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Eigenvalues and eigenfunctions of a non-local boundary value problem of Sturm–Liouville differential equation

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Abstract

In this paper we study the existence and properties of eigenvalues and eigenfunctions of a non-local boundary value problem of the Sturm-Liouville equation.

AMS subject classification:
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1. Introduction

Problems with non-local boundary conditions arise in various fields of mathematical, physical [2]–[7] and biological [8], [9] sciences. The solvability of boundary value problems with non-local conditions for Sturm-Liouville equation has received great attention from many authors and become a very hot research topic.

A. A. Samarskii and A. V. Bitsadze were originators of such problems. They formulated and investigated the non-local boundary problem for an ellibtic equation [1].

Recently, the authors studied in [11] the existence of eigenvalues and eigenfunctions of the boundary value problem of the Sturm- Liouville differential equation

$$-y'' + q(x)y = \lambda^2 y, \quad 0 \le x \le \pi \tag{1.1}$$

with each one of the two non-local conditions

$$y(0) = 0, \quad y(\xi) = 0, \quad \xi \in (0, \pi], \tag{1.2}$$

and

$$y(\eta) = 0, \;\; y(\pi) = 0, \quad \eta \in [0, \pi) \tag{1.3}$$

Consider the non-local boundary value problem of the Sturm-Liouville equation (1.1) with the nonlocal boundary condition

$$y(\eta) = 0 \text{ and } \; y(\xi) = 0, \quad \text{ and } \quad \eta \in [0, \pi), \quad \xi \in (0, \pi]. \tag{1.4}$$

Here we study the existence and some general properties of the eigenvalues and eigenfunctions of the non-local boundary value problems (1.1) and (1.4). Comparison with the local boundary value problem problem of equation (1.1) with the local boundary value problem

$$y(0) = 0, \quad y(\pi) = 0$$

will be given.

2. General properties

Here we prove some results concerning the eigenvalues and eigenfunctions of the non-local problem (1.1)–(1.4).

Lemma 2.1. The eigenvalues of the non-local boundary value problem (1.1) and (1.4) are real.

Proof. Let $y_0(x)$ be the eigenfunction that corresponds to the eigenvalue λ_0 of the problem (1.1) and (1.4), then

$$y_0'' + q(x)\; y_0 = \lambda_0^2 y_0 \quad (0 \le x \le \pi), \tag{2.5}$$

and

$$y_0(\eta) = y_0(\xi) = 0 \tag{2.6}$$

Multiplying both sides of (2.5) by $\bar{y_0}$ and then integrating form η to ξ with respect to x, we have

$$-\bar{y_0}\acute{y_0}|_\eta^\xi + \int_\eta^\xi |\acute{y_0}|^2 dx + \int_\eta^\xi q(x)|y_0|^2 dx = \lambda_0^2 \int_\eta^\xi |y_0|^2 dx.$$

using the boundary condition (2.6), we have

$$\lambda_0^2 = \frac{\int_\eta^\xi [q(x)|y_0|^2 + |\acute{y_0}|^2]dx}{\int_\eta^\xi |y_0|^2 dx}.$$

From which it follows the reality of λ_0^2. ■

Lemma 2.2. The eigenfunctions that corresponds to two different eigenvalues of the non-local boundary value problem (1.1) and (1.4) are orthogonal.

Proof. Let $\lambda_1 \neq \lambda_2$ be two different eigenvalues of the non-local boundary value problem (1.1) and (1.4). Let $y_1(x)$, $y_2(x)$ be the corresponding eigenfunctions, then

$$-y_1'' + q(x)\, y_1 = \lambda_1^2\, y_1 \quad (0 \le x \le \pi), \tag{2.7}$$

$$y_1(\eta) = y_1(\xi) = 0 \tag{2.8}$$

and

$$-y_2'' + q(x)\, y_2 = \lambda_2^2\, y_2 \quad (0 \le x \le \pi), \tag{2.9}$$

$$y_2(\eta) = y_2(\xi) = 0 \tag{2.10}$$

Multiplying both sides of (2.7) by $\bar{y}_2$ and integrating with respect to x, we obtain

$$-\int_\eta^\xi y''_1\, \bar{y}_2\, dx + \int_\eta^\xi q(x)\, y_1\, \bar{y}_2\, dx = \lambda_1^2 \int_\eta^\xi y_1\, \bar{y}_2\, dx. \tag{2.11}$$

By taking the complex conjugate of (2.9) and multiply it by y_1 and integrate the resulting expression with respect to x, we have

$$-\int_\eta^\xi y_1\, \bar{y}''_2\, dx + \int_\eta^\xi q(x)\, y_1\, \bar{y}_2\, dx = \lambda_1^2 \int_\eta^\xi y_1\, \bar{y}_2\, dx. \tag{2.12}$$

Subtracting (2.11) from (2.12) and using the boundary conditions of (2.8) and (2.10) we obtain

$$(\lambda_1^2 - \lambda_2^2) \int_\eta^\xi y_1\, \bar{y}_2\, dx = 0, \quad \lambda_1^2 \neq \lambda_2^2.$$

which completes the proof. ■

3. The asymptotic formulas for the solution

Here we study the asymptotic formulas for the solutions of problem (1.1) and (1.4).

Lemma 1.1 deals with the nature the eigenvalues. Let be $\phi(x, \lambda)$ the solution of equation (1.1) and (1.4) satisfying the initial conditions

$$\phi(\eta, \lambda) = 0, \quad \phi'(\eta, \lambda) = 1 \tag{3.13}$$

and by $\vartheta(x, \lambda)$ the solution of the same equation, satisfying the initial conditions

$$\vartheta(\eta, \lambda) = 1, \quad \vartheta'(\eta, \lambda) = 0 \tag{3.14}$$

We notes that $\phi(x,\lambda)$ and $\vartheta(x,\lambda)$ are linearly independent if and only if $\omega(\lambda) \neq 0$.

$$\omega(\lambda) = \phi(x,\lambda)\vartheta'(x,\lambda) - \phi'(x,\lambda)\vartheta(x,\lambda).$$

The characteristic equation will be

$$\omega(\lambda) = \phi(\xi,\lambda) \tag{3.15}$$

Lemma 3.1. The solution $\phi(x,\lambda)$ of problem (1.1) and (1.4) satisfy the integral equations

$$\phi(x,\lambda) = \frac{\sin\lambda(x-\eta)}{\lambda} + \int_{\eta}^{x} \frac{\sin\lambda(x-\tau)}{\lambda} q(\tau)\phi(\tau,\lambda)d\tau. \tag{3.16}$$

Proof. First we obtain formula (3.16) Indeed,with solution of the form $q(x) = 0$. (1.1) becomes becomes $-y'' = \lambda^2 y$ by means of variation of parameter method, we have

$$\phi(x,\lambda) = C_1(x,\lambda)\cos\lambda x + C_2(x,\lambda)\sin\lambda x \tag{3.17}$$

and the direct calculation of $C_1(x,s)$ and $C_2(x,s)$, we have

$$C_1(x,\lambda) = -\frac{\sin\lambda\eta}{\lambda} - \int_{\eta}^{x} \frac{\sin\lambda\tau}{\lambda} q(\tau)\phi(\tau,\lambda)d\tau, \tag{3.18}$$

$$C_2(x,\lambda) = \frac{\cos\lambda\eta}{\lambda} + \int_{\eta}^{x} \frac{\cos\lambda\tau}{\lambda} q(\tau)\phi(\tau,\lambda)d\tau.$$

substituting from (3.18) into (3.17) equation (3.16) follows. Second we show that the integral representation (3.16) satisfies the problem (1.1) and (3.13). Let $\varphi(x,\lambda)$ be the solution of (1.1), so that

$$q(x)\,\phi(x,\lambda) = \phi''(x,\lambda) + \lambda^2\phi(x,\lambda).$$

We multiply both sides by

$$\frac{\sin\lambda(x-\tau)}{\lambda}$$

and integrating with respect to τ from η to x we obtain

$$\int_{\eta}^{x} \frac{\sin\lambda(x-\tau)}{\lambda}\, q(\tau)\,\phi(\tau,\lambda)\,d\tau = \int_{\eta}^{x} \frac{\sin\lambda(x-\tau)}{\lambda}\,\phi''(\tau,\lambda)\,d\tau \tag{3.19}$$

$$+\lambda^2 \int_{\eta}^{x} \frac{\sin\lambda(x-\tau)}{\lambda}\,\phi(\tau,\lambda)\,d\tau.$$

Integrating by parts twice and using the condition (3.13), we have

$$\int_{\eta}^{x} \frac{\sin\lambda(x-\tau)}{\lambda}\,\phi''(\tau,\lambda)\,d\tau \tag{3.20}$$

$$= \phi(x,\lambda) - \frac{\sin\lambda(x-\eta)}{\lambda} - \lambda\int_{\eta}^{x} \sin\lambda(x-\tau)\,\phi(\tau,\lambda)\,d\tau.$$

By substituting from (3.20) into (3.19) we get the required formula (3.16). ■

Lemma 3.2. Let $\lambda = \sigma + it$. Then there exists $\lambda_0 > 0$, such that $|\lambda| > \lambda_0$ the following inequalities for the solutions $\phi(x,\lambda)$ of boundary value problem (1.1) and (1.4) hold true

$$\phi(x,\lambda) = \frac{\sin\lambda(x-\eta)}{\lambda} + O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^2}\right), \tag{3.21}$$

Proof. We show first that

$$\phi(x,\lambda) = O\left(\frac{e^{|t|(x-\eta)}}{|\lambda|}\right).$$

where the inequality is uniformly with respect to x. Form the integral equation (3.16) we have

$$|\phi(x,\lambda)| \le \frac{e^{|t|(x-\eta)}}{\lambda} + \frac{e^{|t|(x-\eta)}}{\lambda}\int_{\eta}^{x} e^{|t|(x-\eta)}|q(\tau)||\phi(\tau,\lambda)|\,d\tau. \tag{3.22}$$

By using the notation $\phi(x,\lambda)e^{-|t|(x-\eta)} = F(x,\lambda)$, equation (3.22) takes the form

$$|F(x,\lambda)| \le \frac{1}{\lambda} + \frac{1}{\lambda}\int_{0}^{\pi} |q(\tau)||F(\tau,\lambda)|\,d\tau. \tag{3.23}$$

Let $\mu = \max\limits_{0\le x\le\pi} F(x,\lambda)$, so that from (3.23) it follows that

$$\mu \;\le\; \frac{\frac{1}{\lambda}}{1 - \frac{1}{\lambda}\int_0^{\pi}\;|q(\tau)|\;d\tau}.$$

For $|\lambda| > \lambda_0 = \displaystyle\int_0^{\pi} |q(\tau)|d\tau$ it follows from the last inequality that $F(x,\lambda) \le$ constant $/|\lambda|$ and this implies that

$$\phi(x,\lambda) = O\left(\frac{e^{|t|(x-\eta)}}{|\lambda|}\right), \tag{3.24}$$

By the aid of (3.23) we find that

$$\int_{\eta}^{x} \frac{\sin\lambda(x-\tau)}{\lambda}\,q(\tau)\,\phi(\tau,\lambda)\,d\tau = O\left(\frac{e^{|t|(x-\eta)}}{|\lambda|^2}\right). \tag{3.25}$$

From (3.16) and (3.23) it follows that, $\varphi(x, \lambda)$ has the asymptotic formula (3.21). ■

Theorem 3.3. Let $\lambda = \sigma + it$ and suppose that $q(x)$ has a second order piecewise differentiable derivatives on $[0, \pi]$. Then the solution $\phi(x, \lambda)$ of non-local boundary value (1.1) and (1.4) have the following asymptotic formula

$$\phi(x, \lambda) = \frac{\sin \lambda(x-\eta)}{\lambda} - \frac{\alpha_1(x)}{\lambda^2} \cos \lambda(x-\eta) - \frac{\alpha_2(x)}{\lambda^3} \sin \lambda(x-\eta) + O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda^4|}\right) \tag{3.26}$$

where

$$\begin{aligned} \alpha_1(x) &= \frac{1}{2} \int_\eta^x q(t)\, dt, \\ \alpha_2(x) &= \frac{1}{4} \left(\int_\eta^x q(t)\, dt \right)^2 - \frac{[q(x) + q(0)]}{4}. \end{aligned} \tag{3.27}$$

Proof. By substituting from (3.21) into the integral equation (3.16), we have

$$\begin{aligned} \phi(x, \lambda) = {} & \frac{\sin \lambda(x-\eta)}{\lambda} - \frac{\cos \lambda(x-\eta)}{2\lambda^2} \int_\eta^x q(t)\, dt \\ & + \frac{1}{2\lambda^2} \int_\eta^x \cos \lambda(x+\eta-2t)\, q(t)\, dt \\ & + O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^3}\right). \end{aligned} \tag{3.28}$$

Integrating the last integration of (3.28) by parts and noticing that there exists $q'(x)$ such that $q' \in L_1[0, \pi]$

$$\begin{aligned} & \frac{1}{2\lambda^2} \int_\eta^x \cos \lambda(x+\eta-2t)\, q(t)\, dt \\ & \quad = \frac{1}{4} [q(x) + q(\eta)] \frac{\sin \lambda(x-\eta)}{\lambda^3} - \frac{1}{4\lambda^3} \int_\eta^x \sin \lambda(x+\eta-2t)\, q'(t)\, dt \\ & = O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^3}\right). \end{aligned} \tag{3.29}$$

substituting from (3.29) into (3.28), we get

$$\phi(x, \lambda) = \frac{\sin \lambda(x-\eta)}{\lambda} - \frac{\alpha_1(x)}{\lambda^2} \cos \lambda(x-\eta) + O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^3}\right). \tag{3.30}$$

where $\alpha_1(x)$ is defined by (3.27). In order to make $\phi(x, \lambda)$ more precise we repeat this procedure again by substituting from the last result (3.30) into the same integral equation

(3.16), we have

$$\phi(x,\lambda) = \frac{\sin\lambda(x-\eta)}{\lambda} + \int_\eta^x \frac{\sin\lambda(x-t)\sin\lambda(t-\eta)}{\lambda^2}\, q(t)\, dt$$
$$- \int_\eta^x \frac{\sin\lambda(x-t)\cos\lambda(t-\eta)}{\lambda^3}\, q(t)\alpha_1(t)\, dt$$
$$+ \int_\eta^x \frac{\sin\lambda(x-t)}{\lambda}\, q(t) O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^3}\right) dt. \quad (3.31)$$

Now we estimate each term in (3.31). Integrating by parts twice the first term of (3.31), and noticing that $q'' \in L_1[0,\pi]$, we have

$$\int_\eta^x \frac{\sin\lambda(x-t)\sin\lambda(t-\eta)}{\lambda^2}\, q(t)\, dt$$
$$= -\frac{\alpha_1(x)}{\lambda^2}\cos\lambda(x-\eta) + \frac{q(x)+q(\eta)}{4\lambda^3}\sin\lambda(x-\eta) + O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^4}\right). \quad (3.32)$$

Similarly, we have

$$- \int_\eta^x \frac{\sin\lambda(x-t)\cos\lambda(t-\eta)}{\lambda^3}\, q(t)\, \alpha_1(t)\, dt$$
$$= -\frac{1}{2}\int_\eta^x \alpha_1(t)\, q(t)\, dt \frac{\sin\lambda(x-\eta)}{\lambda^3} + O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda|^4}\right) \quad (3.33)$$

Substituting from (3.32) and (3.33) into (3.31) we get the required formula (3.26). ■

Now inserting the values of the functions $\varphi(x,\lambda)$ from the estimate (3.26) into the second of the boundary conditions in (1.4), we obtain the following equation for the determination of the eigenvalues:

Equation (3.21) is the characteristic equation which gives roots of λ

$$\lambda_n^0 = \frac{n\pi}{(\xi-\eta)}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Then the $\omega(\lambda)$ has the same root of the function $\sin\lambda\xi$ (By Rouche's theorem)

$$\lambda_n = \lambda_n^0 + \varepsilon_n, \quad n = 0, 1, 2, \ldots \quad (3.34)$$

Theorem 3.4. Let $q \in L_1(0,\pi)$, then we have the following asymptotic formulas for λ_n of non-local boundary value (1.1) and (1.4)

$$\lambda_n = \frac{n\pi}{(\xi-\eta)} + \frac{\alpha_1}{n\pi} + O\left(\frac{1}{n^2}\right). \quad (3.35)$$

where $\alpha_1(x)$ defined in (3.27).

Proof.

$$\omega(x,\lambda)=\frac{\sin\lambda(\xi-\eta)}{\lambda}-\frac{\alpha_1}{\lambda^2}\cos\lambda(\xi-\eta)-\frac{\alpha_2}{\lambda^3}\sin\lambda(\xi-\eta)+O\left(\frac{e^{|Im\lambda|(\xi-\eta)}}{|\lambda^4|}\right) \quad (3.36)$$

It follows from (3.36) that

$$\sin\lambda(\xi-\eta)-\frac{\alpha_1}{\lambda}\cos\lambda(\xi-\eta)-\frac{\alpha_2}{\lambda^2}\sin\lambda(\xi-\eta)+O\left(\frac{e^{|Im\lambda|(\xi-\eta)}}{|\lambda^3|}\right)=0 \quad (3.37)$$

From equation (3.37), we have

$$\left[1-\frac{\alpha_2}{\lambda^2}\right]\sin\lambda(\xi-\eta)-\frac{\alpha_1}{\lambda}\cos\lambda(\xi-\eta)=0 \quad (3.38)$$

Dividing (3.38) by $\cos\lambda(\xi-\eta)$ we obtain

$$\left[1-\frac{\alpha_2}{\lambda^2}\right]\tan\lambda(\xi-\eta)=\frac{\alpha_1}{\lambda}$$

since imaginary $\lambda=O\left(\frac{1}{n}\right)$, then

$$\tan\lambda_n(\xi-\eta)=\frac{\alpha_1}{\lambda_n}+O\left(\frac{1}{n^2}\right) \quad (3.39)$$

From (3.34), (3.39) after elementary calculation, we obtain

$$\varepsilon_n=\frac{\alpha_1}{n\pi}+O\left(\frac{1}{n^2}\right) \quad (3.40)$$

From (3.34) and (3.40) , we have

$$\lambda_n=\frac{n\pi}{(\xi-\eta)}+\frac{\alpha_1}{n\pi}+O\left(\frac{1}{n^2}\right).$$

■

Corollary 3.5. If $\eta=0$ and $\xi=\pi$, then the eigenvalues of (3.35), we obtain

$$\lambda_n=n+\frac{\alpha_1}{n\pi}+O\left(\frac{1}{n^2}\right).$$

Which meets with the result obtained in [10].

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