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Measure zero stability problem for Jensen type functional equations

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Abstract

In this paper, using the Baire category theorem we investigate the Hyers-Ulam stability problem of Jensen type functional equations on a set of Lebesgue measure zero. As a consequence, we obtain an asymptotic behavior of the equations.

AMS subject classification: 39B22.
Keywords: Hyers-Ulam stability, Jensen equation, Jensen type functional equation, restricted domain.

1. Results

Throughout the paper, we denote by $\mathbb{R}$, X and Y be the set of real numbers, a real normed space and a real Banach space, respectively, $d > 0$ and $\epsilon \geq 0$ be fixed. A mapping $f : X \to Y$ is called *the Jensen type functional equations* if f satisfies the equations

$$f(x + y) + f(x - y) - 2f(x) = 0, \tag{1.1}$$

$$f(x + y) - f(x - y) - 2f(y) = 0 \tag{1.2}$$

for all $x, y \in X$. A mapping $f : X \to Y$ is called an additive mapping if f satisfies $f(x+y) - f(x) - f(y) = 0$ for all $x, y \in X$. The stability problems for functional equations have been originated by Ulam in 1940 (see [31]). One of the first assertions to be obtained is the following result, essentially due to Hyers [17] that gives an answer to the question of Ulam.

Theorem 1.1. Let $\epsilon > 0$ be fixed. Suppose that $f : X \to Y$ satisfies the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x,\ y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \leq \epsilon$$

for all $x \in X$.

The terminology *domain* means a subset of $X \times X$. Let $f : X \to Y$, $d > 0$ and $\epsilon \geq 0$. Among the numerous results on Ulam-Hyers stability theorem for functional equations (e.g. [17], [19], [20], [24], [26], [27], [30]) there are various interesting results which deal with the stability of functional equations in restricted domains ([2], [3], [5], [7], [13], [15], [21], [24]). In particular, J. Chung, D. Kim and J. M. Rassias prove the Ulam-Hyers stability of the Jensen type functional equation (1.1) and (1.2), i.e., the stability of the inequalities

$$\|f(x+y) + f(x-y) - 2f(x)\| \leq \epsilon, \tag{1.3}$$

$$\|f(x+y) - f(x-y) - 2f(y)\| \leq \epsilon \tag{1.4}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq d$.

Theorem 1.2. Suppose that $f : X \to Y$ satisfies the functional inequality (1.3) for all $x, y \in X$ with $\|x\| + \|y\| \geq d$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \leq \frac{5}{2}\epsilon$$

for all $x \in X$.

Theorem 1.3. Suppose that $f : X \to Y$ satisfies the functional inequality (1.4) for all $x, y \in X$ with $\|x\| + \|y\| \geq d$ and

$$\|f(x) + f(-x)\| \leq 3\epsilon$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq d$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{33}{2}\epsilon$$

for all $x \in X$.

It is very natural to ask if the restricted domain $\Omega_d := \{(x, y) : \|x\| + \|y\| \geq d\}$ in Theorem 1.2 and 1.3 can be replaced by a smaller subset $\Gamma_d \subset \Omega_d$ (e.g. a subset of measure 0 in a measure space X)(see [13] for the quadratic functional equation).

In this paper we consider the Ulam-Hyers stability of the Jensen type functional equations (1.1) and (1.2) in restricted domains $\Omega \subset X \times X$ satisfying the condition (C_1) and (C_2): For given $(x, y) \in X$, there exists a $t \in X$ such that

$$(C_1) \qquad \{(x + y, -x + y + t), (x, -x + t), (y, y + t), (0, t)\} \subset \Omega,$$
$$(C_2) \qquad \{(-x + y + t, x + y), (y + t, y), (-x + t, x)\} \subset \Omega,$$

respectively.

From now on we assume that Ω satisfies the condition (C_1) and (C_2). As main results we prove the following.

Theorem 1.4. Suppose that $f : X \to Y$ satisfies the inequality

$$\|f(x + y) + f(x - y) - 2f(x)\| \leq \epsilon \tag{1.5}$$

for all $(x, y) \in \Omega$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \leq 2\epsilon \tag{1.6}$$

for all $x \in X$.

Theorem 1.5. Suppose that $f : X \to Y$ satisfies the inequality

$$\|f(x + y) - f(x - y) - 2f(y)\| \leq \epsilon \tag{1.7}$$

for all $(x, y) \in \Omega$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{3}{2}\epsilon \tag{1.8}$$

for all $x \in X$.

Note that the set $\{(x, y) \in X \times X : \|x\| + \|y\| \geq d\}$ satisfies the condition (C_1) and (C_2). In particular, if $X = \mathbb{R}$ we prove the following.

Theorem 1.6. Let $X = \mathbb{R}$ and $B \subset \mathbb{R}$. Assume that $B^c := \mathbb{R} \setminus B$ is of the first category. Then there is a rotation Γ of $B^2 := B \times B$ such that for any $d \geq 0$ the set $\Gamma_d := \Gamma \cap \{(x, y) \in \mathbb{R}^2 : |x| + |y| \geq d\}$ satisfies the condition (C_1) and (C_2).

As a consequence of Theorem 1.6 we find a set Γ_d of $R_d := \{(x, y) \in \mathbb{R}^2 : |x|+|y| \geq d\}$ of 2-dimensional Lebesgue zero satisfying the condition (C_1) and (C_2), we obtain an asymptotic behavior of $f : \mathbb{R} \to Y$ satisfying

$$\|f(x + y) + f(x - y) - 2f(x)\| \to 0, \tag{1.9}$$

and that of $f : \mathbb{R} \to Y$

$$\|f(x+y) - f(x-y) - 2f(y)\| \to 0, \tag{1.10}$$

as $|x| + |y| \to \infty$ only for (x, y) in a set of Lebesgue measure zero in $\mathbb{R}$.

Theorem 1.7. Suppose that $f : \mathbb{R} \to Y$ satisfies the condition (1.9). Then $f(x) - f(0)$ is an unique additive mapping.

Theorem 1.8. Suppose that $f : \mathbb{R} \to Y$ satisfies the condition (1.10). Then $f(x)$ is an unique additive mapping.

2. Proofs

Proof of Theorem 1.4. Replacing (x, y) by $(x + y, x + y + t)$, (x, y) by $(x, -x + t)$, (x, y) by $(y, y + t)$ and (x, y) by $(0, t)$ in (1.5), respectively. Then we have

$$\|f(2y+t) + f(2x-t) - 2f(x+y)\| \le \epsilon \tag{2.11}$$

$$\|f(t) + f(2x-t) - 2f(x)\| \le \epsilon \tag{2.12}$$

$$\|f(-t) + f(2y+t) - 2f(y)\| \le \epsilon \tag{2.13}$$

$$\|f(t) + f(-t) - 2f(0)\| \le \epsilon \tag{2.14}$$

for all $x, y, t \in X$. Ω satisfies the condition (C_1), it follows from (1.5) that for given $x, y \in X$, there exist $t \in X$ such that (2.1) $\sim$ (2.4). Then we have the following inequality

$$\begin{aligned} &\|2f(x+y) - 2f(x) - 2f(y) + 2f(0)\| \\ &= \|2f(x+y) - f(2y+t) + f(2y+t) - f(2x-t) + f(2x-t) \\ &\quad - 2f(x) + f(t) - f(t) - 2f(y) - f(-t) + f(-t) + 2f(0)\| \\ &\le \|f(2y+t) + f(2x-t) - 2f(x+y)\| + \|f(t) + f(2x-t) - 2f(x)\| \\ &\quad \|f(-t) + f(2y+t) - 2f(y)\| + \|f(t) + f(-t) - 2f(0)\| \le 4\epsilon \end{aligned} \tag{2.15}$$

for all $x, y, t \in X$. Dividing (2.5) by 2 we have

$$\|f(x+y) - f(x) - f(y) + f(0)\| \le 2\epsilon \tag{2.16}$$

for all $x, y \in X$. By theorem 1.1, there exist additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \le 2\epsilon \tag{2.17}$$

for all $x \in X$. This completes the proof. ∎

Proof of Theorem 1.5. Replacing (x, y) by $(-x + y + t, x + y)$, (x, y) by $(-x + t, x)$ and (x, y) by $(y + t, y)$ in (1.6), respectively. Then we have

$$\|f(2y+t) - f(-2x+t) - 2f(x+y)\| \le \epsilon \tag{2.18}$$

$$\|f(t) - f(-2x+t) - 2f(x)\| \le \epsilon \tag{2.19}$$

$$\|f(2y+t) - f(t) - 2f(y)\| \le \epsilon \tag{2.20}$$

for all $x, y, t \in X$. Ω satisfies the condition (C_2), it follows from (1.6) that for given $x, y \in X$, there exist $t \in X$ such that (2.8) $\sim$ (2.10). Then we have the following inequality

$$\begin{aligned}
&\|2f(x+y) - 2f(x) - 2f(y) + 2f(0)\| \\
&= \|2f(x+y) - f(2y+t) + f(2y+t) - f(-2x+t) + f(2x+t) \\
&\quad - 2f(x) + f(t) - f(t) - 2f(y)) + 2f(0)\| \\
&\leq \|f(2y+t) - f(-2x+t) - 2f(x+y))\| + \|f(t) - f(-2x+t) - 2f(x)\| \\
&\quad \|f(2y+t) - f(t) - 2f(y))\| \leq 3\epsilon
\end{aligned} \tag{2.21}$$

for all $x, y, t \in X$. Dividing (2.11) by 2 we have

$$\|f(x+y) - f(x) - f(y)\| \leq \frac{3}{2}\epsilon \tag{2.22}$$

for all $x, y \in X$. By theorem 1.1, there exist additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{3}{2}\epsilon \tag{2.23}$$

for all $x \in X$. This completes the proof. ■

Definition 2.1. A subset K of a topological space E is said to be of the first category if K is a countable union of nowhere dense subsets of E, and otherwise it is said to be of the second category.

Theorem 2.2. (Baire category theorem) Every nonempty open subset of a compact Hausdorff space or a complete metric space is of the second category.

For the proof of the following Lemma 2.3 we refer the reader to [9, Lemma 2.3].

Lemma 2.3. Let B be a subset of $\mathbb{R}$ such that $B^c := \mathbb{R} \setminus B$ is of the first category. Then, for any countable subsets $U \subset \mathbb{R}$, $V \subset \mathbb{R} \setminus \{0\}$ and $M > 0$, there exists $t \geq M$ such that

$$U + tV = \{u + tv : u \in U, v \in V\} \subset B. \tag{2.24}$$

Lemma 2.4. Let

$$R = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

and let $\Gamma = R^{-1}(B \times B)$ be the rotation of $B \times B$ by R^{-1}. Then $\Gamma_d := \Gamma \cap \{(z_1, z_2) \in \mathbb{R}^2 : |z_1| + |z_2| \geq d\}$ satisfies the condition (C_1) and (C_2).

Proof. Let

$$P(x, y, t) = \{(x+y, -x+y+t), (x, -x+t), (y, y+t), (0, t)\} \tag{2.25}$$

for all $x, y, t \in \mathbb{R}$. Then Γ_d satisfies (C_1) if and only if for given $x, y \in \mathbb{R}$, there exists $t \in \mathbb{R}$ such that

$$R(P(x, y, t)) \subset B \times B \tag{2.26}$$

and

$$P(x, y, t) \subset \{(z_1, z_2) \in \mathbb{R}^2 : |z_1| + |z_2| \geq d\}. \tag{2.27}$$

The inclusion (2.16) is equivalent to

$$W_1(x, y, t) := \bigcup_{(z_1,z_2)\in P(x,y,t)} \left\{ \frac{\sqrt{3}}{2} z_1 - \frac{1}{2} z_2,\ \frac{1}{2} z_1 + \frac{\sqrt{3}}{2} z_2 \right\}. \tag{2.28}$$

It is easy to see that

$$W_1(x, y, t) = U + tV \tag{2.29}$$

where

$$\begin{aligned} U = &\left\{0, \frac{\sqrt{3}+1}{2}x, -\frac{\sqrt{3}-1}{2}x, , \frac{\sqrt{3}-1}{2}y, \frac{\sqrt{3}+1}{2}y, \frac{\sqrt{3}+1}{2}x \right. \\ &\left. +\frac{\sqrt{3}-1}{2}y, -\frac{\sqrt{3}-1}{2}x + \frac{\sqrt{3}+1}{2}y \right\}, \\ V = &\left\{ -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\}. \end{aligned}$$

By (2.17) and Lemma 2.3, for given $x, y \in \mathbb{R}$ and $M > 0$ there exists $t \geq M$ such that

$$W_1(x, y, t) \subset B. \tag{2.30}$$

Now, for given x, y and $d \geq 0$ if we choose $M > 0$ such that

$$M \geq d + |x| + |y|, \tag{2.31}$$

then we have

$$P(x, y, t) \subset \{(z_1, z_2) \in \mathbb{R}^2 : |z_1| + |z_2| \geq d\} \tag{2.32}$$

for all $t \geq M$. Thus, it follows from (2.20) and (2.22) that Γ_d satisfies (C_1). Similarly, let

$$Q(x, y, t) = \{(-x + y + t, x + y), (y + t, y), (-x + t, x)\} \tag{2.33}$$

for all $x, y, t \in \mathbb{R}$. Then Γ_d satisfies (C_2) if and only if for given $x, y \in \mathbb{R}$, there exists $t \in \mathbb{R}$ such that

$$R(Q(x, y, t)) \subset B \times B \tag{2.34}$$

and

$$Q(x, y, t) \subset \{(z_1, z_2) \in \mathbb{R}^2 : |z_1| + |z_2| \geq d\}. \tag{2.35}$$

The inclusion (2.16) is equivalent to

$$W_2(x, y, t) := \bigcup_{(z_1,z_2)\in Q(x,y,t)} \left\{ \frac{\sqrt{3}}{2}z_1 - \frac{1}{2}z_2,\ \frac{1}{2}z_1 + \frac{\sqrt{3}}{2}z_2 \right\} = U + tV \subset B, \quad (2.36)$$

for some $t \in \mathbb{R}$, where

$$U = \left\{ -\frac{\sqrt{3}+1}{2}x, \frac{\sqrt{3}-1}{2}x, \frac{\sqrt{3}-1}{2}y, \frac{\sqrt{3}+1}{2}y, -\frac{\sqrt{3}+1}{2}x + \frac{\sqrt{3}-1}{2}y, \frac{\sqrt{3}-1}{2}x + \frac{\sqrt{3}+1}{2}y \right\},$$

$$V = \left\{ \frac{1}{2}, \frac{\sqrt{3}}{2} \right\}.$$

It follows from (2.26) that Γ_d satisfies (C_2). This completes the proof. ■

Remark 2.5. Similarly, appropriate rotation of $2n$-product B^{2n} of B satisfies the condition (C_1) and (C_2) which has $2n$-dimensional Lebesgue measure 0.

Remark 2.6. The set $\mathbb{R}$ of real numbers can be partitioned as

$$\mathbb{R} = B \cup (\mathbb{R} \setminus B)$$

where B is of Lebesgue measure zero and $\mathbb{R} \setminus B$ is of the first category [22, Theorem 1.6]. Thus, in view of Theorem 1.6 we can find a subset $\Gamma_d \subset \{(x, y) \in \mathbb{R}^2 : |x| + |y| \geq d\}$ of Lebesgue measure zero satisfying (C_1) and (C_2).

Now, we obtain the following results.

Theorem 2.7. Suppose that $f : \mathbb{R} \to Y$ satisfies the inequality

$$\|f(x+y) + f(x-y) - 2f(x)\| \leq \epsilon$$

for all $(x, y) \in \Gamma_d$. Then there exists a unique additive mapping $A : \mathbb{R} \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \leq 2\epsilon$$

for all $x \in \mathbb{R}$.

Theorem 2.8. Suppose that $f : \mathbb{R} \to Y$ satisfies the inequality

$$\|f(x+y) - f(x-y) - 2f(y)\| \leq \epsilon$$

for all $(x, y) \in \Gamma_d$. Then there exists a unique additive mapping $A : \mathbb{R} \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{3}{2}\epsilon$$

for all $x \in \mathbb{R}$.

Proof of Theorem 1.7 and Theorem 1.8. The condition (1.9) implies that for each $n \in \mathbb{N}$, there exists $d_n > 0$ such that

$$\|f(x+y) + f(x-y) - 2f(x)\| \le \frac{1}{n} \tag{2.37}$$

for all $(x, y) \in \Gamma_{d_n}$. By Theorem 1.4, there exists a unique additive mapping $A_n : \mathbb{R} \to Y$ such that

$$\|f(x) - A_n(x) - f(0)\| \le \frac{2}{n} \tag{2.38}$$

for all $x \in \mathbb{R}$. Replacing n by positive integers m, k in (2.28) and using the triangle inequality with the results we have

$$\|A_m(x) - A_k(x)\| \le \frac{2}{m} + \frac{2}{k} \le 4 \tag{2.39}$$

for all $x \in \mathbb{R}$. From the additivity of $A_m,\ A_k$, it follows that $A_m = A_k$ for all $m, k \in \mathbb{N}$. Letting $n \to \infty$ in (2.28) we get the result. The proof of Theorem 1.8 is very similar as that of Theorem 1.7. This completes the proof. ■

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