A Predator-Prey Mathematical Model in a Limited Area

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Abstract
A predator-prey mathematical model is a boundary-value problem for a system of two non-linear differential equations in partial derivatives. A stationary state stability is studied. A variational method is used to build a numerical solution. The dependence of an amplitude and a frequency of damped vibrations on parameters, characterizing the mobility of species, is estimated.

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INTRODUCTION
Most published papers represent a predator-prey mathematical model as a Cauchy problem for the system of ordinary differential equations [1-11]. These models do not take into account spatial distribution of species. Actual populations live in limited areas with different environment characteristics in its various parts [12-23]. For
various reasons (for example, in search of food or habitat availability), some species are inclined to migration through area. Models, considering spatial distribution of the species, describe the population density, and species are believed to be distributed in area. Habitat is believed to be solid, allowing to use the system of differential equations in partial derivatives, widely used at the development of mathematical models of continuous environment with linear properties [24-33].

A PREDATOR-PREY POINT MODEL
A mathematical model describing dynamics of two populations interacting under predator-prey basis was offered by Lotka and Volterra. [3, 24]

\[
\begin{align*}
\frac{du}{d\tau} &= c_1u - a_{12}uv, \\
\frac{dv}{d\tau} &= -c_2v + a_{21}uv.
\end{align*}
\]

(1)

In these equations \( u \) and \( v \) - prey and predator populations are shown correspondingly, \( c_1 \) - the specific rate of prey population growth in predator’s absence, \( a_{12} \) - constant, characterizing the rate of predators’ consumption of prey species, \( c_2 \) - the specific rate of predator’s mortality, \( a_{21} \) - constant characterizing the rate of increased number of predators due to preys’ death. A fixed point of the equation system (1) is \( u_*= c_2 / a_{21}, \quad v_*= c_1 / a_{12} \). This fixed point is stable, host harmonic fluctuations of both populations with a frequency of \( \omega = \sqrt{c_1c_2} \) in its neighborhood [24].

A Volterra model explains one of the reasons for fluctuations in the "predator-prey" system, but it usually does not harmonize with experimental data. For example, the papers [14, 17] contain the data on the population of different types of predators and preys. The authors note that the preys’ and predators’ population extremum coincide in the dynamics - a Volterra model does not provide this result. The population change dynamics cannot be accurately described in the predator-prey system, because it is hard to simultaneously estimate the values of all constants in (1) for the real population [12-18]. The most accurate estimates are possible for birth rate \( c_1 \) and mortality rate \( c_2 \) and, consequently, for the fluctuations frequency [15]. According to statistics on species by square or individual areas [15], values \( u_* = c_2 / a_{21} \) and \( v_* = c_1 / a_{12} \) and, consequently, constants \( a_{12} \) and \( a_{21} \) can be estimated.
A PREY-PREDATOR MODEL IN THE LINEAR AREA

Examples of linear areas are pipelines, roadsides, forest clearings [13, 15, 17, 24]. To create models of populations diffusive in space, the continuum mechanics and physics methods are used [24, 28-31]. A predator-prey mathematical model (1) on the interval with the change of variables [24]

\[ u = \frac{c_1}{a_{21}} u_1, \quad v = \frac{c_1}{a_{12}} u_2, \quad \tau = t / c_1 \]

are represented by a system of two equations evolution [24, 31, 32]

\[
\frac{\partial u_1}{\partial t} = D_1 \frac{\partial^2 u_1}{\partial x^2} + u_1 - u_1 u_2, \\
\frac{\partial u_2}{\partial t} = D_2 \frac{\partial^2 u_2}{\partial x^2} - \gamma u_2 + u_1 u_2. 
\]

In these equations \( x \) is a coordinate, \( t \) is time \( u_1 = u_1(t, x) \) and \( u_2 = u_2(t, x) \) is linear population density, \( D_1 \) and \( D_2 \) - parameters characterizing the species’ mobility, \( \gamma = c_2 / c_1 \).

The value of functions \( u_1 = u_1(t, x) \) and \( u_2 = u_2(t, x) \) is set as initial conditions at the initial time:

at \( t = 0 \) \( u_1(x) = u_{10}(x) \), \( u_2(x) = u_{20}(x) \).

Two following options are considered as boundary conditions for a segment of \( l \) length:

\[
\left. \frac{\partial u_1}{\partial x} \right|_{x=0} = 0, \quad \left. \frac{\partial u_1}{\partial x} \right|_{x=l} = 0, \quad \left. \frac{\partial u_2}{\partial x} \right|_{x=0} = 0, \quad \left. \frac{\partial u_2}{\partial x} \right|_{x=l} = 0
\]

and

\[
\left. u_1 \right|_{x=0} = 0, \quad \left. \frac{\partial u_1}{\partial x} \right|_{x=l} = 0, \quad \left. u_2 \right|_{x=0} = 0, \quad \left. \frac{\partial u_2}{\partial x} \right|_{x=l} = 0.
\]

The condition of bringing functions \( u_1 \) and \( u_2 \) to zero on a segment border corresponds to inability of population to exist at this point, and the condition of bringing derivatives \( \frac{\partial u_1}{\partial x} \) and \( \frac{\partial u_2}{\partial x} \) to zero (condition of environment filling [24]) allows the free growth of population.
Total population of preys \( (M_1(t)) \) and predators \( (M_2(t)) \) on the segment at the time point \( t \) are calculated by the formulas

\[
M_1 = \int_{0}^{l} u_1(t, x) \, dx, \quad M_2 = \int_{0}^{l} u_2(t, x) \, dx.
\]

**STABILITY OF SOLUTIONS**

The trivial solution \( u_1 = 0, \ u_2 = 0 \) satisfies the system of equations (2) at boundary conditions (3) or (4) in the stationary case. Disturbance of this equilibrium state is represented as \([30-33] u_1 = \delta u_1, \ u_2 = \delta u_2,\) where \(\delta u_1\) and \(\delta u_2\) are small compared to the unit value: \(0 \leq \delta u_1 \ll 1, \ 0 \leq \delta u_2 \ll 1.\) Then the equations (2) with accuracy of the second order infinitesimal \([30, 31]\) are brought to:

\[
\frac{\partial \delta u_1}{\partial t} = D_1 \frac{\partial^2 \delta u_1}{\partial x^2} + \delta u_1, \\
\frac{\partial \delta u_2}{\partial t} = D_2 \frac{\partial^2 \delta u_2}{\partial x^2} - \gamma \delta u_2. 
\]

The solution of the first equation satisfying the boundary conditions (3) is represented as a trigonometric series

\[
\delta u_1 = \sum_{k=0}^{\infty} A_k(t) \cos k\pi \frac{x}{l}.
\]

Hence, expansion coefficients have to satisfy the equations \([30, 31]\)

\[
\frac{dA_0}{dt} = A_0, \quad \frac{dA_k}{dt} = -D_1 \left( \frac{k\pi}{l} \right)^2 A_k + A_k, \quad k = 1, 2, \ldots
\]

It is seen from the first equation that \( A_0(t) = A_0(0)e^t \) will be an increasing time function. That is solution \( u_1 = 0 \) will be unstable.

The solution of the first equation in (5), satisfying the boundary conditions (4), is represented as a trigonometric series

\[
\delta u_1 = \sum_{k=1}^{\infty} A_k(t) \sin k\pi \frac{x}{l}.
\]

Hence, expansion coefficients have to satisfy the equations

\[
\frac{dA_k}{dt} = \left( 1 - D_1 \left( \frac{k\pi}{2l} \right)^2 \right) A_k, \quad k = 1, 2, \ldots
\]
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It follows that performing the inequality \((2/l/\pi)^2 < D_1\) all coefficients \(A_k\) are to be the decreasing time functions and, accordingly, solution \(u_i = 0\) is stable. This means that the smallest population in the studied model goes extinct at high preys’ mobility. A similar result was obtained in [31] for a single population.

\[ u_1 = \gamma \, , \, u_2 = 1 \] satisfy equations (2) in stationary case for the boundary conditions (3). In the neighborhood of this solution, a solution of equations (2) is represented in the form [30, 31]

\[ u_i = \gamma + \delta u_i, \, u_2 = \gamma + \delta u_2, \]

where \(\delta u_1\) and \(\delta u_2\) are small, compared to the unit value: \(|\delta u_1| \ll 1, |\delta u_2| \ll 1\). Then the equations (2) with accuracy of the second order infinitesimal are brought to

\[
\frac{\partial^2 \delta u_i}{\partial t} = D_1 \frac{\partial^2 \delta u_i}{\partial x^2} - \gamma \delta u_2, \\
\frac{\partial^2 \delta u_2}{\partial t} = D_2 \frac{\partial^2 \delta u_2}{\partial x^2} + \delta u_1.
\]

The solution of these equations is represented as a trigonometric series [30, 31]

\[ u_1 = \sum_{k=0}^{\infty} A_k(t) \cos \frac{k\pi x}{l}, \, u_2 = \sum_{k=0}^{\infty} B_k \cos \frac{k\pi x}{l} \]

The expansion coefficients \(A_k\) and \(B_k\) have to satisfy the system of ordinary differential equations \((k = 0, 1, 2, ... )\)

\[
\frac{dA_k}{dt} = -D_1 \left( \frac{k\pi}{l} \right)^2 A_k - \gamma B_k, \\
\frac{dB_k}{dt} = -D_2 \left( \frac{k\pi}{l} \right)^2 B_k + A_k.
\]

The Jacobian matrix eigenvalues of right-hand side of these equations satisfy quadratic equation

\[ \lambda_k^2 + (D_1 + D_2) \left( \frac{k\pi}{l} \right)^2 \lambda_k + D_1D_2 \left( \frac{k\pi}{l} \right)^4 + \gamma = 0. \]

At \(k = 1, 2, ...\) \(\lambda_k\) will have negative real parts, and \(\lambda_0 = \pm i\sqrt{\gamma}\). That is all the coefficients \(A_k\) and \(B_k\) at \(k = 1, 2, ...\) will be the decreasing time functions, and
coefficients $A_0$ and $B_0$ will change harmonically. That is, at small deviations from fixed solution $u_1 = \gamma, \ u_2 = 1$ periodic fluctuations will appear, and the solution tends to homogeneity through time by the spatial variable.

NUMERICAL SOLUTION

An analytical solution of the nonlinear equations (2) is considered to be impossible. Therefore, various methods of approximations of equations (2) or their solutions are used. The finite-difference approximation of the equations and variational methods based on solutions as a linear combination of analytic functions are most distributed [34-40]. The numerical solution of the equations (2), satisfying the boundary conditions (4), at the interval is being sought as a sum of trigonometric functions

$$u_1 = \sum_{k=1}^{n} A_k(t) \sin \left( k\pi \frac{x}{l} \right), \quad u_2 = \sum_{k=1}^{n} B_k(t) \sin \left( k\pi \frac{x}{l} \right). \quad (6)$$

The system of functions $\sin(k\pi - \pi/2)x/l \ (k = 1, 2,...)$ satisfies the boundary conditions (4), is full and orthogonal at the segment $[0, l]$. After substitution of expressions (6) into equations (2), the multiplication of the latter by $\sin(k\pi - \pi/2)x/l \ (k = 1, 2,...)$ and subsequent integration by the interval $[0, l]$, the system of ordinary differential equations is received for coefficients $A_k$ and $B_k \ (k = 1, 2,...)$

$$\frac{dA_k}{dt} = -D_1 \left( \frac{k\pi - \pi/2}{l} \right)^2 A_k + \frac{2}{l} \int_0^l (u_1 - u_2) \sin(k\pi - \pi/2)x/l \ dx,$$

$$\frac{dB_k}{dt} = -D_2 \left( \frac{k\pi - \pi/2}{l} \right)^2 B_k + \frac{2}{l} \int_0^l (-\gamma u_2 + u_1u_2) \sin(k\pi - \pi/2)x/l \ dx. \quad (7)$$

For one expansion term ($n=1$) in (5) for the segment of single length ($l=1$) coefficients $A_1(t)$ and $B_1(t)$ satisfy the equations (7)

$$\frac{dA_1}{dt} = \left( -D_1 \left( \frac{\pi}{2} \right)^2 + 1 - \frac{8}{3\pi} B_1 \right) A_1,$$

$$\frac{dB_1}{dt} = \left( -D_2 \left( \frac{\pi}{2} \right)^2 - \gamma + \frac{8}{3\pi} A_1 \right) B_1.$$
Non-trivial fixed point of these equations with values

\[
A_i = \frac{8}{3\pi \gamma + D_2 \left( \frac{\pi}{2} \right)^2} \quad \text{and} \quad B_i = \frac{3\pi}{8} \left( 1 - D_1 \left( \frac{\pi}{2} \right)^2 \right)
\]

is implemented, if the inequality \( D_1 < 4/\pi^2 \) is performed.

The eigenvalues of the Jacobian matrix

\[
J = \begin{pmatrix} 0 & -\left( \gamma + D_2 \left( \frac{\pi}{2} \right)^2 \right) \\ 1 - D_1 \left( \frac{\pi}{2} \right)^2 & 0 \end{pmatrix}
\]

in this fixed point will be

\[
\lambda_{1,2} = \pm i\omega, \quad \omega = \sqrt{D_2 \left( \frac{\pi}{2} \right)^2 + \gamma \left( 1 - D_1 \left( \frac{\pi}{2} \right)^2 \right)}.
\]

It follows from these relations in the first approximation, fluctuations occur in the system with frequency \( \omega \). The fluctuations frequency grows with the increased mobility of a predator (parameter \( D_2 \)) and decreases with the increased mobility of a prey (parameter \( D_1 \)). In this case, as it follows from (8), increased predators’ mobility leads to an increase in the amplitude of preys’ fluctuations. The amplitude of predator’s fluctuations does not depend on its mobility in the first approximation.

Figure shows the dependence of the functions \( M_1(t) \) and \( M_2(t) \) from time to time.

This result obtained for the case \( n = 5 \) in (2) with \( \gamma = 0.7, \quad D_1 = 0.01 \) \( \text{and} \quad D_2 = 1 \). In contrast to the model of Volterra (1) the solution of equations (2) tends to the stationary solution, the oscillations are damped.
CONCLUSION
Consideration of the environment inhomogeneity in the Voltaire predator-prey mathematical model leads to results unavailable in the point model. The main of them: the total population depends on species’ mobility both of predators and preys. Preys’ population may go extinct in high species’ mobility. The growth of predators’ mobility decreases the period of fluctuations and increases the preys’ number.

REFERENCES


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