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Induced Planar Homologies in Epipolar Geometry

Georgi Hristov Georgiev[1]

*Faculty of Mathematics and Informatics,
Konstantin Preslavsky University of Shumen,
9700 Shumen, Bulgaria.*

Vencislav Dakov Radulov

*Department of Descriptive Geometry
and Engineering Graphics,
University of Architecture,
Civil Engineering and Geodesy,
1000 Sofia, Bulgaria.*

Abstract

Epipolar geometry is a mathematical tool for reconstruction of a 3D object from its images on two projection planes. In this paper we show that any plane induces a projective transformation of the first projection plane. We obtain the set of fixed points of such a transformation. Particular cases of affine induced transformations are also discussed.

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[1]Corresponding author.

1. Introduction

Reconstruction of a 3D object from two perspective images is a significant task in Photogrammetry, Computer Vision, Robotics and related fields. Epipolar geometry has been a main tool for its solution in the last two decades. Epipolar geometry describes the geometrical relationship between two perspective views of one and the same 3D scene. This 3D reconstruction problem can be solved using triangulation, if the perspective images are obtained by cameras with known intrinsic and extrinsic parameters. This method based on the projection matrices is presented in [1] and [4]. A generalization is given in [5]. The connection between the corresponding points on the two images can be expressed in terms of homogeneous coordinates. In case of known intrinsic parameters and unknown extrinsic parameters, this connection is represented by the essential matrix see [2] and [11]. More generally, the fundamental 3×3-matrix $\mathbf{F}$ is the algebraic representation of epipolar geometry. If $M_i \in \pi_i$ $(i = 1, 2)$ are projections of the point $M \in \mathbb{E}^3$ in the projection planes π_i, $i = 1, 2$, and if their homogeneous coordinates (written as 3×1-vectors) related to corresponding plane coordinate systems are $\mathbf{p}_i$, $i = 1, 2$, then

$$(\mathbf{p}_2)^T \ \mathbf{F}\mathbf{p}_1 = 0.$$

The matrix $\mathbf{F}$ has a rank 2, in particular $\det(\mathbf{F}) = 0$. There are several methods for computing the fundamental matrix. The 8-point algorithm is described in [1], [4], [14]. A new eight-point algorithm is presented in [15]. The fundamental matrix is computed by parametrization in [1], [4], [13], [14], and by the use of the projection matrices in [1], [4], [14]. Another compact algorithm for computing the fundamental matrix is presented in [6].

Techniques based on projective geometry are widely used in epipolar geometry. There exists a homographyy (a projective transformation) between projection planes defined by another plane. This homographyy is studied in [1], [3] and [4] by the use of the extended Euclidean space which can be considered as a projective closure of the Euclidean 3-space (see also [9]).

In this paper, we investigate projective transformations of the first projection plane into itself that are generated by an arbitrary plane and epipolar geometry. First, we describe the set of fixed points of any such a transformation. Second, we find a matrix representation of the same projective transformations. Important particular cases are also discussed. The paper is organized as follows. The next two section are devoted to the basic facts and some properties of perspective projection and epipolar geometry. In Section 4, it is given a constructive definition for a projective transformation of the first projection plane obtained by a plane not passing through projection centers. In addition, the main theorem for fixed points is proved. The last section contains matrix representations of considered transformations.

2. Central projection

A central (or perspective) projection Π_1 of the Euclidean 3-space $\mathbb{E}^3$ is determined by a projection (or image) plane $\pi_1 \subset \mathbb{E}^3$ and a projection center $C' \in \mathbb{E}^3 \setminus \{\pi_1\}$.

If $M \in \mathbb{E}^3$ is an arbitrary point different from C', and if $C' \vee M$ denotes the line passing through C' and M, then the intersecting point

$$M_1 = (C' \vee M) \bigcap \pi_1$$

is the central projection of M, i.e. $M_1 = \Pi_1(M)$ (see Fig. 1). The extended Euclidean space contains all points, lines and planes of $\mathbb{E}^3$ as well as the set of points at infinity, the set of lines at infinity and the unique plane at infinity π_∞ (see [9]). The central projection is also well defined for an arbitrary point at infinity. Obviously, M_1 is a point at infinity if and only if the joining line $C' \vee M$ is parallel to π_1. Moreover, $M_1 = M$ if and only if $M \in \pi_1$. In the same way, the point C' and an arbitrary line $l \subset \mathbb{E}^3$ not passing through C' (or respectively, an arbitrary line u at infinity) determine a unique plane denoted by $C' \vee l$. Then, the central projection of l (or respectively, u) is the intersection line $l_1 = (C' \vee l) \bigcap \pi_1$ (or $u_1 = (C' \vee u) \bigcap \pi_1$), i,e, $\Pi_1(l) = l_1$ (or $\Pi_1(u) = u_1$).

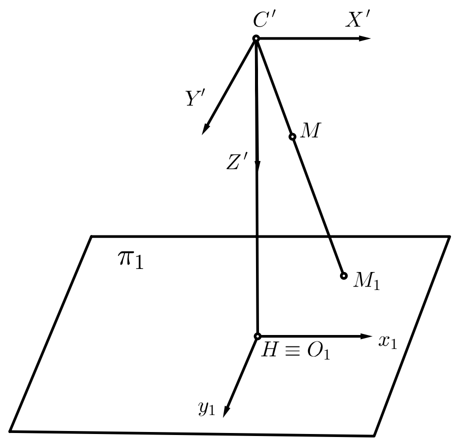

Figure 1: Central projection

Let us consider a Cartesian coordinate system $\{C'; X', Y', Z'\}$ in $\mathbb{E}^3$ with an origin placed at the center C' and coordinate axes X', Y', Z' such that X' and Y' are parallel to the projection plane π_1. This coordinate system is known as a camera frame, and the Z'-axis is the principle ray of the camera. The pedal point H of C' with respect to π_1 is called a principal point, and the distance $|C'H| = d_1$ between C' and π_1 is the focal length (Fig. 1). In the image plane π_1 there is an associated Cartesian coordinate system $\{O_1; x_1, y_1\}$ called the image frame. Its origin O_1 is placed at the point H and the axes

$x_1 \parallel X'$ and $y_1 \parallel Y'$ have the same directions as X' and Y', respectively. Assume that a four-dimensional column vector $\mathbf{m}' = (m'_1, m'_2, m'_3, m'_4)^T$ represents homogeneous coordinates of an arbitrary point $M \in \{\mathbb{E}^3 \setminus C'\} \bigcup \pi_\infty$ with respect to $\{C'; X', Y', Z'\}$ and a column 3×1 vector $\mathbf{m_1} = (m_{11}, m_{12}, m_{13})^T$ represents homogeneous coordinates of the point $M_1 = \Pi_1(M)$ with respect to $\{O_1; x_1, y_1\}$. Then there exists a projection 3×4 matrix $\mathbf{P}_1$ such that

$$\mathbf{m_1} = k_1 \mathbf{P_1} \mathbf{m}', \tag{1}$$

where k_1 is a non-zero real factor. Under the above assumptions, the projection matrix possesses the simplest form, i.e.

$$\mathbf{P_1} = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{2}$$

Recall that, if $M \in \{\mathbb{E}^3 \setminus C'\}$, then $m'_4 \neq 0$, and if $M \in \pi_\infty$ is a point at infinity, then $m'_4 = 0$ and $|m'_1| + |m'_2| + |m'_3| \neq 0$. Similarly,
$|m_{11}| + |m_{12}| + |m_{13}| \neq 0$ and $m_{13} = 0$ if and only if the projection M_1 is a point at infinity.

Let a, b and c be three real constants, and let $\alpha \subset \{\mathbb{E}^3 \bigcup \pi_\infty\}$ be a plane with an equation in homogeneous coordinates $am'_1 + bm'_2 + cm'_3 - m'_4 = 0$ regarding to the camera frame. Then, $C' \notin \alpha$ and the restriction

$$\varphi_1 = \Pi_1|_\alpha : \alpha \longrightarrow \pi_1$$

is a perspectivity with a center C'. This perspectivity is represented by (1) and (2). The 4×3 matrix

$$\mathbf{P}_\alpha = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_1} & 0 \\ 0 & 0 & 1 \\ \frac{a}{d_1} & \frac{b}{d_1} & c \end{bmatrix} \tag{3}$$

is a generalized inverse matrix of $\mathbf{P_1}$, because $\mathbf{P_1}\,\mathbf{P}_\alpha\,\mathbf{P_1} = \mathbf{P_1}$ (see [8]). Moreover, if the point $N_1 \in \pi_1$ has homogeneous coordinates written as a three-dimensional column vector $\mathbf{n_1} = (n_{11}, n_{12}, n_{13})^T$ with respect to $\{O_1; x_1, y_1\}$, then the four-dimensional column vector

$$\mathbf{P}_\alpha\,\mathbf{n_1} = \mathbf{n}' = (n'_1, n'_2, n'_3, n'_4)^T \tag{4}$$

represents homogeneous coordinates of the intersecting point

$$N = \alpha \bigcap \left(C' \vee N_1\right)$$

with respect to $\{C'; X', Y', Z'\}$. Hence, the reverse perspectivity with a center C'

$$\varphi_1^{-1} : \pi_1 \longrightarrow \alpha$$

is fully determined by (3) and (4).

The central projection Π_1, the perspectivity φ_1 and the reverse perspectivity φ_1^{-1} can be also expressed in terms of the camera frame $\{C'; X', Y', Z'\}$. Suppose that $\mathbf{m}' = (m'_1, m'_2, m'_3, m'_4)^T$ and $\mathbf{m'_1} = (m'_{11}, m'_{12}, m'_{13}, m'_{14})^T$ are column vectors representing homogeneous coordinates of the points $M \in \{\mathbb{E}^3 \setminus C'\} \bigcup \pi_\infty$ and $M_1 = \Pi_1(M)$ with respect to the camera frame. Then, $\mathbf{m'_1} = (d\, m'_1, d\, m'_2, d\, m'_3, m'_3)^T$ and the four-dimensional vectors $\mathbf{m}'$ and $\mathbf{m'_1}$ are related by

$$\mathbf{m'_1} = k'_1 \mathbf{P'_1}\, \mathbf{m}', \tag{5}$$

where k'_1 is a non-zero real factor and

$$\mathbf{P'_1} = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 \\ 0 & 0 & d_1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{6}$$

is an extended projection 4×4 matrix. If $M \in \alpha$, then homogeneous coordinates of $M_1 = \varphi_1(M) = \Pi_1(M)$ with respect to the camera frame are determined by (5) and (6). Consider a point $N_1 \in \pi_1$ whose homogeneous coordinates with respect to $\{C'; X', Y', Z'\}$ can be written as a four-dimensional column vector $\mathbf{n'_1} = (n'_{11}, n'_{12}, d_1\, n'_{13}, n'_{13})^T$. If

$$\mathbf{P}_\alpha{}' = \frac{1}{d_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 0 \end{bmatrix} \tag{7}$$

is a generalized inverse of the matrix $\mathbf{P'_1}$ obtained by the plane α, i.e. $\mathbf{P'_1} \mathbf{P}'_\alpha \mathbf{P'_1} = \mathbf{P'_1}$, then the four-dimensional column vector

$$\begin{aligned} \mathbf{P}_\alpha{}' \mathbf{n'_1} = \mathbf{n}' &= (n'_1, n'_2, n'_3, n'_4)^T \\ &= \frac{1}{d_1} \left(n'_{11}, n'_{12}, d_1\, n'_{13}, a\, n'_{11} + b\, n'_{12} + d_1\, c\, n'_{13}\right)^T \end{aligned} \tag{8}$$

represents homogeneous coordinates of the intersecting point

$$N = \alpha \bigcap \left(C' \vee N_1\right)$$

with respect to $\{C'; X', Y', Z'\}$. Hence, $\mathbf{P}_\alpha{}'$ can be considered as an extended projection matrix (related to the camera frame) of the central projection Π_α with a projection center C' and a projection plane α. Moreover, the reverse perspectivity with a center C'

$$\varphi_1^{-1} = \Pi_\alpha|_{\pi_1} : \pi_1 \longrightarrow \alpha$$

is completely determined by (7) and (8) in terms of the camera frame.

3. Epipolar Geometry

In this section, we give a brief overview of the epipolar geometry which will be used in the next sections. Let Π_1 and Π_2 be two central projections determined by the pairs (π_1, C') and (π_2, C''), respectively, and let $C' \neq C''$. We denote by d_1 the distance from the point C' to the plane π_1 and by d_2 the distance from the point C'' to the plane π_2.

If $M \in \{\mathbb{E}^3 \setminus \{C' \vee C''\}\} \bigcup \pi_\infty$ is an arbitrary point, and if $M_1 = \Pi_1(M)$, $M_2 = \Pi_2(M)$ are its central projections, then (M_1, M_2) is a pair of corresponding points associated to M (see Fig. 2). Similarly, if $l \subset \mathbb{E}^3 \bigcup \pi_\infty$ is a straight line not passing through either of the projection centers and if $l_1 = (C' \vee l) \bigcap \pi_1$, $l_2 = (C'' \vee l) \bigcap \pi_2$ are its central projections, then (l_1, l_2) is a pair of corresponding lines associated to the line l.

The joining line $C' \vee C''$ is called a baseline. The intersection points $E_1 = (C' \vee C'') \bigcap \pi_1$ and $E_2 = (C' \vee C'') \bigcap \pi_2$ are known as epipolar points or epipoles. In this paper we consider the case when each of the projection planes π_1 and π_2 is different from the plane at infinity, and neither C' nor C'' is a point at infinity. An epipolar plane is any plane containing the baseline. Epipolar lines are the intersection lines of an epipolar plane with the projection planes.

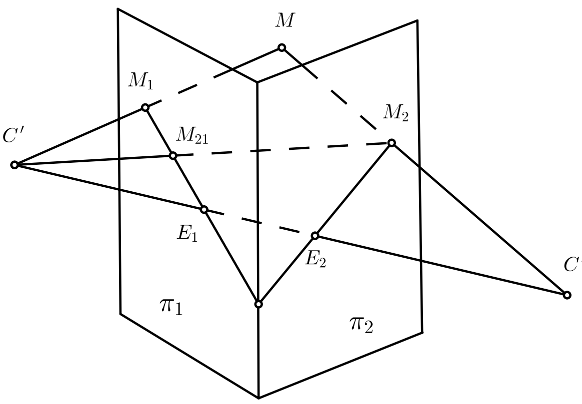

Figure 2: Epipolar geometry.

3.1. Coordinates of a space point

Let us consider the camera frames $\mathcal{F}'_{cam} = \{C'; X', Y', Z'\}$ and $\mathcal{F}''_{cam} = \{C''; X'', Y'', Z''\}$ which are related to the central projections Π_1 and Π_2, respectively. The Cartesian coordinates of the projection centers C' and C'' with respect to the first camera frame $\mathcal{F}_{cam}{}'$ are $(0, 0, 0)$ and (t_1, t_2, t_3). In other words, $\mathbf{t} = \overrightarrow{C'C''} = (t_1, t_2, t_3)^T$ is the representation of the translation vector with respect to the first camera frame. Analogously, the unit vectors $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ along the axes of the first camera frame and the unit vectors $\mathbf{i}'', \mathbf{j}'', \mathbf{k}''$ along the axes of the second camera frame can be related to the first camera frame as follows: $\mathbf{i}' = (1, 0, 0)^T, \mathbf{j}' = (0, 1, 0)^T, \mathbf{k}' = (0, 0, 1)^T, \mathbf{i}'' = (r_{11}, r_{21}, r_{31})^T$, $\mathbf{j}'' = (r_{12}, r_{22}, r_{32})^T, \mathbf{k}'' = (r_{13}, r_{23}, r_{33})^T$. Hence, the space rotation that transforms the positively oriented orthonormal triad $\{\mathbf{i}', \mathbf{j}', \mathbf{k}'\}$ into the positively oriented orthonormal triad $\{\mathbf{i}'', \mathbf{j}'', \mathbf{k}''\}$ is presented by the rotation matrix

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

with respect to the first camera frame. In fact, $\mathbf{Ri}' = \mathbf{i}''$, $\mathbf{Rj}' = \mathbf{j}''$ and $\mathbf{Rk}' = \mathbf{k}''$. It can be noted that any rotation matrix is characterized by $\mathbf{R}^T = \mathbf{R}^{-1}$ and $\det(\mathbf{R}) = 1$.

Suppose that a column vector $\mathbf{m}' = (m'_1, m'_2, m'_3, m'_4)^T$ represents homogeneous coordinates of the point $M \in \{\mathbb{E}^3 \setminus \{C' \vee C''\}\} \bigcup \pi_\infty$ with respect to the first camera frame $\mathcal{F}_{cam}{}'$, and a column vector $\mathbf{m}'' = (m''_1, m''_2, m''_3, m''_4)^T$ represents homogeneous coordinates of the same point M with respect to the second camera frame $\mathcal{F}_{cam}{}''$. Then, $\mathbf{m}'$ and $\mathbf{m}''$ are related by

$$\mathbf{m}' = k'\mathbf{A}\mathbf{m}'' \tag{9}$$

where k' is a non-zero real factor and the square matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{o} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is nonsingular, since $det(\mathbf{A}) = 1$. Its inverse matrix is

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{t} \\ \mathbf{o} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{21} & r_{31} & -\mathbf{i}'' \cdot \mathbf{t} \\ r_{12} & r_{22} & r_{32} & -\mathbf{j}'' \cdot \mathbf{t} \\ r_{13} & r_{23} & r_{33} & -\mathbf{k}'' \cdot \mathbf{t} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\mathbf{i}'' \cdot \mathbf{t}$ denotes a vector dot product. Thus there is another relation between $\mathbf{m}'$ and $\mathbf{m}''$ which can be written in the following matrix form

$$\mathbf{m}'' = k''\mathbf{A}^{-1}\mathbf{m}', \tag{10}$$

where k'' is a non-zero real factor.

3.2. Coordinates of projection points

Let us consider a first image frame $\mathcal{F}_1{}^{im} = \{O_1; x_1, y_1\}$ and a second image frame $\mathcal{F}_2{}^{im} = \{O_2; x_2, y_2\}$ which are related to the central projections Π_1 and Π_2, respectively. As in the previous section homogeneous coordinates of the first projection point $M_1 = \Pi_1(M)$ are denoted by $\mathbf{m_1} = (m_{11}, m_{12}, m_{13})^T$ with respect to $\mathcal{F}_1{}^{im}$ and by

$$\mathbf{m_1'} = (m'_{11}, m'_{12}, m'_{13}, m'_{14})^T = (m'_{11}, m'_{12}, d_1 m'_{13}, m'_{13})^T$$

with respect to the first camera frame $\mathcal{F}_{cam}{}'$. Then, the relation between $\mathbf{m_1}$ and $\mathbf{m_1'}$ is expressed by

$$\mathbf{m_1}' = k'_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d_1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{m_1}, \tag{11}$$

where $k'_1 \neq 0$ is a constant, or equivalently, by

$$\mathbf{m_1} = k_1 \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{m_1}', \tag{12}$$

where $k_1 \neq 0$ is a constant. Moreover, the relation between the coordinate vector $\mathbf{m}'$ from the previous subsection and $\mathbf{m_1}$ is expressed by (1) and (2), Similarly, homogeneous coordinates of the second projection point $M_2 = \Pi_2(M)$ are denoted by $\mathbf{m_2} = (m_{21}, m_{22}, m_{23})^T$ with respect to $\mathcal{F}_2{}^{im}$ and by

$$\mathbf{m_2''} = (m''_{21}, m''_{22}, m''_{23}, m''_{24})^T = (m''_{21}, m''_{22}, d_2 m''_{24}, m''_{24})^T$$

with respect to the second camera frame

$$\mathcal{F}_{cam}{}'' = \{C''; X'', Y'', Z''\}.$$

Then, the relation between $\mathbf{m_2}$ and $\mathbf{m_2''}$ is expressed by

$$\mathbf{m_2}'' = k''_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d_2 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{m_2},$$

where $k''_2 \neq 0$ is a constant, or equivalently, by

$$\mathbf{m_2} = k_2 \begin{bmatrix} d_2 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{m_2}'',$$

where $k_2 \neq 0$ is a constant.

4. Induced Planar Homologies

Let $\alpha \subset \mathbb{E}^3 \bigcup \pi_\infty$ be a plane not passing through the projection centers C' and C''. Then, there exist two perspectivities which are determined by Π_1 , Π_2 and α. The first perspectivity with a center C'

$$\varphi_1^{-1} : \pi_1 \longrightarrow \alpha$$

is described in Section 2, and it is determined by (7) and (8) with respect to the first camera frame. The second perspectivity with a center C''

$$\varphi_2 = \Pi_2|_\alpha : \alpha \longrightarrow \pi_2$$

has a simple matrix representation in terms of the second camera frame

$$\mathbf{m}_2'' = k_2''\mathbf{P}_2''\mathbf{m}'', \quad \mathbf{P}_2'' = \begin{bmatrix} d_2 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad k_2'' \neq 0 \tag{13}$$

where $\mathbf{m}''$ and $\mathbf{m}_2''$ are column 4×1-vectors of homogeneous coordinates of the points $M \in \alpha$ and $M_2 = \varphi_2(M) = \Pi_2(M) \in \pi_2$, respectively, and $\mathbf{P}_2''$ is the extended projection matrix of Π_2 with respect to the second camera frame. Notice that (13) is similar to (5) and (6). Let us consider the restriction

$$\varphi_3 = \Pi_1|_{\pi_2} : \pi_2 \longrightarrow \pi_1$$

which is also a perspectivity with center C'. It is clear that φ_3 can be expressed by (5) and (6) with respect to the first camera frame.

The composition

$$\psi = \varphi_3 \circ \varphi_2 \circ \varphi_1^{-1} \tag{14}$$

is a projective transformation of the first projection plane π_1 into itself. This transformation connected to the plane α can be also defined in direct constructive way. If $M_1 \in \pi_1$ is an arbitrary point, then there exist uniquely determined intersection points $M = (C' \vee M_1) \bigcap \alpha$, $M_2 = (C'' \vee M) \bigcap \pi_2$ and $M_{21} = (C' \vee M_2) \bigcap \pi_1$. Thus, $\psi(M_1) = M_{21}$ (see Fig. 2).

The planar projective transformations are classified by the sets of their fixed points in [7] and [10]. First, we will describe all fixed points of ψ.

Theorem 4.1. Let $\alpha \subset \{\mathbb{E}^3 \bigcup \pi_\infty\}$ be a plane not passing through the projection centers C' and C'', and let g be the intersecting line of the planes α and π_2. Then, the projective transformation

$$\psi : \pi_1 \longrightarrow \pi_1$$

given by (14) has a pencil of fixed lines passing through the epipolar point E_1 and a line of fixed points $g_1 = \varphi_3(g) = (S' \vee g) \bigcap \pi_1$. Moreover, it is fulfilled either ψ has no other fixed point and the points $M_1 \in \pi_1 \setminus \{E_1 \bigcup g_1\}$, $M_{21} = \psi(M_1)$ and E_1 are collinear, or ψ has only fixed points.

Proof. If $E_\alpha = (E_1 \vee E_2) \bigcap \alpha$, then

$$\psi(E_1) = \varphi_3 \circ \varphi_2 \circ \varphi_1^{-1}(E_1) = \varphi_3 \circ \varphi_2(E_\alpha) = \varphi_3(E_2) = E_1.$$

In other words the epipolar point E_1 is a fixed point of the projective transformation ψ. Suppose that a_1 is an arbitrary epipolar line in π_1, i.e. $E_1 \in a_1 \subset \pi_1$ and consider the intersecting lines $b_\alpha = \alpha \bigcap \left(C' \vee a_1\right)$ and $c_2 = \pi_2 \bigcap (C'' \vee a_1)$. Then, we have

$$\psi(a_1) = \varphi_3 \circ \varphi_2 \circ \varphi_1^{-1}(a_1) = \varphi_3 \circ \varphi_2(b_\alpha) = \varphi_3(c_2) = a_1.$$

Hence, all epipolar lines in π_1 form a pencil of fixed lines of ψ. According to the terminology of projective geometry the projective transformation ψ is either a planar homology with vertex E_1, or an elation with vertex E_1. Similarly, we can verify that any point $G_1 \in g_1$ is a fixed point of ψ. In fact, if $G_\alpha = (C' \vee G_1) \bigcap \alpha \in g$, then

$$\psi(G_1) = \varphi_3 \circ \varphi_2 \circ \varphi_1^{-1}(G_1) = \varphi_3 \circ \varphi_2(G_\alpha) = \varphi_3(G_\alpha) = G_1.$$

This means that g_1 is either an axis of a planar homology ψ or an axis of an elation. The last statement follows from the fact that ψ preserves the incidence as a product of three perspectivities. ■

If the plane α does not pass through the point E_2, i.e. $E_2 \notin g$, then $E_1 \notin g_1$ and the projective transformation ψ is called a planar homology (see [4], [10] and [12]). This case is plotted in Fig. 3. If the plane α contains the point E_2, i.e. $E_2 \in g$, then $E_1 \in g_1$ and the projective transformation ψ is called a special plane homology (see [10]) or more often an elation (see, for instance, [4]). In the particular case $\alpha \equiv \pi_2$ the the projective transformation ψ is the identity.

Now, we can name the considered transformations.

Definition 4.2. For any plane $\alpha \subset \{\mathbb{E}^3 \bigcup \pi_\infty\}$ not containing C' and C'', the projective transformation $\psi : \pi_1 \longrightarrow \pi_1$ defined by (14) is called **an induced planar homology** of the first projection plane.

The induced planar homology is an affine transformation of π_1, if the line at infinity of the plane π_1 is a fixed line of ψ. Let us consider two important particular cases.

Corollary 4.3. Let β be a plane containing C' and parallel to π_1, and let $\beta \bigcap \pi_2 = f$. Suppose that $\alpha \neq \pi_2$ is a plane passing through f and not containing C' and C''. Then, the induced planar homology ψ defined by (14) is an affine transformation of π_1. Moreover:

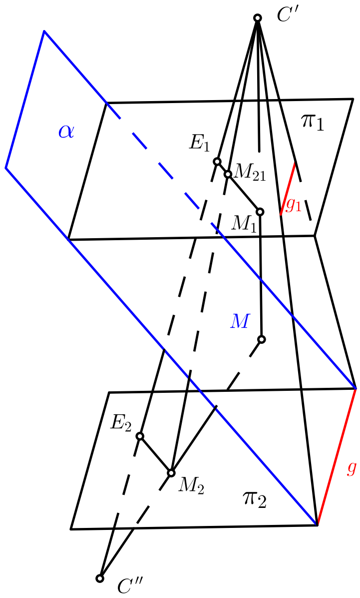

Figure 3: The vertex and the axis of the planar homology

(a) if the epipolar point E_1 is not a point at infinity, or equivalently, the base line $(C' \vee C'')$ is not parallel to π_1, this affine transformation is a homothety with a center E_1,

(b) if E_1 is a point at infinity $\big((C' \vee C'') \parallel \pi_1\big)$, the affine transformation ψ is a translation.

Proof. If $u_1 \subset \pi_1$ is the line at infinity, then

$$\psi(u_1) = \varphi_3 \circ \varphi_2 \circ \varphi_1^{-1}(u_1) = \varphi_3 \circ \varphi_2(f) = \varphi_3(f) = u_1,$$

and for any point at infinity $U_1 \in u_1$ the conditions $(C' \vee U_1) \bigcap \alpha = F_1 \in f$ and

$$\psi(U_1) = \varphi_3 \circ \varphi_2 \circ \varphi_1^{-1}(U_1) = \varphi_3 \circ \varphi_2(F_1) = \varphi_3(F_1) = U_1$$

are fulfilled. This means that the line at infinity of π_1 is the line of fixed points of ψ. Hence, ψ is an affine transformation. If $E_1 \notin u_1$, then the epipolar point E_1 is an isolated fixed point of ψ. Therefore, in this case, the induced planar homology ψ of the extended Euclidean plane π_1 is a homothety with center E_1. Similarly, if $E_1 \in u_1$, then the induced planar homology ψ (whose vertex E_1 is a point at infinity and whose axis u_1 is the line at infinity) is a translation. ∎

5. A Matrix Representation of Induced Planar Homologies

Any projective transformation can be expressed by one matrix equality in terms of homogeneous coordinates. In this section, we derive such an equality for arbitrary induced planar homology.

Theorem 5.1. Let $\alpha \subset \{\mathbb{E}^3 \bigcup \pi_\infty\}$ be a plane given by the equation in homogeneous coordinates $ax_1' + bx_2' + cx_3' - x_4' = 0$ with respect to the first camera frame, and let $at_1 + bt_2 + ct_3 \neq 1$. Then,
(i) the induced planar homology $\psi : \pi_1 \longrightarrow \pi_1$ defined by (14) possesses a representation with respect to the first camera frame $\mathcal{F}_{cam}{}'$

$$\mathbf{m_{21}}' = k_{21}' \begin{bmatrix} t_1q_1d_1 + d_1d_2 & t_1\,q_2\,d_1 & t_1\,q_3\,d_1 & 0 \\ t_2\,q_1\,d_1 & t_2q_2d_1 + d_1d_2 & t_2\,q_3\,d_1 & 0 \\ t_3\,q_1\,d_1 & t_3\,q_2\,d_1 & t_3q_3d_1 + d_1d_2 & 0 \\ t_3\,q_1 & t_3\,q_2 & t_3\,q_3 + d_2 & 0 \end{bmatrix} \mathbf{m_1}', \qquad (15)$$

where

$$\mathbf{m_1}' = \left[m_{11}', m_{12}', d_1 m_{13}', m_{13}'\right]^T$$

and

$$\mathbf{m_{21}}' = \left[m_{211}', m_{212}', m_{213}', m_{214}'\right]^T$$

are homogeneous coordinates of $M_1 \in \pi_1$ and $M_{21} = \psi(M_1)$ related to $\mathcal{F}_{cam}{}'$,

$$\begin{aligned} q_1 &= r_{13} - a(\mathbf{t}\cdot\mathbf{k}'') - a\,d_2, \\ q_2 &= r_{23} - b(\mathbf{t}\cdot\mathbf{k}'') - b\,d_2, \\ q_3 &= r_{33} - c(\mathbf{t}\cdot\mathbf{k}'') - c\,d_2, \\ k_{21}' &\neq 0. \end{aligned}$$

(ii) the induced planar homology ψ has a representation with respect to the first image frame $\mathcal{F}_1{}^{im}$

$$\mathbf{m_{21}} = k_{21} \begin{bmatrix} t_1q_1d_1 + d_1d_2 & t_1\,q_2\,d_1 & t_1\,q_3\,d_1^2 \\ t_2\,q_1\,d_1 & t_2q_2d_1 + d_1d_2 & t_2\,q_3\,d_1^2 \\ t_3\,q_1 & t_3\,q_2 & t_3q_3d_1 + d_1d_2 \end{bmatrix} \mathbf{m_1}, \qquad (16)$$

where $k_{21} \neq 0$, $\mathbf{m_1} = [m_{11}, m_{12}, m_{13}]^T$ and $\mathbf{m_{21}} = [m_{211}, m_{212}, m_{213}]^T$ are homogeneous coordinates of $M_1 \in \pi_1$ and $M_{21} = \psi(M_1) \in \pi_1$ related to $\mathcal{F}_1{}^{im}$.

Proof. In terms of homogeneous coordinates, the planar homology ψ is a linear transformation. Therefore, considering homogeneous coordinates of points on π_1 with respect to the space coordinate system $\mathcal{F}_{cam}{}'$ we can find a 4×4-matrix $\mathbf{B}$ such that $\mathbf{m_{21}}' = \mathbf{B}\,\mathbf{m_1}'$. Since ψ is a composition of three perspectivities, the matrix $\mathbf{B}$ is a product of three extended projection matrices related to the first camera frame. More precisely, we can write

$$\mathbf{m_{21}}' = k_{21}' \mathbf{P_1}' \mathbf{P_2}' \mathbf{P}_\alpha{}' \mathbf{m_1}' \tag{17}$$

where $\mathbf{P_1}'$ is determined by (6), $\mathbf{P}_\alpha{}'$ is expressed by (7), and $\mathbf{P_2}'$ is the extended projection matrix of Π_2 with respect to $\mathcal{F}_{cam}{}'$. Using (9), (10) and (13) we compute

$$\mathbf{P_2}' = \mathbf{A}\,\mathbf{P_2}''\,\mathbf{A}^{-1} = \begin{bmatrix} t_1 r_{13} + d_2 & t_1 r_{23} & t_1 r_{33} & -t_1(\mathbf{t}\cdot\mathbf{k}'' + d_2) \\ t_2 r_{13} & t_2 r_{23} + d_2 & t_2 r_{33} & -t_2(\mathbf{t}\cdot\mathbf{k}'' + d_2) \\ t_3 r_{13} & t_3 r_{23} & t_3 r_{33} + d_2 & -t_3(\mathbf{t}\cdot\mathbf{k}'' + d_2) \\ r_{13} & r_{23} & r_{33} & -\mathbf{t}\cdot\mathbf{k}'' \end{bmatrix}.$$

Replacing the obtained expressions for $\mathbf{P_1}'$, $\mathbf{P}_\alpha{}'$ and $\mathbf{P_2}'$ in (17) we get (15). From (11), (12) and (17) it follows that

$$\mathbf{m}_{21} = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{P_1}' \mathbf{P_2}' \mathbf{P}_\alpha{}' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d_1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{m_1}.$$

Then, the matrix equality (16) is an immediate consequence of the matrix equality (15). ■

We may combine the Corollary 4.3 with Theorem 5.1.

Corollary 5.2. Let the plane α coincides with the plane at infinity π_∞, and let the projection planes π_1 and π_2 are parallel. Then, the induced planar homology ψ of π_1 is an affine transformation that has a representation with respect to the first image frame $\mathcal{F}_1{}^{im}$

$$\mathbf{m_{21}} = k_{21} \begin{bmatrix} d_2 & 0_1 & t_1\, d_1 \\ 0 & d_2 & t_2\, d_1 \\ 0 & 0 & t_3 + d_2 \end{bmatrix} \mathbf{m_1}, \tag{18}$$

where $k_{21} \neq 0$, $(t_1, t_2, t_3) \neq (0, 0, 0)$ are Cartesian coordinates of C'' related to $\mathcal{F}_{cam}{}'$, and $d_2 + t_3 \neq 0$. If $t_3 \neq 0$, then ψ is the homothety whose center E_1 has homogeneous coordinates $(d_1\, t_1, d_1\, t_2, t_3)^T$ related to $\mathcal{F}_1{}^{im}$, and whose ratio is $r = \dfrac{d_2}{d_2 + t_3}$. If $t_3 = 0$, then $|t_1| + |t_2| \neq 0$ and ψ is the translation determined by the two-dimensional vector

$$\mathbf{v} = \left(\frac{t_1\, d_1}{d_2}, \frac{t_2\, d_1}{d_2}\right)$$

related to $\mathcal{F}_1{}^{im}$.

Proof. Since the camera frames $\mathcal{F}_{cam}{}'$ and $\mathcal{F}_{cam}{}''$ have one and the same orientation and $\pi_1 \parallel \pi_2$, $d_2 + t_3 = 0 \iff C' \in \pi_2$. This implies that $d_2 + t_3 \neq 0$, $r_{13} = 0$,

$r_{23} = 0$, $r_{33} = 1$. From $\alpha \equiv \pi_\infty$ it follows that $a = 0$, $b = 0$ and $c = 0$. Then, $q_1 = 0$, $q_2 = 0$, $q_3 = r_{33} = 1$. Substituting these values in (16) we get (18). Thus, ψ is an affine transformation and $\psi(U_1) = U_1$ for any point $U_1 \in u_1 = \pi_1 \bigcap \pi_\infty$. If $t_3 \neq 0$, then the epipolar point $E_1 = (C' \vee C'') \bigcap \pi_1$ with homogeneous coordinates $(d_1\, t_1, d_1\, t_2, t_3)^T$ related to $\mathcal{F}_1{}^{im}$ is not a point at infinity and $\psi(E_1) = E_1$. This means that ψ is a homothety whose center E_1 has Cartesian coordinates $\left(\frac{d_1 t_1}{t_3}, \frac{d_1 t_2}{t_3}\right)$ with respect to $\mathcal{F}_1{}^{im}$. By (18), if the point $M_1 \in \pi_1$ has Cartesian coordinates

$$\left(\frac{d_1 t_1}{t_3} + \frac{1}{t_3}, \frac{d_1 t_2}{t_3}\right),$$

then its image $M_{21} = \psi(M_1)$ has Cartesian coordinates

$$\left(\frac{d_1 t_1}{t_3} + \frac{d_2}{t_3(d_2 + t_3)}, \frac{d_1 t_2}{t_3}\right).$$

Hence

$$\overrightarrow{E_1 M_1} = \left(\frac{1}{t_3}, 0\right),$$

$$\overrightarrow{E_1 M_{21}} = \left(\frac{d_2}{t_3(d_2 + t_3)}, 0\right) = r\,\overrightarrow{E_1 M_1}$$

and $r = \dfrac{d_2}{d_2 + t_3}$. If $t_3 = 0$, then $(t_1, t_2) \neq (0, 0)$ and (18) becomes a matrix representation of the translation of π_1 defined by the nonzero vector

$$\left(\frac{t_1\, d_1}{d_2}, \frac{t_2\, d_1}{d_2}\right).$$

■

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