$$\begin{aligned}&\left[\prod_{j=1}^{n-1} w_{\sigma(j)}\right]_q^m \prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\sigma(l)}-1}(-1)^{\sum_{i=1}^{n-1}k_i}\\&\quad\times\tilde{\mathcal{E}}_{m,q^{w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}}}\left(w_{\sigma(n)}x + w_{\sigma(n)}\sum_{j=1}^{n-1}\frac{k_j}{w_{\sigma(j)}}\right)\end{aligned}$$

are the same for any $\sigma \in \mathbb{S}_n$.

It is not difficult to show that

$$\begin{aligned}&\left[w_nx + w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right]_{q^{w_1w_2\cdots w_{n-1}}}\\&= \frac{[w_n]_q}{\left[(\prod_{j=1}^{n-1} w_j)\right]_q}\left[\sum_{j=1}^{n-1}\left(\prod_{i=1,i\neq j}^{n-1} w_i\right)k_j\right]_{q^{w_n}} q^{w_n\sum_{j=1}^{n-1}\left(\prod_{i=1,i\neq j}^{n-1} w_i\right)k_j}[y + w_nx]_{q^{w_1w_2\cdots w_{n-1}}}.\end{aligned} \tag{2.18}$$

From (2.18), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\left[w_n x + w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right]^m_{q^{w_1w_2\cdots w_{n-1}}} d\mu_{-1}(y) \\
&= \sum_{l=0}^{m}\binom{m}{l}\left(\frac{[w_n]_q}{\left[(\prod_{j=1}^{n-1}w_j)\right]_q}\right)^{m-l}\left[\sum_{j=1}^{n-1}\left(\prod_{i=1,i\neq j}^{n-1}w_i\right)k_j\right]^{m-l}_{q^{w_n}} q^{lw_n\sum_{j=1}^{n-1}\left(\prod_{i=1,i\neq j}^{n-1}w_i\right)k_j} \\
&\quad\times\int_{\mathbb{Z}_p}[y+w_n x]^l_{q^{w_1w_2\cdots w_{n-1}}} d\mu_{-1}(y) \\
&= \sum_{l=0}^{m}\binom{m}{l}\left(\frac{[w_n]_q}{\left[(\prod_{j=1}^{n-1}w_j)\right]_q}\right)^{m-l}\left[\sum_{j=1}^{n-1}\left(\prod_{i=1,i\neq j}^{n-1}w_i\right)k_j\right]^{m-l}_{q^{w_n}} \\
&\quad\times q^{lw_n\sum_{j=1}^{n-1}\left(\prod_{i=1,i\neq j}^{n-1}w_i\right)k_j}\tilde{\mathcal{E}}_{l,q^{w_1w_2\cdots w_{n-1}}}(w_n x).
\end{aligned} \tag{2.19}$$

From (2.13) and (2.19), we get

$$\begin{aligned}
&\left[\prod_{j=1}^{n-1}w_j\right]^m_q\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}(-1)^{\sum_{i=1}^{n-1}k_i}\int_{\mathbb{Z}_p}\left[w_n x + w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right]^m_{q^{w_1w_2\cdots w_{n-1}}} d\mu_{-1}(y) \\
&= \sum_{l=0}^{m}\binom{m}{l}\left[\prod_{j=1}^{n-1}w_j\right]^m_q [w_n]^{m-l}_q\tilde{\mathcal{E}}_{n,q^{w_1w_2\cdots w_{n-1}}}(w_n x)\prod_{s=1}^{n-1}\sum_{k_s=0}^{w_s-1}(-1)^{\sum_{i=1}^{n-1}k_i} \\
&\quad\times\left[\sum_{j=1}^{n-1}\left(\prod_{i=1,i\neq j}^{n-1}w_i\right)k_j\right]^{m-l}_{q^{w_n}} \\
&= \sum_{l=0}^{m}\binom{m}{l}\left[(\prod_{j=1}^{n-1}w_j)\right]^l_q [w_n]^{m-l}_q\tilde{\mathcal{E}}_{n,q^{w_1w_2\cdots w_{n-1}}}(w_n x)\prod_{s=1}^{n-1}\sum_{k_s=0}^{w_s-1}(-1)^{\sum_{i=1}^{n-1}k_i} \\
&\quad\times\left[\sum_{j=1}^{n-1}\left(\prod_{i=1,i\neq j}^{n-1}w_i\right)k_j\right]^{m-l}_{q^{w_n}}.
\end{aligned} \tag{2.20}$$

Therefore, we obtain the following theorem.

Theorem 2.3. Let $w_1, w_2, \ldots, w_n \in \mathbb{N}$ such that $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2, \ldots,$ $w_n \equiv 1 \pmod 2$. Then, the following expressions

$$\sum_{l=0}^{m}\binom{m}{l}\left[(\prod_{j=1}^{n-1} w_{\sigma(j)})\right]_q^l [w_{\sigma(n)}]_q^{m-l}\tilde{\mathcal{E}}_{n,q^{w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}}}(w_n x)$$
$$\times\prod_{s=1}^{n-1}\sum_{k_s=0}^{w_{\sigma(s)}-1}(-1)^{\sum_{i=1}^{n-1}k_i}\left[\sum_{j=1}^{n-1}\left(\prod_{i=1,i\neq j}^{n-1} w_{\sigma(i)}\right)k_j\right]_{q^{w_{\sigma(n)}}}^{m-l}$$

are the same for any $\sigma \in \mathbb{S}_n$.

References

[1] A. Bayad, T. Kim, *Identities involving values of Bernstein, q- Bernoulli, and q-Euler polynomials*, Russ. J. Math. Phys., **18** (2011), 133–143.

[2] L. Carlitz, *q- Bernoulli and Eulerian numbers*, Trans. Amer. Math. Soc., **76** (1954), 332–350.

[3] J.H. Jeong, J.H. Jin, J.W. Park, S. H. Rim , *On the twisted weak q-Euler numbers and polynomials with weight 0*, Proc. Jangjeon Math. Soc., **16** (2013), 157–163.

[4] D.S. Kim, T. Kim, *Identities of symmetry for generalized q-Euler polynomials arising from multivariate fermionic p-adic q-integral on $\mathbb{Z}_p$*, Proc. Jangjeon Math. Soc., **17** (2014), 519–525.

[5] D.S. Kim, T. Kim, *Some symmetric identities for the higher-order q-Euler polynomials related to symmetry group S_3 arising from p-adic q-Ferminoic integral on $\mathbb{Z}_p$*, Filomat, **30** : **7** (2016), 1717–1721.

[6] D.S. Kim, T. Kim, *Three variable symmetric identities involving Carlitz-type q-Euler polynomials*, Math. Sci., **8** (2014), 147–152.

[7] D. S. Kim, T. Kim, S.-H. Rim and J.-J. Seo *A note on symmetric properties of the multiple q-Euler zeta functions and higher-order q-Euler polynomials*, Appl. Math. Sci.(Ruse), **8** (2014), 29–32.

[8] T. Kim, *q-Euler numbers and polynomials associated with p-adic q-integrals*, J. Nonlinear Math. Phys., **14** (2007), 15–27.

[9] T. Kim, *Symmetry p-adic invariant on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials*, J. Difference. Equ., **14** (2008), 1267–1277.

[10] T. Kim, *New approach to q-Euler polynomials of higher-order*, Russ. J. Math. Phys., **17** (2010), 218–225.

[11] T. Kim, *A study on the q-Euler numbers and the fermionic q-integrals of the product of several type q-Bernstein polynomials on $\mathbb{Z}_p$*, Adv. Stud. Contemp. Math., **23** (2013), 5–11.

[12] T. Kim, D. S. Kim , D. V. Dolgy, *Degenerate q-Euler polynomials*, Adv. Difference Equ., **2015** (2015), 13662.

[13] T. Kim, D. S. Kim, H. I. Kwon, J.-J. Seo and D. V. Dolgy, *Some identities of q-Euler polynomials under the symmetric group of degree n*, J. nonlinear Sci. Appl., **9** (2016), 1077–1082.