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An interactive fuzzy programming approach for the two-level linear programming problem with many leaders

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Abstract

In this paper, the two-level linear programming problem with at least two leaders and one follower is considered. To find a satisfactory solution for the problem, an interactive fuzzy programming approach is introduced. The method is aimed so that all the decision makers at the upper level achieve their minimum satisfaction levels together with the suitable satisfaction balance between the decision maker at the lower level and each decision maker at the upper level. So as to exemplify the method, a numerical example is given. Additionally, it is shown that a k-level programming problem can also be changed into a two-level programming problem with *k-1* decision makers at the upper level.

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Keywords: Membership function, Satisfactory solution, Interactive fuzzy programming, Multi-level programming, Optimization.

1. Introduction

The two-level programming problem can be defined as the branch of mathematical programming which is concerned with decision making problems of decentralized organizations with two decision makers ([1],[9],[4],[13], [15] and [16]). In the problem, one decision maker (DM) is at the top level and the other one is at the lower level. The DMs at the top and lower levels are respectively called leader and follower. The leader has higher priority than the follower in order to make the decision. If there exists a cooperative relationship between the leader and the follower and they also come in an agreement to make the decision cooperatively, then the solution obtained under these circumstances is called the satisfactory solution. Otherwise, the solution is named the Stackelberg solution. The vertex enumeration approach [8], the Kuhn-Tucker approach ([5], [6], [8], [11]) and the penalty function approach ([2], [22]) are considered as three broad categories to find Stackelberg solutions to the two-level linear programming problems.

Sakawa et al. [14] introduced an interactive fuzzy programming approach in order to find the satisfactory solutions for the two-level linear problems. In their interactive fuzzy method, the membership functions of the objectives are firstly defined according to the fuzzy goals of the DMs. Then, an initial solution is obtained taking into account the minimal satisfaction degree of the leader. By balance the ratio of the satisfactory degree of the leader to the one of the follower, the satisfactory solution is finally achieved. The satisfactory solution obtained of the approach is also a Pareto optimal solution. This interactive fuzzy programming approach can also be applied to address multi-level programming problems and non-linear two-level programming problems ([14], [18], [19]).

The decentralized two-level programming problem relates to the decentralized organizations with more than one leader or follower. Simaan and Cruz [21] and Anandalingam [2] studied the non-cooperative two-level LPPs with single leader and many followers. In the suggested methods, the leader optimizes the objective of self over a feasible region which is made by the intersection of the inducible regions constructed by the followers separately. Sherali [20] investigated the analysis of the Stackelberg solutions to the non-cooperative two-level linear programming problem with many leaders and single follower.

Sakawa and Nishizaki [19] presented an interactive fuzzy programming approach to solve the cooperative two-level linear programming problem with single leader and multiple followers. In their interactive fuzzy programming approach, the satisfactory solution is achieved in two phases. An interactive fuzzy process was recently introduced in [10] to find satisfactory solution to the two-level linear programming problems with two leaders and one follower. They also demonstrated that a three-level programming problem can be transformed into a two-level programming problem with two DMs at the upper level and one decision maker at the lower level.

In this paper, we apply the interactive fuzzy programming approach to obtain satisfactory solution to the two-level linear programming problem with more than two leaders and one follower when a mutually cooperative relationship exists between the decision makers. In fact, this study relates to the decentralized organization with at least two decision makers at minimum satisfaction levels. In order to achieve an overall satisfaction balance between the decision maker at the lower level and each decision maker at the upper level, the minimum satisfaction levels of the decision makers at the upper level are updated during the algorithm. A numerical example is also given to illustrate the method. It is additionally pointed out that a k-level programming problem can be changed into a two-level programming problem with k-1 decision makers at the upper level. The proposed interactive fuzzy method of this paper can easily be used to solve k-level linear programming problems instead of the rather difficult interactive fuzzy approach introduced in [17].

2. Formulation of the Problem

The general form of the two-level linear programming problem with at least two decision makers at the upper level with a cooperative relationship established among the decision makers is formulated as follows:
P-1

$$\min_{\text{upper level}} Z_1(\mathbf{x}) = \sum_{i=1}^{k} c_{1,i}\mathbf{x}_i$$

$$\min_{\text{upper level}} Z_2(\mathbf{x}) = \sum_{i=1}^{k} c_{2,i}\mathbf{x}_i$$

$$\min_{\text{upper level}} Z_{\text{k-1}}(\mathbf{x}) = \sum_{i=1}^{k} c_{k-1,i}\mathbf{x}_i$$

$$\min_{\text{lower level}} Z_k(\mathbf{x}) = \sum_{i=1}^{k} c_{k,i}\mathbf{x}_i$$

s.t.

$$A_1\mathbf{x}_1 + \cdots + A_k\mathbf{x}_k \leq b,$$

$$\mathbf{x}_1 \geq 0, \ldots, \mathbf{x}_k \geq 0,$$

where, $\mathbf{x}_i$ $i = 1, \ldots, k$ is an n_i-dimensional decision variable, $c_{j,i}$, $j = 1, \ldots, k$, $i = 1, \ldots, k$ is an n_i-dimensional constant row vector, b is an m-dimensional constant column vector, and A_i, $i = 1, \ldots, k$ is an $m \times n_i$ constant matrix.

In the above problem, $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$, $Z_i(\mathbf{x})$, for $i = 1, \ldots, k-1$ represent the objective functions of the upper levels, and the objective function of the lower level is $Z_k(\mathbf{x})$, while $\mathbf{x}_i$, $i = 1, \ldots, k-1$ represent decision variables of the upper levels and $\mathbf{x}_k$ is decision variable of the lower level. First, every DM must define a membership function according to his/her own fuzzy goals in order to apply the interactive fuzzy programming approach to obtain a satisfactory solution to P-1. In this paper, DMi for $i = 1, 2, \ldots, k$ is assumed to select the following linear membership function $\mu_i(Z_i(\mathbf{x}))$ which is a strictly monotonic decreasing function for $Z_i^{\min} \leq Z_i(\mathbf{x}) \leq Z_i^{\max}$.

$$\mu_i(Z_i(\mathbf{x}) = \begin{cases} 0, & Z_i(\mathbf{x}) \geq Z_i^{\max} \\ \dfrac{Z_i(\mathbf{x}) - Z_i^{\max}}{Z_i^{\min} - Z_i^{\max}}, & Z_i^{\min} \leq Z_i(x) \leq Z_i^{\max} \\ 1, & Z_i(\mathbf{x}) \leq Z_i^{\min} \end{cases} \tag{1}$$

Let $\mathbf{X}$ be the feasible region of P-1. In the above definition, $Z_i^{\min}$ and $Z_i^{\max}$ are fuzzy goals for i^{th} decision maker and can be specified by the following rules: for $i = 1, \ldots, k$

$$Z_i^{\min} = min\,\{Z_i(\mathbf{x}) : \mathbf{x} \in \mathbf{X}\} \tag{2}$$

Let $\mathbf{x}^i$ for $i = 1, \ldots, k$ be the solution of i^{th} problem in Eq. 2; subsequently, the suggested method of Zimmermann [23] can be used to find $Z_i^{\max}$ for $i = 1, \ldots, k$, as follows:
for $i = 1, \ldots, k$

$$Z_i^{\max} = \max\left\{Z_j(\mathbf{x}^j)\text{for}j = 1, \ldots, k,\ j \neq i\right\} \tag{3}$$

According to the fuzzy decision making theory of Bellman and Zadeh [7], the following addresses P-1.

$$\text{Maximize}\,\{\min \mu_i(Z_i(\mathbf{x}))\text{for}i = 1, \ldots, k\}$$

s.t.

$$A_1\mathbf{x}_1 + \cdots + A_k\mathbf{x}_k \leq b,$$

$$\mathbf{x}_1 \geq 0, \ldots, \mathbf{x}_k \geq 0.$$

The above problem can be transformed into the following equivalent problem using auxiliary variable λ:
P-2

$$\text{Maximize } \lambda$$

s.t.

$$\mu_1(Z_1(\mathbf{x})) \geq \lambda,$$

$$\mu_2(Z_2(\mathbf{x})) \geq \lambda,$$

$$\mu_k(Z_k(\mathbf{x})) \geq \lambda,$$

$$0 \leq \lambda \leq 1,$$

$$A_1\mathbf{x}_1 + \cdots + A_k\mathbf{x}_k \leq b,$$

$$\mathbf{x}_1 \geq 0, \ldots, \mathbf{x}_k \geq 0.$$

If all the decision makers as DM1, DM2,…, and DM$k-1$ at the upper level are satisfied with the optimal solution $\mathbf{x}^\star$ of the above problem, then it is concluded that $\mathbf{x}^\star$ becomes a satisfactory solution; otherwise, DM1, DM2,…, and DM$k-1$ specify the minimum of the decision makers satisfaction levels with full knowledge of the membership function value of the decision maker at the lower level. If $\hat{\delta}_1, \hat{\delta}_2, \ldots,$ and $\hat{\delta}_{k\text{-}1}$ are the minimum satisfaction levels specified by DM1, DM2,…, and DM$k-1$, respectively, then the following problem must be solved to obtain a solution for which DM1, DM2,…, and DM$k-1$ are satisfied.

P-3

$$\text{Maximize} \qquad \mu_k\,(Z_k(\mathbf{x}))$$

s.t.

$$\begin{aligned} \mu_1(Z_1(\mathbf{x})) &\geq \hat{\delta}_1, \\ \mu_2(Z_2(\mathbf{x})) &\geq \hat{\delta}_2, \\ &\vdots \\ \mu_{k\text{-}1}\,(Z_{k\text{-}1}(\mathbf{x})) &\geq \hat{\delta}_{k\text{-}1} \\ A_1\mathbf{x}_1 + \cdots + A_k\mathbf{x}_k \leq b, \mathbf{x}_1 &\geq 0, \cdots, \mathbf{x}_k \geq 0. \end{aligned}$$

Lemma 2.1. The solutions of P-2 and P-3 are Pareto optimal solutions.

Proof. The proof is given here only for P-3. The proof of P-2 is similar and is thus omitted. Let $\mathbf{x}^l$be an unique optimal solution of P-3 obtained in iteration l. If $\mathbf{x}^l$ is not a Pareto optimal solution of P-1. Therefore, there exists feasible point $\tilde{\mathbf{x}}$ such that $Z_j(\tilde{\mathbf{x}}) < Z_j(\mathbf{x}^l)$ for some j and $Z_i(\tilde{\mathbf{x}}) \leq Z_i(\mathbf{x}^\star)$, for $i = 1, \ldots, k, i \neq j$. Due to the fact that $\mu_i(Z_i(\mathbf{x}))$ is a monotone decreasing function for $Z_i(\mathbf{x})$, accordingly, we have: $\mu_i(Z_i(\tilde{\mathbf{x}})) \geq \mu_i(Z_i(\mathbf{x}^l)) \geq \hat{\delta}_i$, for $i = 1, \ldots, k, i \neq j$ and $\mu_j(Z_j(\tilde{\mathbf{x}})) > \mu_j(Z_j(\mathbf{x}^l)) \geq \hat{\delta}_j$. This is a contradiction to uniqueness optimality of $\mathbf{x}^l$ for P-3. ∎

With assumption that the objective functions at the both levels conflict with one another, the obtained satisfaction level for DM1, DM2,..., and DM*k-1* at $\mathbf{x}^*$ achieved from P-3 decreases the satisfaction level of the DM*k*, and consequently, this reduction may not be desirable for the DM*k* at the lower level who acts in cooperation with DM1, DM2,..., and DM*k-1*. To overcome this problem, Lai [12] introduced the ratio $\Delta = \frac{\mu_{\text{lower level}}}{\mu_{\text{upper level}}}$ to adjust the satisfaction levels between DMs for two-level linear programming problems with one decision maker at both levels.

Since k-1 decision makers exist at the upper level, therefore, *k-1* ratios $\Delta_1 = \frac{\mu_k(Z_k(\mathbf{x}))}{\mu_1(Z_1(\mathbf{x}))}$, $\Delta_2 = \frac{\mu_k(Z_k(\mathbf{x}))}{\mu_2(Z_2(\mathbf{x}))}, \ldots,$ and $\Delta_{k\text{-}1} = \frac{\mu_k(Z_k(\mathbf{x}))}{\mu_{k\text{-}1}(Z_{k\text{-}1}(\mathbf{x}))}$ must be considered to adjust the satisfactory levels between DMs for two-level linear programming problems. Let $[\Delta_{L1}, \Delta_{U1}]$, $[\Delta_{L2}, \Delta_{U2}], \ldots,$ and $[\Delta_{Lk\text{-}1}, \Delta_{Uk\text{-}1}]$ be the desirable domains for $\Delta_1, \Delta_2, \ldots, \Delta_{k\text{-}1}$ specified by DM1, DM2, and DM *k-1*, respectively. Accordingly, a solution $\mathbf{x}$ of P-3 will become satisfactory if the following relations hold true:

$$\Delta_1 \in [\Delta_{L1}, \Delta_{U1}], \Delta_2 \in [\Delta_{L2}, \Delta_{U2}], \ldots, \text{and} \quad \Delta_{k\text{-}1} \in [\Delta_{Lk\text{-}1}, \Delta_{Uk\text{-}1}]. \tag{4}$$

If at least one of the above relations does not hold, then the values of minimal satisfactory levels of the decision makers at the upper level must be updated to obtain a satisfactory solution. Updating the values of minimum satisfaction levels to confirm the above relations is not easy in practice. For example, for k=3, the following procedure, which is derived directly from [12], needs to be considered to update the values of$\hat{\delta}_1$ and $\hat{\delta}_2$.

[Procedure 1 to update the values of $\hat{\delta}_1$ and $\hat{\delta}_2$]

If no feasible solution exists for P-3 for minimal satisfaction levels $\hat{\delta}_1$ and $\hat{\delta}_2$, then DM1 and DM2 decrease their value of $\hat{\delta}_1$ and $\hat{\delta}_2$, respectively.
If $\Delta_{31} < \Delta_{L31}$ and $\Delta_{32} \in [\Delta_{L32}, \Delta_{U32}]$, the value of $\hat{\delta}_1$ is increased by DM1.
If $\Delta_{31} > \Delta_{U31}$ and $\Delta_{32} \in [\Delta_{L32}, \Delta_{U32}]$, DM1 decreases the value of $\hat{\delta}_1$.
If $\Delta_{32} < \Delta_{L32}$ and $\Delta_{31} \in [\Delta_{L31}, \Delta_{U31}]$, DM2 increases the value of $\hat{\delta}_2$.
If $\Delta_{32} > \Delta_{U32}$ and $\Delta_{31} \in [\Delta_{L31}, \Delta_{U31}]$, DM2 decreases the value of $\hat{\delta}_2$.
If $\Delta_{31} < \Delta_{L31}$ and $\Delta_{32} < \Delta_{L32}$, DM1 and DM2 increase the values of $\hat{\delta}_1$ and $\hat{\delta}_2$, respectively.
If $\Delta_{31} > \Delta_{U31}$ and $\Delta_{32} > \Delta_{U32}$, the values of $\hat{\delta}_1$ and $\hat{\delta}_2$ are decreased by DM1 and DM2, respectively.
If $\Delta_{31} < \Delta_{L31}$ and $\Delta_{32} > \Delta_{U32}$, the value of $\hat{\delta}_1$ is increased by DM1 and the value of $\hat{\delta}_2$ is decreased by DM2.
If $\Delta_{31} > \Delta_{U31}$ and $\Delta_{32} < \Delta_{L32}$, the value of $\hat{\delta}_1$ is decreased by DM1and the value of $\hat{\delta}_2$ is increased by DM2.

It is not difficult to show that for a problem of M decision makers at the upper level, 3^M comparisons are needed to update values $\hat{\delta}_M, \ldots, \hat{\delta}_1$. Therefore, doing updates for rather big M is not easy and encompasses some difficulties in practice.

Presenting an easier procedure is the main idea of this paper. To do this, with the assumption that $\mathbf{x}^p$ is the optimal solution of P-3 at iteration p, we subsequently define:

$$[\Delta_L, \Delta_U] = [\Delta_{L1}, \Delta_{U1}] \cap [\Delta_{L2}, \Delta_{U2}] \cap \cdots \cap [\Delta_{Lk-1} \Delta_{Uk-1}], \tag{5}$$

$$\Delta^p_{\max} = \frac{\mu_k(Z_k(\mathbf{x}^p))}{\min\{\mu_1(Z_1(\mathbf{x}^p)), \mu_2(Z_2(\mathbf{x}^p)), \ldots, \mu_{k-1}(Z_{k-1}(\mathbf{x}^p))\}}, \tag{6}$$

$$\Delta^p_{\min} = \frac{\mu_k(Z_k(\mathbf{x}^p))}{\max\{\mu_1(Z_1(\mathbf{x}^p)), \mu_2(Z_2(\mathbf{x}^p)), \ldots, \mu_{k-1}(Z_{k-1}(\mathbf{x}^p))\}}. \tag{7}$$

It is resulted from definitions of Δ^l_{max} and Δ^l_{min} that if $\Delta^l_{max} \in [\Delta_L, \Delta_U]$ and $\Delta^l_{\min} \in [\Delta_L, \Delta_U]$, then $\Delta_i \in [\Delta_{Li}, \Delta_{Ui}]$ for $i = 1, \ldots, k-1$.

According to (5), (6), and (7), DM1, DM2,..., DM*k-1* must update the levels based on the following procedure:

[Procedure 2 to update the minimal satisfactory levels $\hat{\delta}_1, \hat{\delta}_2, \ldots,$ and $\hat{\delta}_{k-1}$]

If there is not a feasible solution to P-3 for a minimal satisfactory level $\hat{\delta}_1, \hat{\delta}_2, \ldots,$ and $\hat{\delta}_{k-1}$, DM1, DM2,..., DM *k-1* decrease their values of $\hat{\delta}_1, \hat{\delta}_2, \ldots,$ and $\hat{\delta}_{k-1}$, respectively.
If $\Delta^p_{\max} > \Delta_U$, the decision maker at the upper level with a minimum value of membership function must increase his/her minimum satisfaction level.
If $\Delta^p_{\min} < \Delta_L$, the decision maker at the upper level with a maximum value of membership function must decrease his/her minimum satisfaction level.
If $\Delta^p_{\max} < \Delta_L$, all decision makers at the upper level must decrease their values of the minimum satisfaction level.

If $\Delta_{\min}^{p} > \Delta_U$, all decision makers at the upper level must increase their values of the minimum satisfaction level.

In iteration p, P-3 has to be resolved for the updated values of $\hat{\delta}_1, \hat{\delta}_2, \ldots,$ and $\hat{\delta}_{k\text{-}1}$, and the ratios of $\Delta_{\max}^{p}$ and $\Delta_{\min}^{p}$ must be checked again at the solution obtained.

At the iteration p, let $\mu_1(z_1^p), \mu_2(Z_2^p), \ldots, \mu_k(Z_k^p)$,

$$\Delta_{max}^{p} = \frac{\mu_k(Z_k^p)}{\min\{\mu_1(z_1^p), \mu_2(Z_2^p), \ldots, \mu_{k-1}(Z_{k-1}^p)\}}$$

and

$$\Delta_{min}^{p} = \frac{\mu_k(Z_k^p)}{\max\{\mu_1(z_1^p), \mu_2(Z_2^p), \ldots, \mu_{k-1}(Z_{k-1}^p)\}}$$

denote the satisfaction degrees of DM1, DM2,…, and DMk, the ratio of satisfaction degree of the follower to the minimum satisfaction degree of the leaders, and the ratio of the follower to the maximum value of the satisfaction degree of the leaders, respectively. The interactive process will be terminated if the following two conditions are satisfied.

[Termination conditions of interactive process]
Condition 1

$$\mu_1(Z_1^p) \geq \hat{\delta}_1, \mu_2(Z_2^p) \geq \hat{\delta}_2, \ldots, \mu_{k\text{-}1}(Z_{k\text{-}1}^p) \geq \hat{\delta}_{k\text{-}1}.$$

Condition 2

$$\Delta_{\max}^{p} \in [\Delta_L, \Delta_U]$$

and

$$\Delta_{\min}^{p} \in [\Delta_L, \Delta_U].$$

3. *k*-level programming problems

In this section, it is illustrated that a k-level programming problem can be transformed into a two-level programming problem with *k-1* decision makers at the upper level. When a mutual cooperative relationship exists between decision makers, the general form of a *k*-level programming problem is as following:
P-4

$$\min_{\text{for DM1}} Z_1(\mathbf{x})$$

$$\min_{\text{for DM2}} Z_2(\mathbf{x})$$

$$\vdots$$

$$\min_{\text{for DM}k} Z_k(\mathbf{x})$$

s.t $\quad \mathbf{x} \in X$

In the above problem, DM1, DM2,..., DM*k-1* have minimum satisfaction levels of $\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_{k\text{-}1}$, respectively. Additionally,

$$\Delta_i = \frac{\mu_{i+1}(Z_{i+1}(\mathbf{x}))}{\mu_i(Z_i(\mathbf{x}))}, i = 1, ..., k-1, \tag{8}$$

is the ratio of decision makers satisfactory degrees in adjacent two levels. If DM*i*, $i = 1, \ldots, k-1$ specifies the lower bound Δ_{Li} and the upper bound Δ_{Ui} of the ratio Δ_i, the following condition can be presented in which the overall satisfactory balance is appropriate.

$$\Delta_i \in [\Delta_{Li}, \Delta_{Ui}], i = 1, \ldots, k-1. \tag{9}$$

According to the above descriptions, the interactive fuzzy process to solve P-4 will be terminated if the following conditions hold true:

Condition 1:

$$\mu_1(Z_1) \geq \hat{\delta}_1, \ldots, \mu_{k\text{-}1}(Z_{k\text{-}1}) \geq \hat{\delta}_{k\text{-}1}.$$

Condition 2:

$$\Delta_1 \in [\Delta_{L1}, \Delta_{U1}], \ldots, \Delta_{k\text{-}1} \in [\Delta_{Lk\text{-}1}, \Delta_{Uk\text{-}1}].$$

Now, to transform the above problem into the two-level programming problem, DM1, DM2,..., DM*k-1* that respectively have minimal satisfactory levels of $\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_{k\text{-}1}$ are considered as decision makers at the upper level and DM*k* who does not have a minimum satisfaction level of self is taken into account as the follower. To achieve an overall satisfactory balance between DM*k* at the lower level and each decision maker at the upper level the ratio of $\frac{\mu_k(Z_k(\mathbf{x})}{\mu_i(Z_i(\mathbf{x})}$, $i = 1, \ldots, k-1$ must belong to the interval $[\tilde{\Delta}_{Li}, \tilde{\Delta}_{Ui}]$. According to (8), and (9) the lower bound $\tilde{\Delta}_{Li}$ and the upper bound $\tilde{\Delta}_{Ui}$, $i = 1, \ldots, k-1$ are computed as following:

$$\tilde{\Delta}_{Lk\text{-}1} = \Delta_{Lk\text{-}1} \quad \text{and} \quad \tilde{\Delta}_{Uk\text{-}1} = \Delta_{Uk\text{-}1}, \tag{10}$$

$$\tilde{\Delta}_{Ln} = \Delta_{Ln} \times \cdots \Delta_{k\text{-}1} \quad \text{and} \quad \tilde{\Delta}_{Un} = \Delta_{Un} \times \cdots \times \Delta_{k\text{-}1}, n = 1, \ldots, k-2. \tag{11}$$

According to the above-mentioned description, P-4 is transformed into the two-level programming problem including *k-1* decision makers at the upper level and one decision maker at the lower level with the following termination conditions which can be solved by the use of the interactive fuzzy process introduced in the previous section.

Condition 1:

$$\mu_1(Z_1) \geq \hat{\delta}_1, \ldots, \mu_{k\text{-}1}(Z_{k\text{-}1}) \geq \hat{\delta}_{k\text{-}1}.$$

Condition 2:

$$\Delta_{\max} \in [\Delta_L, \Delta_U]$$

and

$$\Delta_{\min} \in [\Delta_L, \Delta_U],$$

where

$$[\Delta_L, \Delta_U] = [\tilde{\Delta}_{L1}, \tilde{\Delta}_{U1}] \cap [\tilde{\Delta}_{L2}, \tilde{\Delta}_{U2}] \cap \cdots \cap [\tilde{\Delta}_{Lk\text{-}1}, \tilde{\Delta}_{Uk\text{-}1}].$$

4. Numerical example

The following two-level linear programming problem with three decision makers at the upper level is solved to illustrate the proposed method. All data are derived from [17].
P-5

$$\min_{\text{upper level}} \sum_{i=1}^{10} C_{1\text{i}} x_i$$

$$\min_{\text{upper level}} \sum_{i=1}^{10} C_{2\text{i}} x_i$$

$$\min_{\text{upper level}} \sum_{i=1}^{10} C_{3\text{i}} x_i$$

$$\min_{\text{lower level}} \sum_{i=1}^{10} C_{4\text{i}} x_i$$

s.t $\quad A_1 x_1 + \cdots + A_{10} x_{10} \le b, \qquad x_i \ge 0$ for $i = 1, \ldots, 10.$
The coefficients of the above problem are listed in Table 1.

Four individual minimization problems of the four decision makers are solved at the beginning of the procedure in order to identify membership functions of the fuzzy goals for the objective functions. The individual minima and the corresponding optimal solution are shown in Table 2.

Suppose that the decision makers employ the linear membership function (1) whose parameters are determined by relations (2) and (3). According to the values $(Z_1^{\min}, Z_1^{\max}) = (43.016, 131.994)$, $(Z_2^{\min}, Z_2^{\max}) = (23.387, 96.824)$, $(Z_3^{\min}, Z_3^{\max}) = (28.387, 56.319)$, and $(Z_4^{\min}, Z_4^{\max}) = (-33.594, 60.046)$, maximization P-2 for this problem is written as P-6:

Table 1: The coefficients of P-5

i	1	2	3	4	5	6	7	8	9	10		
c_{1i}	15	−46	1	34	−30	42	−18	39	46	25		
c_{2i}	26	27	6	1	49	−16	−45	18	41	−40		
c_{3i}	14	36	17	−5	−26	17	37	6	25	−4		
c_{4i}	23	14	12	−10	27	−11	−14	−6	−27	−7		
A_i	−43	−4	38	−45	−18	−6	25	46	48	−20	b	13
	25	−39	−38	9	47	−32	26	−45	−1	17		−20
	−39	−3	33	−27	34	−26	20	43	−29	−9		−1
	38	19	−22	−35	39	−21	30	41	34	−38		55
	13	−19	45	17	−47	10	33	−40	−5	−30		−14
	−41	49	−10	−19	−22	−23	−36	−49	−11	4		−102
	−41	38	−9	−11	12	−9	48	14	13	−39		10
	−32	−48	48	30	−16	29	−3	−35	−38	−43		−70
	−38	−11	−48	−5	41	19	−36	−28	11	−34		−83
	−26	−30	38	−36	41	−41	−33	−18	7	22		−49
	−28	40	28	−9	46	23	10	7	−44	6		51
	32	−10	18	−37	−25	36	−9	−26	34	16		18
	−47	−38	38	−7	−40	−35	−27	10	−15	36		−81
	−6	−34	−3	2	7	48	−34	16	18	26		26
	−44	−11	39	−23	−43	0	−42	−28	−29	−9		−123
	11	3	−36	25	12	3	42	25	6	47		89

Table 2: Optimal solutions to the individual problems

x_1 1.171	x_2 0	x_3 0	x_4 0	x_5 0.575
x_6 0	x_7 0.343	x_8 0.918	x_9 0	x_{10} 0
$Z_1^{\min}$43.016				$Z_1^{\max}$131.994
x_1 1.232	x_2 0	x_3 0	x_4 0	x_5 0.167
x_6 0.789	x_7 0.387	x_8 0.808	x_9 0	x_{10} 0
$Z_2^{\min}$23.387				$Z_2^{\max}$98.824
x_1 0.986	x_2 0	x_3 0	x_4 0	x_5 0.688
x_6 0	x_7 0	x_8 1.146	x_9 0	x_{10} 0
$Z_3^{\min}$28.387				$Z_3^{\max}$56.319
x_1 0.705	x_2 0	x_3 0	x_4 0	x_5 0
x_6 0.256	x_7 0	x_8 1.188	x_9 1.399	x_{10} 0
$Z_4^{\min}$−33.594				$Z_4^{\max}$60.046

P-6

$$\text{Maximize} \quad \lambda$$

s.t.

$$\frac{(Z_1(\mathbf{x}) - 131.994)}{(43.016 - 131.994)} \geq \lambda,$$
$$\frac{(Z_2(\mathbf{x}) - 96.824)}{(23.387 - 96.824)} \geq \lambda,$$
$$\frac{(Z_3(\mathbf{x}) - 56.319)}{(28.387 - 56.319)} \geq \lambda,$$
$$\frac{(Z_4(\mathbf{x}) - 60.046)}{(-33.594 - 60.046)} \geq \lambda,$$
$$\mathbf{x} \in S$$

where S denotes the feasible region of P-5. The result of the first iteration including an optimal solution to P-6 is shown in Table 3.

Table 3: Iteration 1 of the interactive fuzzy process

x_1	1.079	x_2	0.527	x_3	0	x_4	1.223	x_5	0.163
x_6	0.179	x_7	0.164	x_8	1.038	x_9	0.117	x_{10}	0.191
Z_1^1	83.84	Z_2^1	57.09	Z_3^1	41.225	Z_4^1	9.377		
$\mu_1(Z_1^1)$	0.541	$\mu_2(Z_2^1)$	0.541	$\mu_3(Z_3^1)$	0.543	$\mu_4(Z_4^1)$	0.543		

Suppose that leaders are not satisfied with the solution obtained in iteration 1 and taking the result of the first iteration into account, decision makers DM1, DM2 and DM3 specify the minimal satisfactory levels at $\hat{\delta_1} = 0.7, \hat{\delta_2} = 0.6$ and $\hat{\delta_3} = 0.6$, respectively. Moreover, suppose that DM1, DM2 and DM3 specify $[\Delta_{L1}, \Delta_{U1}] = [0.6, 0.8]$,$[\Delta_{L2}, \Delta_{U2}] = [0.6, 0.9]$ and $[\Delta_{L3}, \Delta_{U3}] = [0.5, 0.9]$ considering the result of the first iteration. According to (5), $[\Delta_L, \Delta_U] = [0.6, 0.8]$. The problem with the minimal satisfactory levels 3 is therefore written as P-7:

P-7

$$\text{Maximize} \quad \mu_4(Z_4(\mathbf{x}))$$

s.t.

$$\frac{(Z_1(\mathbf{x}) - 131.994)}{(43.016 - 131.994)} \geq 0.7,$$
$$\frac{(Z_2(\mathbf{x}) - 96.824)}{(23.387 - 96.824)} \geq 0.6,$$
$$\frac{(Z_3(\mathbf{x}) - 56.319)}{(28.387 - 56.319)} \geq 0.6,$$
$$\mathbf{x} \in S.$$

Table 4: Iteration 2 of the interactive fuzzy process

x_1	1.092	x_2	0.599	x_3	0.141	x_4	0.805	x_5	0.428
x_6	0.408	x_7	0.124	x_8	0.993	x_9	0	x_{10}	0.505
Z_1^2	69.753	Z_2^2	52.754	Z_3^2	39.558	Z_4^2	22.983		
$\mu_1(Z_1^2)$	0.7	$\mu_2(Z_2^2)$	0.6	$\mu_3(Z_3^2)$	0.6	$\mu_4(Z_4^2)$	0.398		
Δ_{max}^2	0.66							Δ_{min}^2	0.566

The result of the second iteration including an optimal solution to P-7 is shown in Table 4.

At the second iteration, the satisfactory degrees $\mu_1(Z_1^2)$=0.7 of DM1, $\mu_2(Z_2^2)$=0.6 of DM2 and $\mu_3(Z_3^2)$ =0.6 of DM3 become equal to their minimal satisfactory levels 0.7, 0.6, and 0.6, respectively. But the ratio Δ_{min}^2=0.566 is not in the interval of [0.6,0.8]. Consequently, this solution does not satisfy the second condition of termination of the interactive process. Since $\Delta_{min}^2 < \Delta_L$, the satisfactory level of DM1 must therefore be reduced. Suppose DM1 updates its minimal satisfaction level at $\hat{\delta_1}$=0.67, then, the problem with the revised minimal satisfactory levels 3 is formulated as P-8 and the result of the third iteration is shown in Table 5.

P-8

$$\text{Maximize} \qquad \mu_4(Z_4(\mathbf{x}))$$

s.t.

$$\frac{(Z_1(\mathbf{x}) - 131.994)}{(43.016 - 131.994)} \geq 0.67,$$
$$\frac{(Z_2(\mathbf{x}) - 96.824)}{(23.387 - 96.824)} \geq 0.6,$$
$$\frac{(Z_3(\mathbf{x}) - 56.319)}{(28.387 - 56.319)} \geq 0.6,$$
$$\mathbf{x} \in S.$$

Table 5: Iteration 3 of the interactive fuzzy process

x_1	1.059	x_2	0.576	x_3	0.002	x_4	1.118	x_5	0.218
x_6	0.2	x_7	0.178	x_8	1.034	x_9	0	x_{10}	0.239
Z_1^3	72.36	Z_2^3	52.74	Z_3^3	39.572	Z_4^3	14.582		
$\mu_1(Z_1^3)$	0.67	$\mu_2(Z_2^3)$	0.6	$\mu_3(Z_3^3)$	0.6	$\mu_4(Z_4^3)$	0.485		
Δ_{max}^3	0.809							Δ_{min}^3	0.724

At the third iteration, the ratio $\Delta_{\max}^3 = 0.809$ is not in the interval of [0.6,0.8]. Therefore, this solution does not satisfy the second condition of termination of the interactive process. Since $\Delta^3_{\max} > \Delta_U$, DM2 and DM3 who have minimum values of membership functions must increase their minimum satisfaction levels. Suppose DM1 and DM2 respectively specify $\hat{\delta}_1 = 0.61$ and $\hat{\delta}_2 = 0.62$ as their minimum satisfaction levels, then, the problem with the revised minimal satisfactory levels 3 is formulated as P-9 and the result of the fourth iteration is listed in Table 6.

P-9

$$\text{Maximize} \qquad \mu_4(Z_4(\mathbf{x}))$$

s.t.

$$\frac{(Z_1(\mathbf{x}) - 131.994)}{(43.016 - 131.994)} \geq 0.67,$$
$$\frac{(Z_2(\mathbf{x}) - 96.824)}{(23.387 - 96.824)} \geq 0.61,$$
$$\frac{((Z_3(\mathbf{x}) - 56.319)}{(28.387 - 56.319)} \geq 0.62,$$
$$\mathbf{x} \in S.$$

Table 6: Iteration 4 of the interactive fuzzy process

x_1	1.126	x_2	0.584	x_3	0.11	x_4	0.771	x_5	0.4
x_6	0.424	x_7	0.095	x_8	0.995	x_9	0	x_{10}	0.523
Z^4_1	72.328	Z^4_2	52.006	Z^4_3	39.004	Z^4_4	22.859		
$\mu_1(Z^4_1)$	0.671	$\mu_2(Z^4_2)$	0.61	$\mu_3(Z^4_3)$	0.62	$\mu_4(Z^4_4)$	0.397		
Δ^4_{max}	0.651							Δ^4_{min}	0.592

At the fourth iteration, Δ^4_{min} is not in the interval of [0.6, 0.8]. As a consequence, the solution does not satisfy the second condition of termination of the interactive process. Since $\Delta^4_{min} < \Delta_L$, DM1 must decrease its minimum satisfaction level. If DM1 specify $\hat{\delta}_1 = 0.66$ as his/her minimum satisfaction level, P-3 for the revised minimum satisfaction levels is then formulated as P-10 and the results of the fifth iteration are shown in Table 7.

P-10

$$\text{Maximize} \qquad \mu_4(Z_4(\mathbf{x}))$$

s.t.

$$\frac{(Z_1(\mathbf{x}) - 131.994)}{(43.016 - 131.994)} \geq 0.66,$$
$$\frac{(Z_2(\mathbf{x}) - 96.824)}{(23.387 - 96.824)} \geq 0.61,$$

$$\frac{(Z_3(\mathbf{x}) - 56.319)}{(28.387 - 56.319)} \geq 0.62,$$
$$\mathbf{x} \in S.$$

Table 7: Iteration 5 of the interactive fuzzy process

x_1	1.117	x_2	0.576	x_3	0.063	x_4	0.875	x_5	0.33
x_6	0.355	x_7	0.113	x_8	1.009	x_9	0	x_{10}	0.434
Z_1^5	73.249	Z_2^5	52.054	Z_3^5	39.024	Z_4^5	20.092		
$\mu_1(Z_1^5)$	0.66	$\mu_2(Z_2^5)$	0.61	$\mu_3(Z_3^5)$	0.62	$\mu_4(Z_4^5)$	0.427		
$\Delta_{\max}^5$	0.7							$\Delta_{\min}^5$	0.647

At the fifth iteration, since the satisfactory degrees $\mu_1(Z_1^5) = 0.66$ of DM1, $\mu_2(Z_2^5) = 0.61$ of DM2 and $\mu_3(Z_3^5) = 0.62$ of DM3 become equal to their minimal satisfactory levels 0.66, 0.61, and 0.62, respectively, and $\Delta_{\max}^5 = 0.7$ and $\Delta_{min}^5 = 0.647$ are in the interval of [0.6,0.7], it is therefore concluded that the obtained solution is a satisfactory solution and the interactive fuzzy stops.

5. Conclusion

In this paper, a method based on interactive fuzzy programming approach is introduced to obtain a Pareto satisfactory solution to the two-level linear programming problems with at least two decision makers at the upper level when a mutual cooperative relationship exists between the decision makers. In the method, All the leaders are satisfied with the satisfactory solution and appropriate coordination exists between the satisfaction degree of the follower and that of each leader. It is shown that this interactive fuzzy process can also be applied to the multi-level linear programming problems.

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