Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 5 (2016), pp. 4065–4074

http://www.ripublication.com/gjpam.htm

Some symmetric properties about Carlitz's type (h, q)-tangent zeta function

C. S. Ryoo
Department of Mathematics,
Hannam University, Daejeon 306-791, Korea

Abstract

Our aim in this paper is to discover special symmetric properties for (h, q)-tangent polynomials. We are going to find a symmetric identity for (h, q)-tangent zeta function. From property of the (h, q)-tangent zeta function, we derive some symmetric properties of (h, q)-tangent polynomials by combing the basic Definitions or Theorems(1.1-1.7) and some formulas.

AMS subject classification: 11B68, 11S40, 11S80.
Keywords: Symmetric identities of (h, q)-tangent polynomials, symmetric properties of (h, q)-tangent zeta function.

1. Introduction

The area of the Euler, Bernoulli and Genocchi polynomials have been studied by many authors. Those polynomials possess many interesting properties and are of great importance in pure mathematics, for example, number theory, mathematical analysis and in the calculus of finite differences. Those polynomials also have various applications in other branches of science. The tangent polynomials are defined by

$$F(t, x) = \left(\frac{2}{e^{2t} + 1}\right) e^{xt} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}, \quad |t| < \pi.$$

The tangent numbers, $T_n = T_n(0)$, implies that they are rational numbers (see [3]).

Recently, many authors also have been researched the (h, q)-tangent numbers and polynomials because those numbers and polynomials are very important in several filed of mathematics and physics. Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers.

Let q be a complex number with $|q| < 1$ and w be the p^N-th root of unity. Then we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q}.$$

Note that $\lim_{q \to 1} [x]_q = x$ for any x.

Definition 1.1. Let q be a complex number with $|q| < 1$. The (h, q)-tangent polynomials are defined as

$$\sum_{n=0}^{\infty} T_{n,q}^{(h)}(x) \frac{t^n}{n!} = 2 \sum_{m=0}^{\infty} (-1)^m q^{hm} e^{[2m+x]_q t},$$

where we use the technical method's notation by replacing $\left(T_q^{(h)}\right)^n (x)$ by $T_{n,q}^{(h)}(x)$, symbolically. In the special case $x = 0$, $T_{n,q}^{(h)}(0) = T_{n,q}^{(h)}$ are called the n-th (h, q)-tangent numbers. The following elementary properties of the (h, q)- tangent numbers $T_{n,q}^{(h)}$ and polynomials $T_{n,q}^{(h)}(x)$ are readily derived form Definition 1.1 (see, for details, [4]). We, therefore, choose to omit details involved.

Theorem 1.2. Let $q \in \mathbb{C}$ with $|q| < 1$. Then we have

$$T_{n,q}^{(h)}(x) = 2 \sum_{m=0}^{\infty} (-1)^m q^{hm} [x + 2m]_q^n.$$

Theorem 1.3. Let $q \in \mathbb{C}$ with $|q| < 1$. Then we have

$$\begin{aligned} T_{n,q}^{(h)}(x) &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{xl} T_{l,q}^{(h)} \\ &= \left(q^x T_q^{(h)} + [x]_q\right)^n. \end{aligned}$$

Theorem 1.4. Let $x, y \in \mathbb{C}$. Then we have

$$T_{n,q}^{(h)}(x + y) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{xl} T_{l,q}^{(h)}(y).$$

Theorem 1.5. (Property of complement) Let n be a positive integer. Then one has

$$T_{n,q^{-1}}^{(h)}(2 - x) = (-1)^n q^{n+h} T_{n,q}^{(h)}(x)$$

Definition 1.6. For $s \in \mathbb{C}$ and $\mathrm{Re}(s) > 0$, the Hurwitz-type $(h, q)q$-tangent zeta function are defined by

$$\zeta_q^{(h)}(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n q^{hn}}{[2n + x]_q^s}.$$

Note that $\zeta_q^{(h)}(s,x)$ is a meromorphic function on $\mathbb{C}$. Obverse that, if $q \to 1$, then $\zeta_q^{(h)}(s,x) = \zeta_T(s,x)$ which is the Hurwitz tangent zeta functions (see [3]). Relation between $\zeta_q^{(h)}(s,x)$ and $T_{k,q}^{(h)}(x)$ is given by the following theorem.

Theorem 1.7. For $k \in \mathbb{N}$, we have

$$\zeta_q^{(h)}(-k,x) = T_{k,q}^{(h)}(x).$$

Observe that $\zeta_q^{(h)}(-k,x)$ function interpolates $T_{k,q}^{(h)}(x)$ numbers at non-negative integers.

In Section 2, we obtain some interesting symmetric properties of (h,q)-tangent polynomials. In Section 3, we derive some symmetric identities about (h,q)-tangent polynomials.

2. Symmetric properties about Carlitz's type (h, q)-tangent zeta function

In this section, by using the similar method of [1, 2], expect for obvious modifications, we are going to obtain the main results of Carlitz's type (h,q)-tangent polynomials. We also establish some interesting symmetric identities for Carlitz's type (h,q)-tangent polynomials by using Carlitz's type (h,q)-tangent zeta function. Let w_1, w_2 be any positive odd integers. Our main result of symmetry of Carlitz's type (h,q)-tangent zeta function is given the following theorem, which is symmetric in w_1 and w_2.

Theorem 2.1. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and w_1, w_2: odd positive integers. Then one has

$$\begin{aligned}[w_1]_q^s \sum_{i=0}^{w_2-1} (-1)^i q^{hw_1 i} \zeta_{q^{w_2}}^{(h)}\left(s, w_1 x + \frac{2w_1 i}{w_2}\right) \\ = [w_2]_q^s \sum_{j=0}^{w_1-1} (-1)^j q^{hw_2 j} \zeta_{q^{w_1}}^{(h)}\left(s, w_2 x + \frac{2w_2 j}{w_1}\right).\end{aligned}$$

Proof. Observe that $[xy]_q = [x]_{q^y}[y]_q$ for any $x, y \in \mathbb{C}$. By substitute $w_1 x + \dfrac{2w_1 i}{w_2}$ for x in Definition 1.6, replace q by q^{w_2}, we derive

$$\begin{aligned}\zeta_{q^{w_2}}^{(h)}\left(s, w_1 x + \frac{2w_1 i}{w_2}\right) &= 2\sum_{n=0}^{\infty} \frac{(-1)^n q^{hw_2 n}}{[w_1 x + \frac{2w_1 i}{w_2} + 2n]_{q^{w_2}}^s} \\ &= 2[w_2]_q^s \sum_{n=0}^{\infty} \frac{(-1)^n q^{hw_2 n}}{[w_1 w_2 x + 2w_1 i + 2w_2 n]_q^s}.\end{aligned}$$

Since for any non-negative integer m and odd positive integer w_1, there exist unique non-negative integer r such that $m = w_1 r + j$ with $0 \le j \le w_1 - 1$. Hence, this can be written as

$$\zeta_{q^{w_2}}^{(h)}\left(s, w_1x+\frac{2w_1 i}{w_2}\right) = 2[w_2]_q^s \sum_{\substack{w_1r+j=0\\0\le j\le w_1-1}}^{\infty} \frac{(-1)^{w_1r+j}q^{hw_2(w_1r+j)}}{[2w_2(w_1r+j)+w_1w_2x+2w_1i]_q^s}$$
$$= 2[w_2]_q^s \sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty} \frac{(-1)^{w_1r+j}q^{hw_2(w_1r+j)}}{[w_1w_2(2r+x)+2w_1i+2w_2j]_q^s}.$$

It follows from the above equation that

$$\begin{aligned}
&[w_1]_q^s \sum_{i=0}^{w_2-1}(-1)^i q^{hw_1 i}\zeta_{q^{w_2}}^{(h)}\left(s, w_1x+\frac{2w_1 i}{w_2}\right)\\
&= 2[w_1]_q^s[w_2]_q^s \sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty} \frac{(-1)^{r+i+j}q^{h(w_1w_2r+w_1i+w_2j)}}{[w_1w_2(2r+x)+2w_1i+2w_2j]_q^s}.
\end{aligned} \tag{2.1}$$

From the similar method, we can have that

$$\zeta_{q^{w_1}}^{(h)}\left(s, w_2x+\frac{2w_2 j}{w_1}\right) = 2\sum_{n=0}^{\infty}\frac{(-1)^n q^{hw_1 n}}{[w_2x+\frac{2w_2j}{w_1}+2n]_{q^{w_1}}^s}$$
$$= 2[w_1]_q^s\sum_{n=0}^{\infty}\frac{(-1)^n q^{hw_1 n}}{[w_1w_2x+2w_2j+2w_1n]_q^s}.$$

After some calculations in the above, we have

$$\begin{aligned}
&[w_2]_q^s \sum_{j=0}^{w_1-1}(-1)^j q^{hw_2 j}\zeta_{q^{w_1}}^{(h)}\left(s, w_2x+\frac{2w_2 j}{w_1}\right)\\
&= 2[w_1]_q^s[w_2]_q^s \sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty} \frac{(-1)^{r+i+j}q^{h(w_1w_2r+w_1i+w_2j)}}{[w_1w_2(2r+x)+2w_1i+2w_2j]_q^s}.
\end{aligned} \tag{2.2}$$

Thus, we complete the proof of the theorem from (2.1) and (2.2). ■

In Theorem 2.1, we get the following formulas for the (h, q)-tangent zeta function.

Corollary 2.2. Let $w_2 = 1$ in Theorem 2.1. Then we get

$$\zeta_q^{(h)}(s,x) = [w_1]_q^{-s}\sum_{j=0}^{w_1-1}(-1)^j q^{hj}\zeta_{q^{w_1}}^{(h)}\left(s, \frac{x+2j}{w_1}\right).$$

Corollary 2.3. Let $w_1 = 2$, $w_2 = 1$ in Theorem 2.1. Then we have

$$\zeta_{q^2}^{(h)}\left(s, \frac{x}{2}\right) - q^h \zeta_{q^2}^{(h)}\left(s, \frac{x+1}{2}\right) = [2]_q^s \zeta_q(s, x).$$

Theorem 2.4. Let w_1, w_2 be any odd positive integer. Then for non-negative integers n, one has

$$[w_2]_q^n \sum_{i=0}^{w_2-1} (-1)^i q^{hw_1 i} T_{n,q^{w_2}}^{(h)}\left(w_1 x + \frac{2w_1 i}{w_2}\right)$$
$$= [w_1]_q^n \sum_{i=0}^{w_1-1} (-1)^i q^{hw_2 i} T_{n,q^{w_1},w^{w_1}}^{(h)}\left(w_2 x + \frac{2w_2 i}{w_1}\right).$$

Proof. For $n \in \mathbb{N}$, we have

$$\zeta_q^{(h)}(-n, x) = T_{n,q}^{(h)}(x), \text{ (see Theorem 1.7)}.$$

By substituting $T_{n,q}^{(h)}(x)$ for $\zeta_q^{(h)}(s, x)$ in Theorem 2.1, we can derive that

$$[w_1]_q^{-n} \sum_{i=0}^{w_2-1} (-1)^i q^{hw_1 i} \zeta_{q^{w_2}}^{(h)}\left(-n, w_1 x + \frac{2w_1 i}{w_2}\right)$$
$$= [w_1]_q^{-n} \sum_{i=0}^{w_2-1} (-1)^i q^{hw_1 i} T_{n,q^{w_2}}^{(h)}\left(w_1 x + \frac{2w_1 i}{w_2}\right),$$

and

$$[w_2]_q^{-n} \sum_{j=0}^{w_1-1} (-1)^j q^{hw_2 j} \zeta_{q^{w_1}}^{(h)}\left(-n, w_2 x + \frac{2w_2 j}{w_1}\right)$$
$$= [w_2]_q^{-n} \sum_{j=0}^{w_1-1} (-1)^j q^{hw_2 j} T_{n,q^{w_1}}^{(h)}\left(w_2 x + \frac{2w_2 j}{w_1}\right).$$

Thus, we can complete the proof of the theorem from Theorem 2.1. ■

Considering $w_1 = 1$ in the Theorem 2.4, we obtain as below equation.

$$T_{n,q}^{(h)}(x) = [w_2]_q^n \sum_{j=1}^{w_2-1} (-1)^j q^{hj} T_{n,q^{w_2}}\left(\frac{x+2j}{w_2}\right).$$

We obtain another result by applying the addition theorem for the (h, q)-tangent polynomials (Theorem 1.4).

Theorem 2.5. Let w_1, w_2 be any odd positive integer. Then we have

$$\sum_{l=0}^{n}\binom{n}{l}[w_1]_q^l[w_2]_q^{n-l}T_{n-l,q^{w_2}}^{(h)}(w_1x)\mathcal{T}_{n,l,q^{w_1}}^{(h)}(w_2)$$
$$=\sum_{l=0}^{n}\binom{n}{l}[w_2]_q^l[w_1]_q^{n-l}T_{n-l,q^{w_1}}^{(h)}(w_2x)\mathcal{T}_{n,l,q^{w_2}}^{(h)}(w_1),$$

where $\mathcal{T}_{n,l,q}^{(h)}(k)=\sum_{i=0}^{k-1}(-1)^i q^{(h+2n-2l)i}[2i]_q^l$ is called as the sums of powers.

Proof. From the Theorem 1.4, we have

$$\sum_{i=0}^{w_2-1}(-1)^i q^{hw_1 i}T_{n,q^{w_2}}^{(h)}\left(w_1x+\frac{2w_1i}{w_2}\right)$$
$$=\sum_{i=0}^{w_2-1}(-1)^i q^{hw_1 i}\sum_{l=0}^{n}\binom{n}{l}q^{2w_1(n-l)i}T_{n-l,q^{w_2}}^{(h)}(w_1x)\left(\frac{[w_1]_q}{[w_2]_q}\right)^l[2i]_{q^{w_1}}^l$$
$$=\sum_{l=0}^{n}\binom{n}{l}\left(\frac{[w_1]_q}{[w_2]_q}\right)^l T_{n-l,q^{w_2}}^{(h)}(w_1x)\sum_{i=0}^{w_2-1}(-1)^i q^{hw_1i}q^{2(n-l)w_1i}[2i]_{q^{w_1}}^l.$$

Therefore, we obtain that

$$\begin{aligned}&[w_2]_q^n\sum_{i=0}^{w_2-1}(-1)^i q^{hw_1i}T_{n,q^{w_2}}^{(h)}\left(w_1x+\frac{2w_1i}{w_2}\right)\\&=\sum_{l=0}^{n}\binom{n}{l}[w_1]_q^l[w_2]_q^{n-l}T_{n-l,q^{w_2}}^{(h)}(w_1x)\mathcal{T}_{n,l,q^{w_1}}^{(h)}(w_2),\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}&[w_1]_q^n\sum_{i=0}^{w_1-1}(-1)^i q^{hw_2i}T_{n,q^{w_1}}^{(h)}\left(w_2x+\frac{2w_2i}{w_1}\right)\\&=\sum_{l=0}^{n}\binom{n}{l}[w_2]_q^l[w_1]_q^{n-l}T_{n-l,q^{w_1}}^{(h)}(w_2x)\mathcal{T}_{n,l,q^{w_2}}^{(h)}(w_1).\end{aligned}\tag{2.4}$$

Hence, we complete the proof of the theorem by comparing (2.3) and (2.4). ■

3. Some symmetric identities about (h, q)-tangent polynomials

In this section, we derive the symmetric results by using definition and theorem of (h, q)-tangent polynomials.

Theorem 3.1. Let n, m be any non-negative integers. Then we obtain that

$$\sum_{k=0}^{n}\binom{n}{k}T_{m+k,q}^{(h)}(x+y)q^{(n-k)x}[-x]_q^{n-k} = \sum_{k=0}^{m}\binom{m}{k}T_{k+n,q}^{(h)}(y)q^{(n+k)x}[x]_q^{m-k}.$$

Proof. Observe that

$$[x]_q u + q^x[y+2m]_q(u+v) = [x+y+2m]_q(u+v) - [x]_q v. \tag{3.1}$$

From Definition 1.1, we easily see that

$$2\sum_{m=0}^{\infty}(-1)^m q^{hm} e^{[x+y+2m]_q(u+v)} = 2e^{[x]_q(u+v)}\sum_{m=0}^{\infty}(-1)^m q^{hm} e^{q^x[y+2m]_q(u+v)}.$$

Since $[x+y]_q = [x]_q + q^x[y]_q$, we can find out

$$e^{-[x]_q v}2\sum_{m=0}^{\infty}(-1)^m q^{hm} e^{[x+y+2m]_q(u+v)} = e^{[x]_q u}2\sum_{m=0}^{\infty}(-1)^m q^{hm} e^{q^x[y+2m]_q(u+v)}. \tag{3.2}$$

Note that

$$2\sum_{m=0}^{\infty}(-1)^m q^{hm} e^{[x+y+2m]_q(u+v)} = \sum_{n=0}^{\infty}T_{n,q}^{(h)}(x+y)\frac{(u+v)^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}T_{n+m,q}^{(h)}(x+y)\frac{u^m}{m!}\frac{v^n}{n!}.$$

From the above equation, the left-hand side of (3.2) can be expressed as

$$\begin{aligned}
&e^{-[x]_q v} 2\sum_{m=0}^{\infty}(-1)^m q^{hm} e^{[x+y+2m]_q(u+v)} \\
&= \left(\sum_{n=0}^{\infty}\left(-[x]_q\right)^n \frac{v^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} T_{n+m,q}^{(h)}(x+y)\frac{u^m}{m!}\frac{v^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} T_{k+m,q}^{(h)}(x+y)(-[x]_q)^{n-k}\right)\frac{u^m}{m!}\frac{v^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} q^{(n-k)x} T_{k+m,q}^{(h)}(x+y)[-x]_q^{n-k}\right)\frac{u^m}{m!}\frac{v^n}{n!}.
\end{aligned} \tag{3.3}$$

Observe that

$$\begin{aligned}
2\sum_{m=0}^{\infty}(-1)^m q^{hm} e^{q^x[y+2m]_q(u+v)} &= \sum_{n=0}^{\infty} q^{nx} T_{n,q}^{(h)}(y)\frac{(u+v)^n}{n!} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q^{(n+m)x} T_{n+m,q}^{(h)}(y)\frac{v^n}{n!}\frac{u^m}{m!}.
\end{aligned}$$

From the above equation, the right-hand side of (3.2) can be expressed as follows:

$$\begin{aligned}
&e^{[x]_q u} 2\sum_{m=0}^{\infty}(-1)^m q^{hm} e^{q^x[y+2m]_q(u+v)} \\
&= \left(\sum_{m=0}^{\infty}[x]_q^m \frac{u^m}{m!}\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q^{(n+m)x} T_{n+m,q}^{(h)}(y)\frac{v^n}{n!}\frac{u^m}{m!}\right) \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{m}\binom{m}{k} q^{(k+n)x} T_{n+k,q}^{(h)}(y)[x]_q^{m-k}\right)\frac{v^n}{n!}\frac{u^m}{m!}.
\end{aligned} \tag{3.4}$$

By comparing the coefficients of $\frac{v^n u^m}{n!m!}$ in (3.3) and (3.4), we assert that the theorem is right. ∎

It is obvious that Theorem 1.4 is a special case of Theorem 3.1 by setting $n = 0$ and replacing m by n. As another special case, in view of (2.11), we discover that for any non-negative integers m, n,

$$(-1)^m \sum_{k=0}^{m}\binom{m}{k} q^{n+k} T_{n+k,q}^{(h)}(x) = (-1)^n \sum_{k=0}^{n}\binom{n}{k} q^{1-m-k-h} T_{m+k,q^{-1}}^{(h)}(1-x). \tag{3.5}$$

Theorem 3.2. Let k, n, m be non-negative integers. Then we have

$$(-1)^m \sum_{k=0}^{m+1} \binom{m+1}{k}(k+n+1)q^{k+n}T^{(h)}_{k+n,q}(x)$$
$$+ (-1)^n \sum_{k=0}^{n+1} \binom{n+1}{k}(k+m+1)q^{-(k+m+h-1)}T^{(h)}_{k+m,q^{-1}}(1-x) = 0.$$

Proof. Since $k\binom{m+1}{k} = (m+1)\binom{m}{k-1}$ for any non-negative integers k and m then

$$(-1)^m \sum_{k=0}^{m+1} \binom{m+1}{k}(k+n+1)q^{k+n}T^{(h)}_{k+n,q}(x)$$
$$= (-1)^m(n+1) \sum_{k=0}^{m+1} \binom{m+1}{k}q^{k+n}T^{(h)}_{k+n,q}(x) \tag{3.6}$$
$$+ (-1)^m(m+1) \sum_{k=0}^{m+1} \binom{m}{k-1}q^{k+n}T^{(h)}_{k+n,q}(x).$$

It follows from (3,5) that

$$(-1)^m(n+1) \sum_{k=0}^{m+1} \binom{m+1}{k}q^{k+n}T^{(h)}_{k+n,q}(x)$$
$$= (-1)^{n+1}(n+1) \sum_{k=0}^{n} \binom{n}{k}q^{1-k-m-h-1}T^{(h)}_{k+m+1,q^{-1}}(1-x) \tag{3.7}$$
$$= (-1)^{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k}kq^{1-k-m-h}T^{(h)}_{k+m,q^{-1}}(1-x),$$

and

$$(-1)^m(m+1) \sum_{k=0}^{m+1} \binom{m}{k-1}q^{k+n}T^{(h)}_{k+n,q}(x)$$
$$= (-1)^m(m+1) \sum_{k=0}^{m} \binom{m}{k}q^{k+n+1}T^{(h)}_{k+n+1,q}(x) \tag{3.8}$$
$$= (-1)^{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k}(m+1)q^{1-k-m-h}T^{(h)}_{k+m,q^{-1}}(1-x),$$

Thus, putting (3.7) and (3.8) to the right hand side of (3.6) gives the desired result and this completes the proof. ∎

References

[1] J. Y. Kang, C. S. Ryoo, On Symmetric Property for q-Genocchi Polynomials and Zeta Function, Int. Journal of Math. Analysis, **8(1)** (2014), 9–16.

[2] Y. He, Symmetric identities for Carlitz's q-Bernoulli numbers and polynomials, Advances in Difference Equations, **246**(2013), 10 pages.

[3] C. S. Ryoo, A note on the Tangent Numbers and Polynomials, Adv. Studies Theor. Phys., **7(9)**(2013), 447–454.

[4] C. S. Ryoo, On the (h, q)-tangent numbers and polynomials, Adv. Studies Theor. Phys., **7(19)**(2013), 915–921.

[5] C. S. Ryoo, On Carlitz's type twisted tangent Numbers and Polynomials, Applied Mathematical Sciences, **8(76)** (2014), 3763–3769.