Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 5 (2016), pp. 4349–4361

http://www.ripublication.com/gjpam.htm

Numerical Treatment of a Generalized Black-Scholes Model for Options Pricing in an Illiquid Financial Market with Transactions Costs

Francis Agana

Financial Mathematics,
Pan African University Institute for Basic Sciences,
Technology and Innovation, Nairobi, Kenya

Oluwole Daniel Makinde

Faculty of Military Science,
Stellenbosch University,
Private Bag X2, Saldanha 7395, South Africa

David Mwangi Theuri

Mathematics Department,
Jomo-Kenyatta University of Agriculture and Technology,
Nairobi, Kenya

Abstract

The classical Black-Scholes equation was derived under strict and unrealistic market assumptions. The existence of transactions costs, price slippage and large traders in a financial market that is not perfectly liquid impact heavily on options prices and therefore cannot be ignored. Particularly, the existence of transactions costs and large traders in an illiquid financial market require the implementation of very powerful dynamic hedging strategies which have some feedback effects on the market volatility and drift of the underlying asset and therefore options prices. In such scenarios, the underlying asset needs to be remodeled to incorporate these feedback effects and an adjusted volatility and drift derived in order to accurately quantify stock and options prices. This, invariably, leads to a generalized and nonlinear Black-Scholes Partial Differential Equation (PDE) with an adjusted volatility which is a function of the second derivative of options prices. In this paper, our goal is to derive a European option pricing equation in an illiquid market with transactions costs. The model problem is tackled numerically using a semi-discretization finite difference method known as the method of lines. The effects of embedded parameter on prices of options are graphically and quantitatively discussed.

AMS subject classification:
Keywords: Optin pricing, Nonlinear Black-Scholes equation, transactions costs, Illiquid market, Semi-discretization method.

1. Introduction

The classical Black-Scholes Equation (BSE) [11] for option pricing was derived under the assumption of a frictionless and perfectly liquid market and therefore only suitable in a liquid market where there are no transactions costs such as fees or taxes. The model was equally derived under the assumption that trading takes place continuously, that all parties have access to any information, ask price equals bid price and credit of any size are available at anytime to an investor. As Barles & Soner [4] observed, the assumption of continuous trading in a market with non-zero transactions costs would be very expensive and therefore invalidating the Black-Scholes arbitrage argument. Empirical evidence also shows that large trading on stocks for various reasons, in a market that is not perfectly liquid, has non negligible impact on stocks and on the cost of replicating an option [12].

The Black-Scholes model (BSM) provides not only a rational option pricing formula but also a hedging portfolio that replicates the contingent claim under strict market assumptions [15]. However, in real life situations, these assumptions are never fulfilled. In arbitrage pricing, the risk tolerance of an investor is irrelevant because the latter is never exposed to any risk. If an arbitrage opportunity exists, investors with different levels of risk aversion would follow the same strategies to profit from the arbitrage [21]. The value of an option is the expected cost of creating a hedging portfolio that replicates the payout of an option [5]. The BSM creates a hedging portfolio which exactly replicates the options payoff on the maturity date but the creation of such a portfolio in an illiquid market with non-zero transactions costs is a difficult task. Non-zero transactions costs greatly influences the trading strategy of an investor and changes the expected cost of replication but its effect is bounded [4, 18]. In such a situation, finding an exact replicating portfolio becomes complicated and therefore many researchers tend to find a surreplicating portfolio which is cheaper than an exact replication [4] in certain situations. Similarly, because financial markets are not perfectly liquid, large trading in the underlying asset for hedging purposes has some feedback effects on the underlying and thus on options [3, 12]. In general, transactions costs, price slippage and large trading in illiquid markets are extremely important factors that influence the prices of options [1, 9, 10, 17].

There have been several successful attempts at remodeling the classical Black-Scholes PDE, by separately taking into account the effect of transactions costs, price slippage and market illiquidity on options prices. This, invariably, results in the extension of the classical and linear BSE into a strongly nonlinear equation where both the volatility and the drift in the equation on time or on the underlying asset price and its derivatives [4, 19]. Leland [18] proposed a modified option replication strategy that features transactions costs whose error is not correlated with the market and thus derived a modified BSE with an adjusted volatility. Barles & Soner [4] showed that the value of the portfolio replicating the long position in continuous time tends to the stock price for any finite transactions costs. Frey & Patie [14], on the other hand, studied the feedback effect of large trading in illiquid markets. Liu & Yong [19] also examined the effects of price impact in an illiquid market in replicating a European contingent claim. Several other authors [8, 13] have also studied the impact of the market on options prices and derived various modified nonlinear BSEs in a more realistic financial market. In our work, we shall study the combined effects of transactions costs and large trading in illiquid markets on options prices. We derive a generalized and nonlinear Black-Scholes PDE in an illiquid financial market with transactions costs. Our results extends that of Barles & Soner [4] and that of Liu & Yong [19] which are obtained as special cases. We also present a numerical solution to our result which arises from the combined effects of transactions costs and large trading in an illiquid market.

Our paper is organized as follows. In section 2, we derive our nonlinear model. In section 3, we provide a numerical analysis based on semi-discretization finite difference method coupled with the fourth order Runge-Kutta-Fehlberg-integration scheme [6, 7, 20] for the nonlinear model of European calls and put options. In section 4, the pertinent results are presented and discussed. Section 5 contains the concluding remarks.

2. Model Problem

The stochastic processes in this paper are modeled on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and are assumed to be $\{\mathcal{F}_t\}_{t\geq 0}$-adapted. We consider a financial market with mainly two continuously trading assets; A risky underlying asset called stock which is traded on exchanges and a risk-free asset called Bond. Our market is formulated in such a way that the Bond can be traded in any large quantity without affecting its price(i.e. perfectly liquid market for Bonds) whiles the stock is a non-dividend paying asset and is traded in an illiquid market with transactions costs for any traded share. The price process of the risk-free asset is given by;

$$dB(t) = B(t)r(t, S(t))dt \tag{2.1}$$

where $r(t, S(t))$ is the interest rate which is dependent on time t and on the underlying stock price S. We assume the existence of a large trader in the stock's market whose trading strategy incurs transactions costs and impacts on the underlying asset's price. In accordance with economic theory, our model is such that the price of the stock falls(rises) when the large trader sells(buys) shares in the underlying asset. The price process of the stock is therefore given by;

$$\frac{dS(t)}{S(t)} = \mu(t, S(t))dt + \sigma(t, S(t))dW(t) + \lambda(t, S(t))dN(t) + \kappa(t, S(t))dN(t) \tag{2.2}$$

where $\mu(t, S(t))$ and $\sigma(t, S(t))$ are respectively the drift and the volatility of the stock in the absence of transactions costs and feedback effects from the market. $W(t)$ is a standard one-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The terms $\lambda(t, S(t)) \geq 0$ and $\kappa(t, S(t)) \geq 0$ are positive real functions that describe, respectively, the feedback effects of the illiquid stock market and the impact of transactions costs on the drift and the volatility of the underlying asset price. $N(t)$ represents the hedgers trading strategy for replication purposes. In a similar argument as in [19], the hedgers trading strategy is a Itô process given by;

$$\begin{aligned} dN(t) &= \eta(t)dt + \zeta(t)dW(t) \quad t \geq 0 \\ N(0) &= N_0 \end{aligned} \tag{2.3}$$

where $\eta(t)$ and $\zeta(t)$ are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes to be determined. For clarity, we shall suppress the parameter notations (t) and $(t, S(t))$ in our subsequent equations. Equation(2.2) then becomes;

$$\frac{dS}{S} = [\mu + \lambda\eta + \kappa\eta]dt + [\sigma + \lambda\zeta + \kappa\zeta]dW(t) \tag{2.4}$$

The trader's budget equation or wealth $\Pi(t)$ thus follows the Stochastic Differential Equation (SDE);

$$d\Pi = [r\Pi + NS(\mu - r + \lambda\eta + \kappa\eta)]dt + NS(\sigma + \lambda\zeta + \kappa\zeta)dW(t) \tag{2.5}$$

Hedging a contingent claim entails finding a self financing portfolio Π at time t such that at the maturity date $T \geq t$, the portfolio equals the payoff($h(S(T))$) of the contingent claim(i.e. $\Pi(T) = h(S(T))$). It is noteworthy that the feedback effects or price impact from transaction costs and large trading has direct consequences on the option's payoff. Therefore in order to perfectly hedge the corresponding contingent claim, we have to solve equations (2.4), (2.3) & (2.5) subject to the terminal boundary condition $\Pi(T) = h(S(T))$;

$$\begin{aligned} dN(t) &= \eta dt + \zeta dW(t) \\ N(0) &= N_0 \\ \frac{dS}{S} &= [\mu + \lambda\eta + \kappa\eta]dt + [\sigma + \lambda\zeta + \kappa\zeta]dW(t) \\ d\Pi &= [r\Pi + NS(\mu - r + \lambda\eta + \kappa\eta)]dt + NS(\sigma + \lambda\zeta + \kappa\zeta)dW(t) \\ \Pi(T) &= h(S(T)) \end{aligned} \tag{2.6}$$

The market liquidity profile $\lambda(t, S(t))$ and the transactions costs determinants $\kappa(t, S(t))$ in equation(2.6) can be determined analytically using empirical evidence [19, 2] or determined using statistical methods [16]. We present, formally, our result in a form of generalized and nonlinear BSE which arises from the relaxation of the unrealistic assumptions of a frictionless and perfectly liquid financial market in the BSM. Every other assumption in the BSM is maintained. The theorem below incorporates the effects of transactions costs and price impact from illiquid market in options prices.

Theorem 2.1. Suppose that $\hat{\mu}, r, \hat{\sigma}, \kappa \geq 0, \lambda \geq 0 : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous and bounded functions $\forall\ (t, S(t)) \in [0, T] \times \mathbb{R}$. Then there exist a unique solution $V(t, S(t))$ to the generalized and nonlinear Black-Scholes PDE for option pricing

$$
\begin{aligned}
\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2\left[1 - [\lambda(t, S(t)) + \kappa(t, S(t))]S\frac{\partial^2 V}{\partial S^2}\right]^2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \\
V(t, S(T)) = h(S(T))
\end{aligned} \tag{2.7}
$$

where the positive functions $\lambda(t, S(t))$ and $\kappa(t, S(t))$ are respectively the price impact function in an illiquid market and the transactions costs determinant. The price process of the underlying stock $S(t)$ in an illiquid market with transactions costs follows the SDE;

$$
\frac{dS(t)}{S(t)} = \hat{\mu}(t, S(t))dt + \hat{\sigma}(t, S(t))dW(t) \tag{2.8}
$$

where

$$
\begin{aligned}
\hat{\sigma}(t, S(t)) &= \frac{\sigma}{1 - (\lambda + \kappa)S\frac{\partial^2 V}{\partial S^2}} \\
\hat{\mu}(t, S(t)) &= \left[\frac{\mu(\lambda + \kappa)\frac{\partial^2 V}{\partial t \partial S}}{1 - (\lambda + \kappa)S\frac{\partial^2 V}{\partial S^2}} + \frac{\sigma(\lambda + \kappa)S^2\frac{\partial^3 V}{\partial S^3}}{2\left(1 - (\lambda + \kappa)S\frac{\partial^2 V}{\partial S^2}\right)^2}\right]
\end{aligned} \tag{2.9}
$$

Here σ and μ are the constant volatility and drift assumed in the BSM. r is the interest rate for investing in a Bond.

Proof. Suppose that (Π, S, N) is a solution to the system(2.6) and suppose further that $\Pi(t) = V(t, S(t))$ the price of a European option. Then by Itô's lemma, we have;

$$
\begin{aligned}
d\Pi &= [r\Pi + NS(\mu - r + \lambda\eta + \kappa\eta)]dt + NS(\sigma + \lambda\zeta + \kappa\zeta)dW = dV \\
dV &= \left(\frac{\partial V}{\partial t} + S\frac{\partial V}{\partial S}[\mu + (\lambda + \kappa)\eta] + \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2}(\sigma + (\lambda + \kappa)\zeta)^2\right)dt \\
&\quad + S\frac{\partial V}{\partial S}[\sigma + (\lambda + \kappa)\zeta]dW(t)
\end{aligned} \tag{2.10}
$$

and by identification of similar components, we deduce that

$$
N(t) = \frac{\partial V(t, S(t))}{\partial S} \tag{2.11}
$$

whence

$$
rV - rS\frac{\partial V}{\partial S} = \frac{\partial V}{\partial t} + \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2}(\sigma + (\lambda + \kappa)\zeta)^2 \tag{2.12}
$$

We apply Itô lemma to equation (2.12)

$$
\begin{aligned}
dN(t) &= d\left(\frac{\partial V(t,S(t))}{\partial S}\right)\\
\eta dt + \zeta dW(t) &= \left(\frac{\partial^2 V}{\partial t \partial S} + S\frac{\partial^2 V}{\partial S^2}[\mu + (\lambda+\kappa)\eta] + \frac{1}{2}S^2\frac{\partial^3 V}{\partial S^3}(\sigma + (\lambda+\kappa)\zeta)^2\right)dt\\
&\quad + S\frac{\partial^2 V}{\partial S^2}[\sigma + (\lambda+\kappa)\zeta]dW(t)
\end{aligned}
\tag{2.13}
$$

also by identification, we have

$$
\begin{aligned}
\zeta(t) &= \frac{\sigma S\frac{\partial^2 V}{\partial S^2}}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}}\\
\eta(t) &= \frac{1}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}}\left(\frac{\partial^2 V}{\partial t\partial S} + \mu S\frac{\partial^2 V}{\partial S^2} + \frac{\sigma S^2\frac{\partial^3 V}{\partial S^3}}{2\left(1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}\right)^2}\right)
\end{aligned}
\tag{2.14}
$$

Thus the underlying stock in the presence of transactions costs and price impact in an illiquid market is modeled as;

$$
\begin{aligned}
\frac{dS(t)}{S(t)} &= (\mu + \lambda\eta + \kappa\eta)dt + (\sigma + \lambda\zeta + \kappa\zeta)dW(t)\\
&= \left[\frac{\mu + (\lambda+\kappa)\frac{\partial^2 V}{\partial t\partial S}}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}} + \frac{\sigma(\lambda+\kappa)S^2\frac{\partial^3 V}{\partial S^3}}{2\left(1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}\right)^2}\right]dt + \left[\frac{\sigma}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}}\right]dW(t)\\
\frac{dS(t)}{S(t)} &= \hat{\mu}(t,S(t))dt + \hat{\sigma}(t,S(t))dW(t)
\end{aligned}
\tag{2.15}
$$

Where

$$
\begin{aligned}
\hat{\sigma}(t,S(t)) &= \frac{\sigma(t,S(t))}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}} \quad ,\\
\hat{\mu}(t,S(t)) &= \left[\frac{\mu(t,S(t)) + (\lambda+\kappa)\frac{\partial^2 V}{\partial t\partial S}}{1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}} + \frac{\sigma(t,S(t))(\lambda+\kappa)S^2\frac{\partial^3 V}{\partial S^3}}{2\left(1-(\lambda+\kappa)S\frac{\partial^2 V}{\partial S^2}\right)^2}\right]
\end{aligned}
\tag{2.16}
$$

are respectively the modified volatility and the modified drift in order to incorporate transactions costs and illiquid market price impact and thus extending and generalizing the BSE. The parameters σ and μ are respectively the constant volatility and drift in the classical BSM. ■

Our result extends the analysis of [4],[14] and [19] and we obtain their result as a special case of our model. Theorem (2.1) is a generalization of the BSM by relaxing the unrealistic assumption of a frictionless and perfectly liquid financial market. Most importantly our model and result is an extension of the results of Barles & Soner [4], Frey & Patie [14] and that of Liu & Yong [19] which can be obtained as special cases of theorem (2.1). For instance, if we assume the existence of price impact of large trading in an illiquid financial market with no transactions costs(i.e. $\kappa(t, (S(t)) = 0)$, equation (2.7) reduces to that obtained in [19] and in [14] with $\lambda(t, S(t)) = \rho$ the liquidity parameter. Also if we assume that the financial market is perfectly liquid(i.e. $\lambda(t, S(t)) = 0)$ but trading in the underlying asset attracts transactions costs, our model then reduces to that of Barles & Soner [4] where $\kappa(t, S(t)) = \dfrac{Sk^2 e^{r(T-t)}}{2}$ with k being the proportional transactions costs. The classical BSE is equally obtain when we assume a frictionless(i.e. $\kappa(t, S(t)) = 0)$ and perfectly liquid market(i.e. $\lambda(t, S(t)) = 0$).

Our hedging strategy is very similar to that of Liu & Young [19] as the impact of transactions costs is absorbed in the market impact parameter $\lambda(t, S(t))$ and therefore does not change the replication strategy employed in an imperfectly liquid market. Thus from equation (2.11), we deduce that the large trader at time $t = 0$ trades $N(0) = \dfrac{\partial V(0, S(0))}{\partial S}$ at a discrete time t and subsequently follows continuous trading strategy defined by $N(t)$.

We shall now present a numerical solution of the generalized and nonlinear Black-Scholes PDE in equation (2.7) subject to the terminal boundary conditions below;

$$\left.\begin{aligned} V(T, S(T)) &= \max(S(T) - K, 0) \\ V(t, L) &= L - Ke^{-r(T-t)}, (\forall L > S(T)) \\ V(t, 0) &= 0 \end{aligned}\right\} \text{(For Call Options)} \tag{2.17}$$

and

$$\left.\begin{aligned} V(T, S(T)) &= \max(K - S(T), 0) \\ V(t, L) &= 0, (\forall L > S) \\ V(t, 0) &= Ke^{-r(T-t)} \end{aligned}\right\} \text{(For Put Options)} \tag{2.18}$$

where K is the strike price and T the expiry time of the European option.

3. Numerical Procedure

Equations (2.7) subject to conditions(2.17) & (2.18) may be considered as a terminal boundary value problem and solved numerically using a semi-discretization finite difference method known as the method of lines [20][6]. A partition of the spatial interval

$0 \leq S \leq L$ into N equal parts is introduced such that the grid size $\Delta S = L/N$ and grid points $S_i = (1-i)\Delta S$, $1 \leq i \leq N+1$. Here, $L > K$ is the maximum price attained by the underlying asset. A finite difference discretization on a uniform grid of Cartesian mesh is employed. Both first and second spatial derivatives in equation (2.7) are approximated with second-order central finite differences. Let $V_i(t)$ be the approximation of $V(t, S_i)$ and consider $\lambda_i(t, S(t)) = \rho$ and $\kappa_i(t, S(t)) = \dfrac{S_i k^2 e^{r(T-t)}}{2}$. Assuming constant interest rate, then the semi-discrete system for the problem becomes

$$\begin{aligned} &\frac{\partial V_i}{\partial t} + \frac{\sigma^2 S_i^2}{2\left[1 - [\lambda + \kappa] S_i \left(\frac{V_{i+1} - 2V_i + V_{i-1}}{(\Delta S)^2}\right)\right]^2} \left(\frac{V_{i+1} - 2V_i + V_{i-1}}{(\Delta S)^2}\right) \\ &\quad + r S_i \left(\frac{V_{i+1} - V_{i-1}}{2\Delta S}\right) - r V_i = 0 \end{aligned} \tag{3.19}$$

with the terminal conditions for the European option given as:

$$\left.\begin{aligned} V_i(T) &= \max(S_i - K, 0), \text{ For Call Option} \\ V_i(T) &= \max(K - S_i, 0), \text{ For Put Option} \end{aligned}\right\}, \quad 1 \leq i \leq N+1 \tag{3.20}$$

The equations corresponding to the first and the last grid points are modified to incorporate the boundary conditions as follows:

$$\left.\begin{aligned} V_1 &= 0 \\ V_{N+1} &= L - K e^{-r(T-t)} \end{aligned}\right\} \text{ (For Call Options)} \tag{3.21}$$

and

$$\left.\begin{aligned} V_1 &= K e^{-r(T-t)} \\ V_{N+1} &= 0 \end{aligned}\right\} \text{ (For Put Options)} \tag{3.22}$$

Equation (3.19) only has one independent variable, so it is first order nonlinear Ordinary Differential Equation (ODE) with known terminal conditions. Therefore the resulting problem can be easily solved iteratively using the forth order Runge-Kutta-Fehlberg integration technique [7]. From the process of numerical computation, the effects of transactions costs and large trading in an illiquid market on the spatio-temporal structure of option price value are obtained and presented graphically and in tabular form below:

4. Results and Discussion

Our result extends the BSM in an imperfectly liquid financial market with transactions costs. The result generalizes that of Frey & Patie [14], Liu & Young [19] and that of Barles & Soner [4] which in turn are obtain as special cases of theorem (2.1). From tables (1) & (2), it is obvious that options valuation in an illiquid financial market in the presence of transactions costs differs from the one predicted by the classical BSM.

Table 1: Effects of Increasing transactions costs and increasing liquidity parameter on the Valuation of a European Call Option where $S = 40$, $K = 40$, $L = 70$, $T = 1$, $\sigma = 0.3$, $r = 0.05$

Call Option Prices				
$\kappa \setminus \rho$	0.00	0.01	0.02	0.03
0.00	5.2610002	5.3099366	5.3585060	5.40671429
0.01	5.2715813	5.3204343	5.3689216	5.41704892
0.02	5.3032086	5.3518128	5.4000552	5.44794122
0.03	5.3555365	5.4037317	5.4515709	5.49905913

Table 2: Effects of Increasing transactions costs and increasing liquidity parameter on the Valuation of a European Put Option where $S = 40$, $K = 40$, $L = 70$, $T = 1$, $\sigma = 0.3$, $r = 0.05$

Put Option Prices				
$\kappa \setminus \rho$	0.00	0.01	0.02	0.03
0.00	3.3164669	3.3654220	3.4140102	3.4622374
0.01	3.3270523	3.3759239	3.4244300	3.4725763
0.02	3.3586922	3.4073153	3.4555766	3.5034817
0.03	3.4110415	3.4592557	3.5071140	3.5546214

Interestingly, as the market become more and more liquid (i.e. $\lambda(t, S(t)) \to 0$) and as the transactions costs become smaller(i.e. $\kappa(t, S(t)) \to 0$), the prices estimated by theorem (2.1) converge very fast to that of the BSM. On the other hand, as the liquidity or transaction cost parameter increases, both call and put options increase in value. This is because an illiquid market or a financial market in the presence of transactions costs has direct impact on the volatility function of the underlying stock and therefore on options which affirms the results of [13] and [4]. A similar trend is observed in an illiquid market with transaction costs(see tables (1) and (2)). This means that in such a market, options are even more valuable. Hence a trader, in an illiquid market, generally needs to buy(short) more asset and to borrow(lend) more to replicate a call(put) option. The excess cost a trader incurs is found to be significant even with small price impact which further invalidates the BSM in an illiquid market. We also found that the incurred transaction costs for a put option seem to be higher than that of an identical call option with the same adjusted volatility and therefore the need to remodel the put-call parity which holds in the case of the BSM. Both call and put prices increase with an increase in the strike price with or without transaction costs and price impact from the market. The spatio-temporal behaviour of European Call and Put options in the presence of transactions costs and price impact from an illiquid market are illustrated in figures (1) and (2). It is observed that an

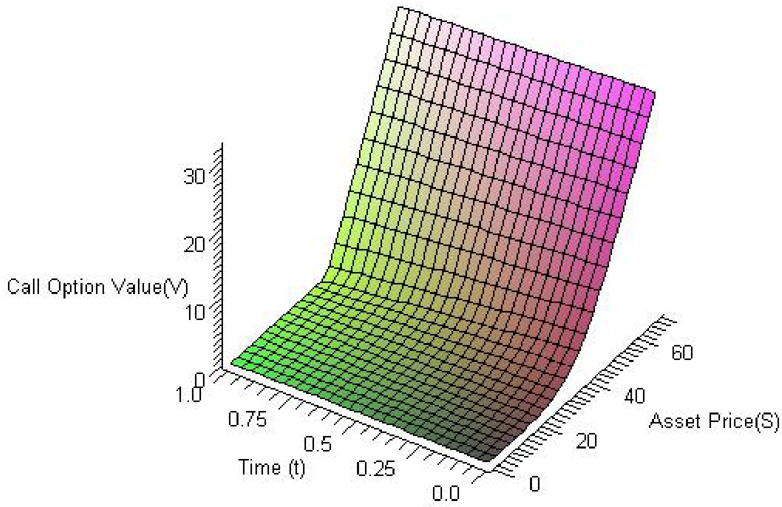

Figure 1: Call Option Value for $\sigma = 0.3$, $r = 0.05$, $T = 1$year, $K = 40$, $L = 70$, $S = 40$, $\lambda = 0.02$, $k = 0.02$

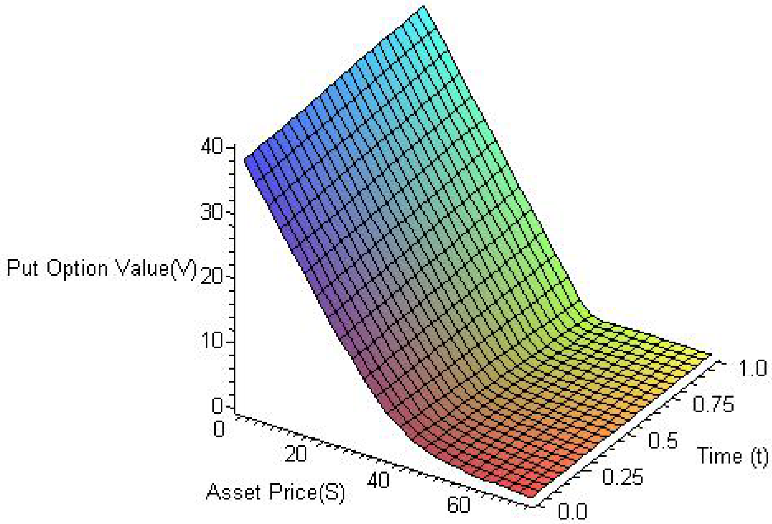

Figure 2: Put Option Value for $\sigma = 0.3$, $r = 0.05$, $T = 1$year, $K = 40$, $L = 70$, $S = 40$, $\lambda = 0.02$, $k = 0.02$

increase in the price of the underlying asset increases the call option value and decreases the put option value as expected. Figures (3) and (4) illustrate the effect of transaction costs and liquidity parameter on the call and put options. The value of both the call and put options increase with a rise in transaction costs and in the market liquidity parameter; hence the option holder will benefit greatly since the call and put options are more likely to end up in-the-money. Meanwhile, as the call and put get more and more in-the-money, the excess cost increases and therefore the adjusted volatility.

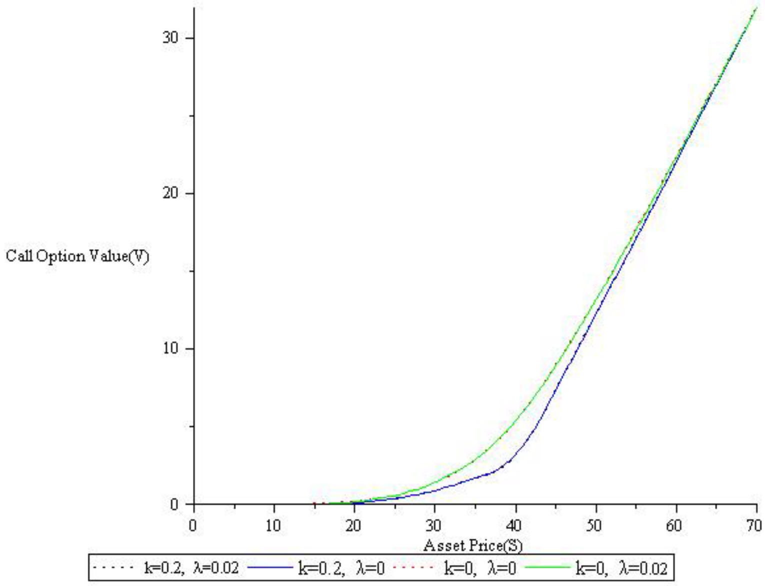

Figure 3: Call Option Value for $\sigma = 0.3$, $r = 0.05$, $T = 1$year, $K = 40$, $L = 70$, $S = 40$

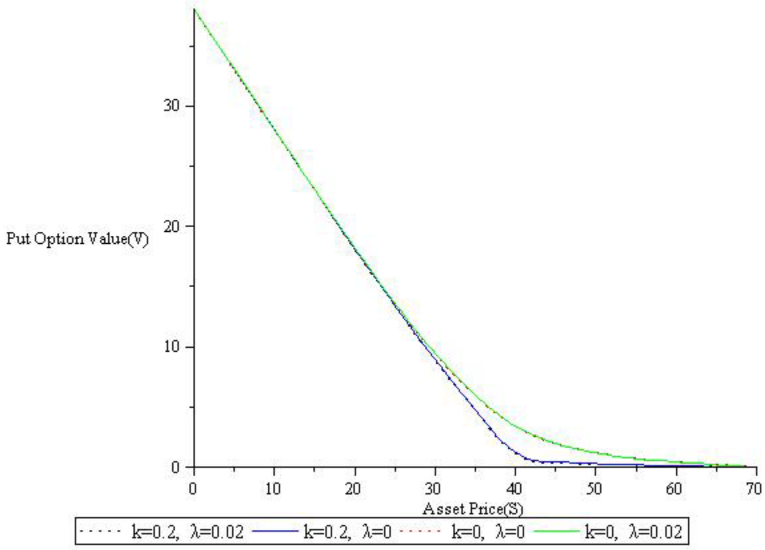

Figure 4: Put Option Value for $\sigma = 0.3$, $r = 0.05$, $T = 1$year, $K = 40$, $L = 70$, $S = 40$

5. Conclusion

In this paper, we derived a generalized and nonlinear Black-Scholes PDE arising in an imperfectly liquid market with transaction costs and presented its spatio-temporal numerical solution. The semi-discretization finite difference method known as method of lines was employed to tackle the problem. Our results revealed that European options become more volatile due to a rise in transaction costs and the price impact from an illiquid market. Our results can be extended to an American option in an illiquid market with transaction and the numerical approach can be applied to other related option pricing problems arising from financial mathematics.

References

[1] Julia Ankudinova and Matthias Ehrhardt. On the numerical solution of nonlinear black–scholes equations. *Computers & Mathematics with Applications*, 56(3):799–812, 2008.

[2] Kerry Back. Asymmetric information and options. *Review of financial studies*, 6(3):435–472, 1993.

[3] Peter Bank and Dietmar Baum. Hedging and portfolio optimization in financial markets with a large trader. *Mathematical Finance*, 14(1):1–18, 2004.

[4] Guy Barles and Halil Mete Soner. Option pricing with transaction costs and a nonlinear black-scholes equation. *Finance and Stochastics*, 2(4):369–397, 1998.

[5] Tomas Björk. *Arbitrage theory in continuous time*. Oxford university press, 2004.

[6] R.L. Burden and J.D. Faires. *Numerical Analysis*. Cengage Learning, 2010.

[7] Julyan HE Cartwright and Oreste Piro. The dynamics of runge–kutta methods. *International Journal of Bifurcation and Chaos*, 2(03):427–449, 1992.

[8] Umut Çetin, H Mete Soner, and Nizar Touzi. Option hedging for small investors under liquidity costs. *Finance and Stochastics*, 14(3):317–341, 2010.

[9] Rafael Company, Lucas Jódar, and José-Ramón Pintos. Numerical analysis and computing for option pricing models in illiquid markets. *Mathematical and Computer Modelling*, 52(7):1066–1073, 2010.

[10] Joseph Eyang?an Esekon. Analytic solution of a nonlinear black-scholes equation. *International Journal of Pure and Applied Mathematics*, 82(4):547–555, 2013.

[11] Myron Scholes Fischer Black. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[12] Rüdiger Frey. Perfect option hedging for a large trader. *Finance and Stochastics*, 2(2):115–141, 1998.

[13] Rüdiger Frey. Market illiquidity as a source of model risk in dynamic hedging. *Model Risk*, pages 125–136, 2000.

[14] Rüdiger Frey and Pierre Patie. Risk management for derivatives in illiquid markets: A simulation study. In *Advances in finance and stochastics*, pages 137–159. Springer, 2002.

[15] J. Hull. *Options, Futures, and Other Derivatives*. Options, Futures, and Other Derivatives. Prentice Hall, 2012.

[16] Anne-Kerstin Kampovsky and Siegfried Trautmann. Price impact and profit of xetra-traders: Does profitability increase with trade size?

[17] Albert S Kyle. Continuous auctions and insider trading. *Econometrica: Journal of the Econometric Society*, pages 1315–1335, 1985.

[18] Hayne E Leland. Option pricing and replication with transactions costs. *The journal of finance*, 40(5):1283–1301, 1985.

[19] Hong Liu and Jiongmin Yong. Option pricing with an illiquid underlying asset market. *Journal of Economic Dynamics and Control*, 29(12):2125–2156, 2005.

[20] L Chris G Rogers and Denis Talay. *Numerical methods in finance*, volume 13. Cambridge University Press, 1997.

[21] Kwok Yue-Kuen. Mathematical models of financial derivatives, 1998.