Some identities of symmetry for the degenerate $q$-Bernoulli polynomials under symmetry group of degree $n$

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Abstract

Recently, Kim-Kim introduced some interesting identities of symmetry for q-Bernoulli polynomials under symmetry group of degree n. In this paper, we study the degenerate q-Euler polynomials and derive some identities of symmetry for these polynomials arising from the p-adic q-integral on \( \mathbb{Z}_p \).

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1. Introduction

Let \( p \) be a fixed odd prime number. Throughout this paper, \( \mathbb{Z}_p \), \( \mathbb{Q}_p \) and \( \mathbb{C}_p \) will denote the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers and the completion of the algebraic closure of \( \mathbb{Q}_p \). The \( p \)-adic norm \( | \cdot |_p \) is normalized as \( |p|_p = \frac{1}{p} \). Let \( q \in \mathbb{C}_p \) be an indeterminate such that \( |1 - q|_p < p^{-\frac{1}{p-1}} \). The \( q \)-analogue of the number \( x \) is defined by \( [x]_q = \frac{1 - q^x}{1 - q} \). Let \( f(x) \) be uniformly differentiable function on \( \mathbb{Z}_p \). Then the \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) is defined by Kim to be

\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)\mu_q(x + p^NZ_p)
\]

(1)

In [1], L. Carlitz considered \( q \)-analogue of Bernoulli numbers which are given by recurrence relation to be

\[
\beta_{0,q} = 1, \quad q(\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 
1, & \text{if } n = 1 \\
0, & \text{if } n > 1,
\end{cases}
\]

(2)

with the usual convention about replacing \( \beta_q^n \) by \( \beta_{n,q} \). He defined \( q \)-Bernoulli polynomials as

\[
\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^l x \beta_{l,q}[x]_q^{n-l}, \quad (\text{see } [1, 13]).
\]

(3)

In [13], Kim proved that the Carlitz’s \( q \)-Bernoulli polynomials are represented as the \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) which are given by

\[
\int_{\mathbb{Z}_p} [x + y]^n d\mu_q(y) = \beta_{n,q}(x), \quad (n \geq 0).
\]

(4)
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When $x = 0$, $\beta_{n,q} = \beta_{n,q}(0)$ are the Carlitz $q$-Bernoulli numbers.

In [2], L. Carlitz also introduced the degenerate Bernoulli polynomials which are given by the generating function to be

$$
\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} B_n^*(x) \frac{t^n}{n!}.
$$

(5)

Note that $\lim B_n^*(x) = B_n(x)$, where $B_n(x)$ are ordinary Bernoulli polynomials (see [1-10]). When $x = 0$, $B_n^*(0) = B_n^*(0)$ are called the degenerate Bernoulli numbers. Recently, Kim-Kim introduced (fully) degenerate Bernoulli polynomials which are derived from the $p$-adic invariant integral on $\mathbb{Z}_p$ as follows:

$$
\int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{x+y}{\lambda}} d\mu_1(x) = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.
$$

(6)

where $\lambda, t \in \mathbb{C}_p$ with $|\lambda|_p \leq 1$, $|t|_p < p^{-\frac{1}{p-1}}$, and

$$
\lim \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x).
$$

The (fully) degenerate Bernoulli polynomials are defined by the generating function to be

$$
\frac{\log(1 + \lambda t)}{\lambda(1 + \lambda t)^{\frac{1}{\lambda}} - \lambda} (1 + \lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.
$$

(7)

Note that Kim’s degenerate Bernoulli polynomials are slightly different from the Carlitz’s degenerate Bernoulli polynomials.

From (6) and (7), we note that

$$
\lambda^n \int_{\mathbb{Z}_p} \binom{x+y}{\lambda} \frac{t^n}{n!} d\mu_1(x) = B_n(x) \quad (n \geq 0),
$$

(8)

where $\binom{x+y}{\lambda} = x(x-1) \cdots (x-n+1)$, $n \geq 1$.

In [16], Kim considered degenerate $q$-Bernoulli polynomials which are given by the generating function to be

$$
\int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{x+y}{n}} d\mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,q}(x) \frac{t^n}{n!}.
$$

(9)

When $x = 0$, $\beta_{n,q} = \beta_{n,q}(0)$ are called (fully) degenerate $q$-Bernoulli numbers. Note that $\lim \beta_{n,q} = \beta_{n,q}(x)$, $n \geq 0$.

In this paper, we give some identities of symmetry for the degenerate $q$-Bernoulli polynomials under symmetry group of degree $n$ arising from the $p$-adic $q$-integral on $\mathbb{Z}_p$. 

2. Identities of symmetry for the degenerate $q$-Bernoulli polynomials

We assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda|_p \leq 1$, $|t|_p < p^{-\frac{1}{p-1}}$. In this section, let $w_1, w_2, \ldots, w_n$ be positive integers. For $N \in \mathbb{N}$, we have

$$
\int_{\mathbb{Z}_p} (1 + \lambda t) \frac{1}{q} \left[ w_1 w_2 \cdots w_{n-1} y + w_1 \cdots w_n x + w_n \sum_{j=1}^{n-1} \left( \prod_{i=1, i \neq j}^{n-1} w_i \right) k_j \right]_q d\mu_q w_1 w_2 \cdots w_{n-1} (y)
$$

$$
= \lim_{N \to \infty} \frac{1}{[w_n p^N]_q w_1 \cdots w_{n-1}} \sum_{k=0}^{w_n p^N-1} \sum_{y=0}^{w_n p^N-1} \frac{1}{q} \left[ \left( \sum_{j=1}^{n-1} w_j \right) (k_n + w_n y) + \sum_{j=1}^{n} w_j x + w_n \sum_{j=1}^{n-1} \left( \prod_{i=1, i \neq j}^{n-1} w_i \right) k_j \right]_q
\times (1 + \lambda t)
$$

$$
= \lim_{N \to \infty} \frac{1}{[w_1 \cdots w_n p^N]_q} \prod_{k=0}^{w_n p^N-1} \sum_{y=0}^{w_n p^N-1} \frac{1}{q} \left[ \left( \sum_{j=1}^{n-1} w_j \right) (k_n + w_n y) + \sum_{j=1}^{n} w_j x + w_n \sum_{j=1}^{n-1} \left( \prod_{i=1, i \neq j}^{n-1} w_i \right) k_j \right]_q
\times (1 + \lambda t)
$$

(10)

From (10), we note that

$$
\frac{1}{[w_1 \cdots w_{n-1}]_q} \prod_{i=1}^{n-1} \sum_{k=0}^{w_i-1} q \sum_{j=1}^{n-1} \left( \prod_{i=1, i \neq j}^{n-1} w_i \right) k_j
\times \int_{\mathbb{Z}_p} (1 + \lambda t) \frac{1}{q} \left[ w_1 w_2 \cdots w_{n-1} y + w_1 \cdots w_n x + w_n \sum_{j=1}^{n-1} \left( \prod_{i=1, i \neq j}^{n-1} w_i \right) k_j \right]_q d\mu_q w_1 w_2 \cdots w_{n-1} (y)
$$

$$
= \lim_{N \to \infty} \frac{1}{[w_1 \cdots w_n p^N]_q} \prod_{k=0}^{w_n p^N-1} \sum_{k_n=0}^{w_n p^N-1} \sum_{y=0}^{w_n p^N-1} q \left[ \left( \sum_{j=1}^{n-1} w_j \right) (k_n + w_n y) + \sum_{j=1}^{n} w_j x + w_n \sum_{j=1}^{n-1} \left( \prod_{i=1, i \neq j}^{n-1} w_i \right) k_j \right]_q
\times (1 + \lambda t)
$$

(11)

It is easy to show that (11) is invariant under any permutation in the symmetry group of degree $n$. Therefore, by (11), we obtain the following theorem.
Theorem 2.1. Let $w_1, w_2, \ldots, w_n$ be positive integers. Then, the following expressions

\[
\frac{1}{[w_{\sigma(1)} \cdots w_{\sigma(n-1)}]_q} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_{\sigma(l)}-1} q \left( \prod_{i \neq j}^{n-1} \frac{1}{w_{\sigma(i)} - w_{\sigma(j)}} k_j \right)
\]

\[
\int_{\mathbb{Z}_p} (1 + \lambda t) \frac{1}{[w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(n-1)}]_q} \sum_{j=1}^{n-1} w_{\sigma(j)} (\prod_{i \neq j}^{n-1} w_{\sigma(i)} k_j) q
\]

are the same for any permutation $\sigma$ in the symmetry group of order $n$.

It is not difficult to show that

\[
[w_1 w_2 \cdots w_{n-1} y + w_1 w_2 \cdots w_n x + w_n \sum_{j=1}^{n-1} (\prod_{i \neq j}^{n-1} w_i) k_j]_q
\]

\[
= [w_1 w_2 \cdots w_{n-1}]_q \left[ y + w_n x + \frac{w_n}{w_1} k_1 + \cdots + \frac{w_n}{w_{n-1}} k_{n-1} \right] q^{w_1 w_2 \cdots w_{n-1}} (12)
\]

From (12), we note that

\[
\int_{\mathbb{Z}_p} (1 + \lambda t) \frac{1}{[w_1 w_2 \cdots w_{n-1}]_q} \left[ y + w_n x + \frac{w_n}{w_1} k_1 + \cdots + \frac{w_n}{w_{n-1}} k_{n-1} \right] q^{w_1 w_2 \cdots w_{n-1}} d \mu_q^{w_1 w_2 \cdots w_{n-1}} (y)
\]

\[
= \int_{\mathbb{Z}_p} (1 + \lambda t) \frac{[w_1 w_2 \cdots w_{n-1}]_q}{[w_1 \cdots w_{n-1}]_q} \left[ y + w_n x + \frac{w_n}{w_1} k_1 + \cdots + \frac{w_n}{w_{n-1}} k_{n-1} \right] q^{w_1 w_2 \cdots w_{n-1}} d \mu_q^{w_1 w_2 \cdots w_{n-1}} (y)
\]

\[
\times d \mu_q^{w_1 w_2 \cdots w_{n-1}} (y)
\]

\[
= \sum_{m=0}^{\infty} [w_1 \cdots w_{n-1}]_q \beta_m \frac{\lambda}{[w_1 w_2 \cdots w_{n-1}]_q} q^{w_1 w_2 \cdots w_{n-1}} \left( w_n x + \frac{w_n}{w_1} k_1 + \cdots + \frac{w_n}{w_{n-1}} k_{n-1} \right) t^n
\]

\[
= \frac{t^n}{n!}.
\]

Therefore, by Theorem 2.1 and (13), we obtain the following theorem.

Theorem 2.2. For $m \geq 0$, $w_1, w_2, \ldots, w_n \in \mathbb{N}$, the following expressions

\[
[w_{\sigma(1)} \cdots w_{\sigma(n-1)}]_q^{m-1} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_{\sigma(l)}-1} q \left( \prod_{i \neq j}^{n-1} \frac{1}{w_{\sigma(i)} - w_{\sigma(j)}} k_j \right)
\]

\[
\times \beta_m \frac{\lambda}{[w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(n-1)}]_q} q^{w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(n-1)}} \left( w_{\sigma(n)} x + \frac{w_{\sigma(n)}}{w_{\sigma(1)}} k_1 + \cdots + \frac{w_{\sigma(n)}}{w_{\sigma(n-1)}} k_{n-1} \right)
\]

are the same for any permutation $\sigma$ in the symmetry group of order $n$. 
are the same for any permutation $\sigma$ in the symmetry group of order $n$.

From (9), we note that

$$\sum_{n=0}^{\infty} \beta_{n,\lambda,q}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{[x+y]_q}{\lambda}} d\mu_q(y)$$

$$= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \frac{[x+y]_q}{\lambda} \frac{d\mu_q(x)}{n!} \lambda^n t^n$$

$$= \sum_{n=0}^{\infty} \lambda^n \int_{\mathbb{Z}_p} \frac{[x+y]_q}{\lambda} \frac{d\mu_q(x)}{n!} \lambda^n t^n$$

By comparing the coefficients on the both sides of (14), we get

$$\beta_{n,\lambda,q} = \lambda^n \int_{\mathbb{Z}_p} \frac{[x+y]_q}{\lambda} \frac{d\mu_q(x)}{n!} \lambda^n \beta_{m,q}(x).$$

where $\beta_{m,q}(x)$ are called Carlitz’s $q$-Bernoulli polynomials.

Now, we observe that

$$[y + w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j}] q^{w_1 \cdots w_{n-1}}$$

$$= \frac{[w_n]q}{[w_1 \cdots w_{n-1}]q} \left[ \sum_{j=1}^{n-1} \left( \prod_{i=1 \atop i \neq j}^{n-1} w_i \right) k_j \right] q^{w_n} + q^{w_n} \sum_{j=1}^{n-1} \left( \prod_{i=1 \atop i \neq j}^{n-1} w_i \right) k_j [y + w_n x] q^{w_1 \cdots w_{n-1}}.$$
By (15), we get
\[
\beta_{m, \frac{\lambda}{w_1 \cdots w_{n-1}} q^{w_1 \cdots w_{n-1}}} (w_n^x + \frac{w_n}{w_1} k_1 + \cdots + \frac{w_n}{w_{n-1}} k_{n-1}) \\
= \left( \frac{\lambda}{[w_1 \cdots w_{n-1}]_q} \right)^m \int_{Z_p} \left( \frac{\lambda}{[w_1 \cdots w_{n-1}]_q} \right)^{-l} \left[ y + w_n^x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j} \right]_q^l d\mu_{q^{w_1 \cdots w_{n-1}}}(y) \\
= \left( \frac{\lambda}{[w_1 \cdots w_{n-1}]_q} \right)^m \sum_{l=0}^m S_1(l, l) [w_1 \cdots w_{n-1}]_q^l \lambda^{-l} \\
\times \int_{Z_p} \left[ y + w_n^x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j} \right]_q d\mu_{q^{w_1 \cdots w_{n-1}}}(y).
\]

From (16), we can derive the following equation:
\[
\int_{Z_p} \left[ y + w_n^x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j} \right]_q^l d\mu_{q^{w_1 \cdots w_{n-1}}}(y) \\
= \sum_{s=0}^l \binom{l}{s} \left( \frac{[w_n]_q}{[w_1 \cdots w_{n-1}]_q} \right)^{l-s} \left[ \sum_{j=1}^{n-1} \left( \prod_{i=1, i \neq j}^{n-1} w_i \right) k_j \right]_q^{l-s} \frac{w_n^s \sum_{j=1}^{n-1} \left( \prod_{i=1, i \neq j}^{n-1} w_i \right) k_j}{q^{w_n}} \\
\times \beta_{s, q^{w_1 \cdots w_{n-1}}} (w_n^x).
\]

By (17) and (18), we get
\[
\beta_{m, \frac{\lambda}{w_1 \cdots w_{n-1}} q^{w_1 \cdots w_{n-1}}} (w_n^x + \frac{w_n}{w_1} k_1 + \cdots + \frac{w_n}{w_{n-1}} k_{n-1}) \\
= \sum_{s=0}^p \sum_{p=0}^s \binom{p}{s} S_1(m, p) \lambda^{-p} [w_1 \cdots w_{n-1}]_q^{p-m} [w_n]_q^{s-p} \left[ \sum_{j=1}^{n-1} \left( \prod_{i=1, i \neq j}^{n-1} w_i \right) k_j \right]_q^{p-s} \frac{w_n^s \sum_{j=1}^{n-1} \left( \prod_{i=1, i \neq j}^{n-1} w_i \right) k_j}{q^{w_n}} \\
\times q \beta_{s, q^{w_1 \cdots w_{n-1}}} (w_n^x).
\]
Theorem 2.3.

Let $\sum_{j=1}^{n-1} \prod_{i \neq j}^{n-1} (w_i) k_j$

Therefore, by (20) and (21), we obtain the following theorem.

where $w_n = \sum_{j=1}^{n-1} \prod_{i \neq j}^{n-1} (w_i) k_j$

From (19), we note that

$\sum_{l=1}^{m} \sum_{k_i=0}^{\lambda q - m - p} \left( \sum_{s=0}^{\lambda} \left( \prod_{i=1}^{n-1} w_i \right) k_j \right)^{p-s} \times \beta_{q, w_n} w_n (w_n x)$

$x \times K_{n, q, w_n} (w_1, \cdots, w_{n-1} | p - s, s), \quad (20)$

where

$K_{n, q, w_n} (w_1, \cdots, w_{n-1} | i, t) = \prod_{l=1}^{n-1} \sum_{k_i=0}^{\lambda q - m - p} \left[ \sum_{j=1}^{n-1} \prod_{i \neq j}^{n-1} (w_i) k_j \right]^{i}. \quad (21)$

Therefore, by (20) and (21), we obtain the following theorem.

**Theorem 2.3.** Let $m \geq 0$ and $w_1, w_2, \ldots, w_n \mathbb{N}$, Then the following expressions

$\sum_{p=0}^{m} \sum_{s=0}^{\lambda} \left( \sum_{s=0}^{\lambda} \left( \prod_{i=1}^{n-1} w_i \right) k_j \right)^{p-s} \times \beta_{q, w_n} w_n (w_n x)$

are the same for any permutation $\sigma$ in the symmetry group of order $n$.

Note that some identities of Bernoulli and Euler polynomials are studied by several authors (see [1-19]).
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References


