Measure zero stability problem for Jensen type functional equations

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Abstract

In this paper, using the Baire category theorem we investigate the Hyers-Ulam stability problem of Jensen type functional equations on a set of Lebesgue measure zero. As a consequence, we obtain an asymptotic behavior of the equations.

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1. Results

Throughout the paper, we denote by $\mathbb{R}$, $X$ and $Y$ be the set of real numbers, a real normed space and a real Banach space, respectively, $d > 0$ and $\varepsilon \geq 0$ be fixed. A mapping $f : X \rightarrow Y$ is called the Jensen type functional equations if $f$ satisfies the equations

\begin{align}
    f(x + y) + f(x - y) - 2f(x) &= 0, \\
    f(x + y) - f(x - y) - 2f(y) &= 0
\end{align}
for all \( x, y \in X \). A mapping \( f : X \to Y \) is called an additive mapping if \( f \) satisfies 
\[ f(x + y) - f(x) - f(y) = 0 \]
for all \( x, y \in X \). The stability problems for functional equations have been originated by Ulam in 1940 (see [31]). One of the first assertions to be obtained is the following result, essentially due to Hyers [17] that gives an answer to the question of Ulam.

**Theorem 1.1.** Let \( \epsilon > 0 \) be fixed. Suppose that \( f : X \to Y \) satisfies the functional inequality 
\[ \| f(x + y) - f(x) - f(y) \| \leq \epsilon \]
for all \( x, y \in X \). Then there exists a unique additive mapping \( A : X \to Y \) satisfying 
\[ \| f(x) - A(x) \| \leq \epsilon \]
for all \( x \in X \).

The terminology *domain* means a subset of \( X \times X \). Let \( f : X \to Y, d > 0 \) and \( \epsilon \geq 0 \). Among the numerous results on Ulam-Hyers stability theorem for functional equations (e.g. [17], [19], [20], [24], [26], [27], [30]) there are various interesting results which deal with the stability of functional equations in restricted domains ([2], [3], [5], [7], [13], [15], [21], [24]). In particular, J. Chung, D. Kim and J. M. Rassias prove the Ulam-Hyers stability of the Jensen type functional equation (1.1) and (1.2), i.e., the stability of the inequalities

\[ \| f(x + y) + f(x - y) - 2f(x) \| \leq \epsilon, \quad (1.3) \]
\[ \| f(x + y) - f(x - y) - 2f(y) \| \leq \epsilon \quad (1.4) \]

for all \( x, y \in X \) with \( \| x \| + \| y \| \geq d \).

**Theorem 1.2.** Suppose that \( f : X \to Y \) satisfies the functional inequality (1.3) for all \( x, y \in X \) with \( \| x \| + \| y \| \geq d \). Then there exists a unique additive mapping \( A : X \to Y \) such that 
\[ \| f(x) - A(x) - f(0) \| \leq \frac{5}{2} \epsilon \]
for all \( x \in X \).

**Theorem 1.3.** Suppose that \( f : X \to Y \) satisfies the functional inequality (1.4) for all \( x, y \in X \) with \( \| x \| + \| y \| \geq d \) and 
\[ \| f(x) + f(-x) \| \leq 3 \epsilon \]
for all \( x, y \in X \) with \( \| x \| + \| y \| \geq d \). Then there exists a unique additive mapping \( A : X \to Y \) such that 
\[ \| f(x) - A(x) \| \leq \frac{33}{2} \epsilon \]
for all \( x \in X \).
It is very natural to ask if the restricted domain $\Omega_d := \{(x, y) : \|x\| + \|y\| \geq d\}$ in Theorem 1.2 and 1.3 can be replaced by a smaller subset $\Gamma_d \subset \Omega_d$ (e.g. a subset of measure 0 in a measure space $X$)(see [13] for the quadratic functional equation).

In this paper we consider the Ulam-Hyers stability of the Jensen type functional equations (1.1) and (1.2) in restricted domains $\Omega \subset X \times X$ satisfying the condition (C1) and (C2): For given $(x, y) \in X$, there exists a $t \in X$ such that

\begin{align*}
(C1) \quad & \{(x + y, -x + y + t), (x, -x + t), (y, y + t), (0, t)\} \subset \Omega, \\
(C2) \quad & \{(-x + y + t, x + y), (y + t, y), (-x + t, x)\} \subset \Omega,
\end{align*}

respectively.

From now on we assume that $\Omega$ satisfies the condition (C1) and (C2). As main results we prove the following.

**Theorem 1.4.** Suppose that $f : X \to Y$ satisfies the inequality

$$\|f(x + y) + f(x - y) - 2f(x)\| \leq \epsilon$$

(1.5)

for all $(x, y) \in \Omega$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \leq 2\epsilon$$

(1.6)

for all $x \in X$.

**Theorem 1.5.** Suppose that $f : X \to Y$ satisfies the inequality

$$\|f(x + y) - f(x - y) - 2f(y)\| \leq \epsilon$$

(1.7)

for all $(x, y) \in \Omega$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{3}{2}\epsilon$$

(1.8)

for all $x \in X$.

Note that the set $\{(x, y) \in X \times X : \|x\| + \|y\| \geq d\}$ satisfies the condition (C1) and (C2). In particular, if $X = \mathbb{R}$ we prove the following.

**Theorem 1.6.** Let $X = \mathbb{R}$ and $B \subset \mathbb{R}$. Assume that $B^c := \mathbb{R} \setminus B$ is of the first category. Then there is a rotation $\Gamma$ of $B^2 := B \times B$ such that for any $d \geq 0$ the set $\Gamma_d := \Gamma \cap \{(x, y) \in \mathbb{R}^2 : |x| + |y| \geq d\}$ satisfies the condition (C1) and (C2).

As a consequence of Theorem 1.6 we find a set $\Gamma_d$ of $R_d := \{(x, y) \in \mathbb{R}^2 : |x| + |y| \geq d\}$ of 2-dimensional Lebesgue zero satisfying the condition (C1) and (C2), we obtain an asymptotic behavior of $f : \mathbb{R} \to Y$ satisfying

$$\|f(x + y) + f(x - y) - 2f(x)\| \to 0,$$  

(1.9)
and that of \( f : \mathbb{R} \to Y \)
\[
\| f(x + y) - f(x) - 2f(y) \| \to 0, \tag{1.10}
\]
as \(|x| + |y| \to \infty\) only for \((x, y)\) in a set of Lebesgue measure zero in \(\mathbb{R}\).

**Theorem 1.7.** Suppose that \( f : \mathbb{R} \to Y \) satisfies the condition (1.9). Then \( f(x) - f(0) \) is an unique additive mapping.

**Theorem 1.8.** Suppose that \( f : \mathbb{R} \to Y \) satisfies the condition (1.10). Then \( f(x) \) is an unique additive mapping.

## 2. Proofs

**Proof of Theorem 1.4.** Replacing \((x, y)\) by \((x + y, x + y + t)\), \((x, y)\) by \((-x + t, x, y)\) by \((y, y + t)\) and \((x, y)\) by \((0, t)\) in (1.5), respectively. Then we have
\[
\| f(2y + t) + f(2x - t) - 2f(x + y) \| \leq \epsilon \tag{2.11}
\]
\[
\| f(t) + f(2x - t) - 2f(x) \| \leq \epsilon \tag{2.12}
\]
\[
\| f(-t) + f(2y + t) - 2f(y) \| \leq \epsilon \tag{2.13}
\]
\[
\| f(t) + f(-t) - 2f(0) \| \leq \epsilon \tag{2.14}
\]
for all \(x, y, t \in X\). \(\Omega\) satisfies the condition \((C_1)\), it follows from (1.5) that for given \(x, y \in X\) such that (2.1) \sim (2.4). Then we have the following inequality
\[
\| 2f(x + y) - 2f(x) - 2f(y) + 2f(0) \| \leq 4\epsilon \tag{2.15}
\]
for all \(x, y, t \in X\). Dividing (2.5) by 2 we have
\[
\| f(x + y) - f(x) - f(y) + f(0) \| \leq 2\epsilon \tag{2.16}
\]
for all \(x, y \in X\). By theorem 1.1, there exist additive mapping \(A : X \to Y\) such that
\[
\| f(x) - A(x) - f(0) \| \leq 2\epsilon \tag{2.17}
\]
for all \(x \in X\). This completes the proof.

**Proof of Theorem 1.5.** Replacing \((x, y)\) by \((-x + y + t, x + y)\), \((x, y)\) by \((-x + t, x)\) and \((x, y)\) by \((y + t, y)\) in (1.6), respectively. Then we have
\[
\| f(2y + t) - f(-2x + t) - 2f(x + y) \| \leq \epsilon \tag{2.18}
\]
\[
\| f(t) - f(-2x + t) - 2f(x) \| \leq \epsilon \tag{2.19}
\]
\[
\| f(2y + t) - f(t) - 2f(y) \| \leq \epsilon \tag{2.20}
\]
for all \( x, y, t \in X \). \( \Omega \) satisfies the condition \((C_2)\), it follows from (1.6) that for given \( x, y \in X \), there exist \( t \in X \) such that (2.8) \( \sim (2.10) \). Then we have the following inequality

\[
\|2f(x + y) - 2f(x) - 2f(y) + 2f(0)\| \\
= \|2f(x + y) - f(2y + t) + f(2y + t) - f(-2x + t) + f(2x + t) - 2f(x) + f(2x + t) - f(t) - 2f(y) + 2f(0)\| \\
\leq \|f(2y + t) - f(-2x + t) - 2f(x + y)\| + \|f(t) - f(-2x + t) - 2f(x)\| \\
\|f(2y + t) - f(t) - 2f(y)\| \leq 3\epsilon
\]

for all \( x, y, t \in X \). Dividing (2.11) by 2 we have

\[
\|f(x + y) - f(x) - f(y)\| \leq \frac{3}{2}\epsilon
\]

for all \( x, y \in X \). By theorem 1.1, there exist additive mapping \( A : X \to Y \) such that

\[
\|f(x) - A(x)\| \leq \frac{3}{2}\epsilon
\]

for all \( x \in X \). This completes the proof. \( \blacksquare \)

**Definition 2.1.** A subset \( K \) of a topological space \( E \) is said to be of the first category if \( K \) is a countable union of nowhere dense subsets of \( E \), and otherwise it is said to be of the second category.

**Theorem 2.2. (Baire category theorem)** Every nonempty open subset of a compact Hausdorff space or a complete metric space is of the second category.

For the proof of the following Lemma 2.3 we refer the reader to [9, Lemma 2.3].

**Lemma 2.3.** Let \( B \) be a subset of \( \mathbb{R} \) such that \( B^c := \mathbb{R} \setminus B \) is of the first category. Then, for any countable subsets \( U \subset \mathbb{R}, \ V \subset \mathbb{R} \setminus \{0\} \) and \( M > 0 \), there exists \( t \geq M \) such that

\[
U + tV = \{u + tv : u \in U, v \in V\} \subset B.
\]

**Lemma 2.4.** Let

\[
R = \begin{pmatrix}
\sqrt{3} & -1 \\
2 & \frac{2}{3} \\
1 & \frac{\sqrt{3}}{2} \\
\frac{2}{3} & \frac{1}{2}
\end{pmatrix}
\]

and let \( \Gamma = R^{-1}(B \times B) \) be the rotation of \( B \times B \) by \( R^{-1} \). Then \( \Gamma_d := \Gamma \cap \{(z_1, z_2) \in \mathbb{R}^2 : |z_1| + |z_2| \geq d\} \) satisfies the condition \((C_1)\) and \((C_2)\).

**Proof.** Let

\[
P(x, y, t) = \{(x + y, -x + y + t), (x, -x + t), (y, y + t), (0, t)\}
\]
Then \( \Gamma_d \) satisfies \((C_1)\) if and only if for given \( x, y \in \mathbb{R} \), there exists \( t \in \mathbb{R} \) such that
\[
R(P(x, y, t)) \subset B \times B
\] (2.26)
and
\[
P(x, y, t) \subset \{(z_1, z_2) \in \mathbb{R}^2 : |z_1| + |z_2| \geq d\}.
\] (2.27)
The inclusion (2.16) is equivalent to
\[
W_1(x, y, t) := \bigcup_{(z_1, z_2) \in P(x, y, t)} \left\{ \frac{\sqrt{3}}{2} z_1 - \frac{1}{2} z_2, \frac{1}{2} z_1 + \frac{\sqrt{3}}{2} z_2 \right\}.
\] (2.28)
It is easy to see that
\[
W_1(x, y, t) = U + tV
\] (2.29)
where
\[
U = \left\{ 0, \frac{\sqrt{3} + 1}{2} x, -\frac{\sqrt{3} - 1}{2} x, \frac{\sqrt{3} - 1}{2} y, \frac{\sqrt{3} + 1}{2} y, \frac{\sqrt{3} + 1}{2} x + \frac{\sqrt{3} - 1}{2} y \right\},
\]
\[
V = \left\{ -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\}.
\]
By (2.17) and Lemma 2.3, for given \( x, y \in \mathbb{R} \) and \( M > 0 \) there exists \( t \geq M \) such that
\[
W_1(x, y, t) \subset B.
\] (2.30)
Now, for given \( x, y \) and \( d \geq 0 \) if we choose \( M > 0 \) such that
\[
M \geq d + |x| + |y|,
\] (2.31)
then we have
\[
P(x, y, t) \subset \{(z_1, z_2) \in \mathbb{R}^2 : |z_1| + |z_2| \geq d\}
\] (2.32)
for all \( t \geq M \). Thus, it follows from (2.20) and (2.22) that \( \Gamma_d \) satisfies \((C_1)\). Similarly, let
\[
Q(x, y, t) = \{(-x + y + t, x + y), (y + t, y), (-x + t, x)\}
\] (2.33)
for all \( x, y, t \in \mathbb{R} \). Then \( \Gamma_d \) satisfies \((C_2)\) if and only if for given \( x, y \in \mathbb{R} \), there exists \( t \in \mathbb{R} \) such that
\[
R(Q(x, y, t)) \subset B \times B
\] (2.34)
and
\[
Q(x, y, t) \subset \{(z_1, z_2) \in \mathbb{R}^2 : |z_1| + |z_2| \geq d\}.
\] (2.35)
The inclusion (2.16) is equivalent to
\[ W_2(x, y, t) := \bigcup_{(z_1, z_2) \in Q(x, y, t)} \left\{ \frac{\sqrt{3}}{2} z_1 - \frac{1}{2} z_2, \frac{1}{2} z_1 + \frac{\sqrt{3}}{2} z_2 \right\} = U + tV \subset B, \tag{2.36} \]
for some \( t \in \mathbb{R} \), where
\[
U = \left\{ \frac{-\sqrt{3} + 1}{2} x, \frac{-\sqrt{3} - 1}{2} x, \frac{-\sqrt{3} - 1}{2} y, \frac{-\sqrt{3} + 1}{2} y, -\frac{-\sqrt{3} + 1}{2} x \right. \\
\left. + \frac{\sqrt{3} - 1}{2} y, \frac{\sqrt{3} - 1}{2} x + \frac{\sqrt{3} + 1}{2} y \right\},
\]
\[
V = \left\{ \frac{1}{2}, \frac{\sqrt{3}}{2} \right\}.
\]
It follows from (2.26) that \( \Gamma_d \) satisfies \((C_2)\). This completes the proof. \( \blacksquare \)

**Remark 2.5.** Similarly, appropriate rotation of \( 2n \)-product \( B^{2n} \) of \( B \) satisfies the condition \((C_1)\) and \((C_2)\) which has \( 2n \)-dimensional Lebesgue measure 0.

**Remark 2.6.** The set \( \mathbb{R} \) of real numbers can be partitioned as
\[
\mathbb{R} = B \cup (\mathbb{R} \setminus B)
\]
where \( B \) is of Lebesgue measure zero and \( \mathbb{R} \setminus B \) is of the first category [22, Theorem 1.6]. Thus, in view of Theorem 1.6 we can find a subset \( \Gamma_d \subset \{(x, y) \in \mathbb{R}^2 : |x| + |y| \geq d \} \) of Lebesgue measure zero satisfying \((C_1)\) and \((C_2)\).

Now, we obtain the following results.

**Theorem 2.7.** Suppose that \( f : \mathbb{R} \rightarrow Y \) satisfies the inequality
\[
\| f(x + y) + f(x - y) - 2f(x) \| \leq \epsilon
\]
for all \( (x, y) \in \Gamma_d \). Then there exists a unique additive mapping \( A : \mathbb{R} \rightarrow Y \) such that
\[
\| f(x) - A(x) - f(0) \| \leq 2\epsilon
\]
for all \( x \in \mathbb{R} \).

**Theorem 2.8.** Suppose that \( f : \mathbb{R} \rightarrow Y \) satisfies the inequality
\[
\| f(x + y) - f(x - y) - 2f(y) \| \leq \epsilon
\]
for all \( (x, y) \in \Gamma_d \). Then there exists a unique additive mapping \( A : \mathbb{R} \rightarrow Y \) such that
\[
\| f(x) - A(x) \| \leq \frac{3}{2} \epsilon
\]
for all \( x \in \mathbb{R} \).

**Proof of Theorem 1.7 and Theorem 1.8.** The condition (1.9) implies that for each \( n \in \mathbb{N} \), there exists \( d_n > 0 \) such that

\[
\| f(x + y) + f(x - y) - 2f(x) \| \leq \frac{1}{n} \quad (2.37)
\]

for all \((x, y) \in \Gamma_{d_n}\). By Theorem 1.4, there exists a unique additive mapping \( A_n : \mathbb{R} \rightarrow Y \) such that

\[
\| f(x) - A_n(x) - f(0) \| \leq \frac{2}{n} \quad (2.38)
\]

for all \( x \in \mathbb{R} \). Replacing \( n \) by positive integers \( m, k \) in (2.28) and using the triangle inequality with the results we have

\[
\| A_m(x) - A_k(x) \| \leq \frac{2}{m} + \frac{2}{k} \leq 4 \quad (2.39)
\]

for all \( x \in \mathbb{R} \). From the additivity of \( A_m, A_k \), it follows that \( A_m = A_k \) for all \( m, k \in \mathbb{N} \). Letting \( n \to \infty \) in (2.28) we get the result. The proof of Theorem 1.8 is very similar as that of Theorem 1.7. This completes the proof. ■

**References**


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